Masonry strength domain by homogenization in generalized plane state

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SUMMARY. This paper deals with the evaluation of the strength domain for non-periodic masonry using a random media micromechanical approach. The strength domain of the homogenized continuum is evaluated through the use of the hierarchy theory related to partitions with increasing size and using a failure criterion based on the mean stress state of each phase. The generalized plane state formulation, employed in the numerical models, allows to take into account the out-of-plane stresses and their effects on failure mechanisms. As a benchmark, the proposed procedure is applied to a sample of periodic masonry subjected to biaxial stress states: the strength domains obtained are in good agreement with the experimental ones; furthermore, the numerical model correctly reproduces the main failure mechanisms. Then, the application of the procedure to an actual non-periodic masonry allows to evaluate its strength surface and to verify the convergence of the domains with the increase in size of the portions of the partitions.

1 INTRODUCTION

During the past thirty years, several investigations were carried out with the aim to determine the behavior of various kinds of masonry in different states of stress. The complex aspects of such a kind of study are due to the combined effect of the anisotropic overall response and the nonlinearities of the constituents and of the interface. These surveys were directed on one hand to determine masonry failure criteria in an analytical form, i.e. with the use of mathematical relations, employing limit analysis or models based on elastic stress distributions [1, 2, 3]; on the other hand, they exploited the results of experimental tests. For what concerns the latter kind of investigation, it is well-known that the most thorough testing programme on biaxial behavior of masonry was carried out by Page [4, 5]. The tests and the subsequent elaborations [6] allowed to establish failure surfaces for brickwork loaded in orthogonal biaxial compression and tension-compression states of principal stresses.

Furthermore, the qualitative observation of experimental results [7] permitted to single out different kinds of failure mechanisms. In particular, when one of the principal stresses prevailed, failure was reached in a plane normal to the panel, otherwise, failure occurred by splitting on a plane parallel to the panel.

For this reason, many kinds of mathematical frames commonly used to study masonry behavior, such as plane stress, appear inadequate, because they cannot consider the presence of out-of-plane stresses and so they do not allow to model some of the failure mechanisms. On the contrary, the use of generalized plane state can be suitable to describe a micromechanical model for masonry material, as suggested by several authors [8, 9].

In this paper, starting form the homogenization approach proposed by Cluni and Gusella [10, 11], a method for the determination of the masonry strength domain is presented.

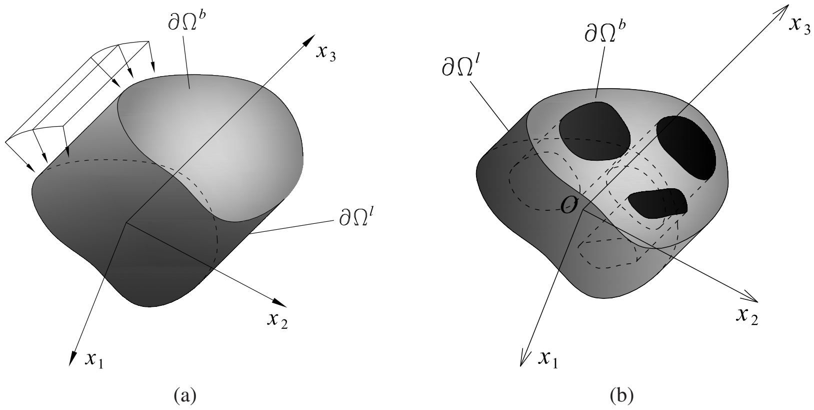


Figure 1: Reference solid for the generalized plane state for homogeneous (a) and heterogeneous (b) material.

2 MODELING STRATEGIES FOR THE BEHAVIOR OF MASONRY MATERIAL: THE GENERALIZED PLANE STATE OF AXIAL STRAIN

The generalized plane state is a kinematic two-dimensional problem, provided with three-dimensional components at the constitutive level, that can be specialized in several plane problems. The reference solid is a finite length cylinder Ω placed in a $\mathbb{R}^3$ Euclidean space, which refers to a coordinate system x_1, x_2, x_3 (Fig. 1), with x_3 as the longitudinal axis. In the following we will take into account both the cases of homogeneous (Fig. 1(a)) and multi-phase solid, the latter made up of n phases whose distribution varies randomly in the x_1x_2-plane but not in the x_3 direction (Fig. 1(b)). A linear elastic and isotropic behavior is assumed for the materials.

The foundamental hypothesis is that the stress tensor is independent from the x_3 direction. This implies, through the compatibility and kinematic equation, that the general form of the displacement field is

$$u_1 = -\frac{A}{2}x_3^2 + Dx_2x_3 + Fx_3 + f(x_1, x_2) \tag{1}$$

$$u_2 = -\frac{B}{2}x_3^2 - Dx_1x_3 + Hx_3 + g(x_1, x_2) \tag{2}$$

$$u_3 = (Ax_1 + Bx_2 + C)x_3 + h(x_1, x_2). \tag{3}$$

where the constants A, B, C, D, F, H and the functions f, g and h depend on the boundary conditions imposed on the bases and determine different kinds of plane problems [12].

The case of generalized plane state of axial strain, considered in the following to analyze masonry panels, is characterized by $A = B = D = 0$, $F = H = 0$ and $h = 0$: these assumptions determine the absence of the tangential components in the longitudinal direction for stress and strain tensors and make two generic cross sections remain plane and parallel.

The solving equations of the elastic problem are reduced to 10 (2 equilibrium equations, 4 kinematic conditions and 4 constitutive laws) with 11 unknowns (stress and strain components and displacements field). The further equation is obtained by the overall equilibrium condition in the x_3

direction; in the case of heterogeneous solid, it can be written as a summation over the n phases

$$\sum_{i=1}^{n} \int_{A_i} \sigma_{33}^{(i)} dA_i = 0 \tag{4}$$

where A_i indicates the area of the i^{th} phase .

Using the (4) together with the constitutive relations and the compatibility equation

$$\varepsilon_{33} = C$$

the value of the constant C is

$$C = -\frac{\sum_{i=1}^{n} \frac{\lambda_i}{2(\lambda_i+\mu_i)} \int_{A_i} \left(\sigma_{11}^{(i)} + \sigma_{22}^{(i)}\right) dA_i}{\sum_{i=1}^{n} \frac{\mu_i(3\lambda_i+2\mu_i)}{\lambda_i+\mu_i} A_i}. \tag{5}$$

Unlike the most commonly used plane stress state, this kind of mathematical frame allows to take into account the out-of-plane behavior of the masonry panel. In particular, the presence of the σ_{33} stresses assumes a relevant rule in the modeling of some failure mechanisms, as will be shown in the following.

3 EVALUATION OF NON-PERIODIC MASONRY STRENGTH DOMAIN

3.1 *Homogenization approach*

The estimation of the strength domain of the homogenized continuum is performed through the approach proposed by He [13] as an extension of the one proposed by Huet [14] in the elastic field.

Let us consider a partition P_δ of the solid Ω constituted by portions Ω_i with a nominal size d, where $i = 1 \ldots n_\delta$, such that

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_{n_\delta}, \qquad \text{with} \qquad \Omega_i \cap \Omega_j = \varnothing \qquad \forall\, i \neq j. \tag{6}$$

For each portion Ω_i, the apparent strength domains $\mathscr{D}_{\sigma i}^{app}$ (for natural BC) and $\mathscr{D}_{\varepsilon i}^{app}$ (for essential BC) can be evaluated. Being $\mathscr{D}_{\sigma\delta}^{app}$ and $\mathscr{D}_{\varepsilon\delta}^{app}$ the apparent strength domains of the heterogeneous solid corresponding to the partition P_δ, the following relations are given

$$\mathscr{D}_{\sigma\delta}^{app} \supseteq \bigcap_{i=1}^{n_\delta} \mathscr{D}_{\sigma i}^{app} \qquad \mathscr{D}_{\varepsilon\delta}^{app} \subseteq \sum_{i=1}^{n_\delta} \gamma_i\, \mathscr{D}_{\varepsilon i}^{app} \tag{7}$$

where γ_i is the volume ratio of the portion Ω_i over Ω such that $0 \leq \gamma_i \leq 1$ and $\gamma_1+\gamma_2+\ldots+\gamma_{n_\delta} = 1$. Moreover, in presence of two partitions, $P_{\delta'}$, with $n_{\delta'}$ portions of size d', and $P_{\delta''}$ with $n_{\delta''}$ portions of size $d'' > d'$, the next hierarchy of inclusions is demonstrated [13]

$$\bigcap_{\mathbf{x}\in\Omega} \mathscr{D}(\mathbf{x}) \subseteq \mathscr{D}_{\sigma\delta'}^{app} \subseteq \mathscr{D}_{\sigma\delta''}^{app} \subseteq \mathscr{D}_{\varepsilon\delta''}^{app} \subseteq \mathscr{D}_{\varepsilon\delta'}^{app} \subseteq \langle \mathscr{D} \rangle \tag{8}$$

where $\mathscr{D}(\mathbf{x})$ is the local strength domain of a generic point and the two bounding values

$$\bigcap_{\mathbf{x}\in\Omega} \mathscr{D}(\mathbf{x}) \qquad \langle \mathscr{D} \rangle = \frac{1}{\text{vol}(\Omega)} \int_{\Omega} \mathscr{D}(\mathbf{x}) d\Omega \tag{9}$$

represent the limits of Sachs and Taylor respectively, analogous to the limits of Voigt and Reuss in elastic analysis.

3.2 *Failure criteria for the masonry material and its constituents*

The application of the approach proposed in 3.1 requires the definition of a failure criterion for the masonry material. This is a problematic matter because, generally, all phases participate to the overall strength, each with a peculiar, different behavior.

The method proposed in this paper consists of considering that the overall strength limit of the specimen is reached when, with the increase of the boundary loads or of the boundary displacements, the mean stress tensor of one of the phases satisfies its own yield condition. It should be noted that, in this way, the local evolution of microcracks and/or plastic strains developed before the global collapse is taken into account in an overall way. On the other hand, the method allows to perform elastic analyses in generalized plane state of axial deformation with significant computational advantages and, at the same time, adequate results, as will be shown in the following.

Owing to the linearity of the method, which allows effects superposition, and since the problem is kinematically plane, each specimen has been studied under only six different kinds of boundary conditions ($\boldsymbol{\sigma}_\xi^0$ and $\boldsymbol{\varepsilon}_\xi^0$, with $\xi = 1, 2, 3$):

$$\boldsymbol{\sigma}_1^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\sigma}_3^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{10}$$

$$\boldsymbol{\varepsilon}_{\mathbf{1}}^{\mathbf{0}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_3^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

Then, the general state of stress $\langle \boldsymbol{\sigma}_\eta \rangle$ has been obtained from the variation of the a, b, c coefficients in the expression

$$\langle \boldsymbol{\sigma}_\eta \rangle = \kappa \big[a \langle \boldsymbol{\sigma}_{1\eta}^0 \rangle + b \langle \boldsymbol{\sigma}_{2\eta}^0 \rangle + c \langle \boldsymbol{\sigma}_{3\eta}^0 \rangle \big] \tag{12}$$

where κ is a small factor that provides a state of stress far enough from the crisis.

Being $\langle \boldsymbol{\sigma}_\eta^{(i)} \rangle$ the mean stress state of the i^{th} phase and $\lambda_\eta^{(i)}$ a stress state multiplicative factor by which the strength condition of the phase is satisfied

$$F^{(i)}(\lambda_\eta^{(i)} \langle \boldsymbol{\sigma}_\eta^{(i)} \rangle) = 0, \tag{13}$$

the ultimate strength value of a heterogeneous material portion determined by the application of a certain set of boundary conditions is defined by the following overall mean stress tensor

$$\boldsymbol{\sigma}_\eta^f = \min_i \{\lambda_\eta^{(i)}\} \cdot \langle \boldsymbol{\sigma}_\eta \rangle. \tag{14}$$

For what concerns the choice of the strength condition for each phase (yield function F in (13)), a failure criterion in which both the isotropic and the deviatoric parts of the stress tensor participate to the ultimate strength, proposed by Lubliner et al. [15], has been used. The yield function calculated on the mean stress tensor of the i^{th} phase is

$$F(\langle \boldsymbol{\sigma} \rangle) = \frac{1}{1-\alpha} \Big[\alpha I_1 + \sqrt{3 J_2} + \beta \hat{\sigma}_{\max} \Big] - \sigma_c \tag{15}$$

where I_1 and J_2 are the first invariant of the isotropic part and the second invariant of the deviatoric part of the stress tensor respectively, α and β are parameters which depend on uniaxial tension and compression (σ_t and σ_c) and equibiaxial compression (σ_b) of each single phase, as defined below

$$\alpha = \frac{\sigma_b - \sigma_c}{2\sigma_b - \sigma_c} \qquad \beta = \frac{\sigma_c}{\sigma_t}\big(1-\alpha\big) - \big(1+\alpha\big) \tag{16}$$

and $\hat{\sigma}_{\max}$ is the maximum value of the principal stress tensor.

4 NUMERICAL RESULTS: STRENGTH DOMAIN FOR PERIODIC MASONRY

As a benchmark of the method, the experimental tests of masonry under biaxial stress states carried out by Page [4, 5] and then elaborated by Dhanasekar et al. [6] have been used.

The numerical model has been developed in ABAQUS environment, using bidimensional elements with 3 and 4 nodes with a generalized plane state formulation [16]. The choice of the mechanical parameters has been based on the experimental data and the numerical elaborations of Shieh-Beygi and Pietruszczak [17] for the missing values (Tab. 1).

Phase	E [MPa]	ν	σ_c [MPa]	σ_t [MPa]	σ_b [MPa]
Brick	6740	0.167	15.41	1.5	17.0
Mortar	1700	0.20	5.08	0.5	6.0

Table 1: Mechanical characteristics of the masonry constituents - E: Young's modulus; ν: Poisson's ratio; σ_c: failure stress in uniaxial compression; σ_t: failure stress in uniaxial tension; σ_b: failure stress in equibiaxial compression. See [15] for details.

Three models, at different values of bed joints slope ($90°$, $67.5°$ and $45°$ from the vertical direction), have been studied with the application of natural and essential boundary conditions.

The results obtained using the basic cases of boundary conditions, $\boldsymbol{\sigma}_1^0$, $\boldsymbol{\sigma}_2^0$, $\boldsymbol{\varepsilon}_1^0$, $\boldsymbol{\varepsilon}_2^0$, have been superposed to have several states of stress and to plot the whole curves, which appear very close each other owing to the dimension of the analyzed models, greater than those of the RVE (defined according to Hill [18]).

In Fig. 2 experimental and numerical strength domains are compared. The numerical results are in good agreement with the experimental ones. It should be noticed that the proposed method allows to recognize some experimental failure mechanisms: ultimate stress in biaxial compression is reached owing to the failure of the brick phase, while in tension-compression the failure occurs for the one or the other phase. Obviously, biaxial tension cases are characterized by the failure of the weaker phase, i.e. the mortar.

5 NUMERICAL RESULTS: STRENGTH DOMAIN FOR NON-PERIODIC MASONRY

5.1 *Statement of the problem*

The numerical approach previously introduced has been applied to a quasi-periodic masonry wall of a monumental and historical structure sited in the center of Italy (Fig. 3(a)).

To obtain a consistent separation of the phases, a photographic image of the wall (Fig. 3(a)) was elaborated through DIP (Digital Image Processing) techniques [19] (Fig. 3(b)). Then, a regular mesh with 4-nodes square elements has been created through an automatic numerical procedure, in which the algorithm assigns each 5 mm square cell of a prefixed grid to either one phase or the other according to the local concentration of the phases. Fig. 3(c) shows as an example a portion before and after the meshing process.

The wall, having 2000 mm square size, has been subdivided using four partitions named P_1, P_2, P_3 and P_4 made of 25, 16, 9 and 4 portions respectively (Fig. 4); the term P_5 indicates the entire panel.

5.2 *Evaluation of the overall strength domain*

The procedure explained in section 3.2 has been used to estimate the overall strength domain of the homogenized continuum. In particular all basic cases of boundary conditions $\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0$ and

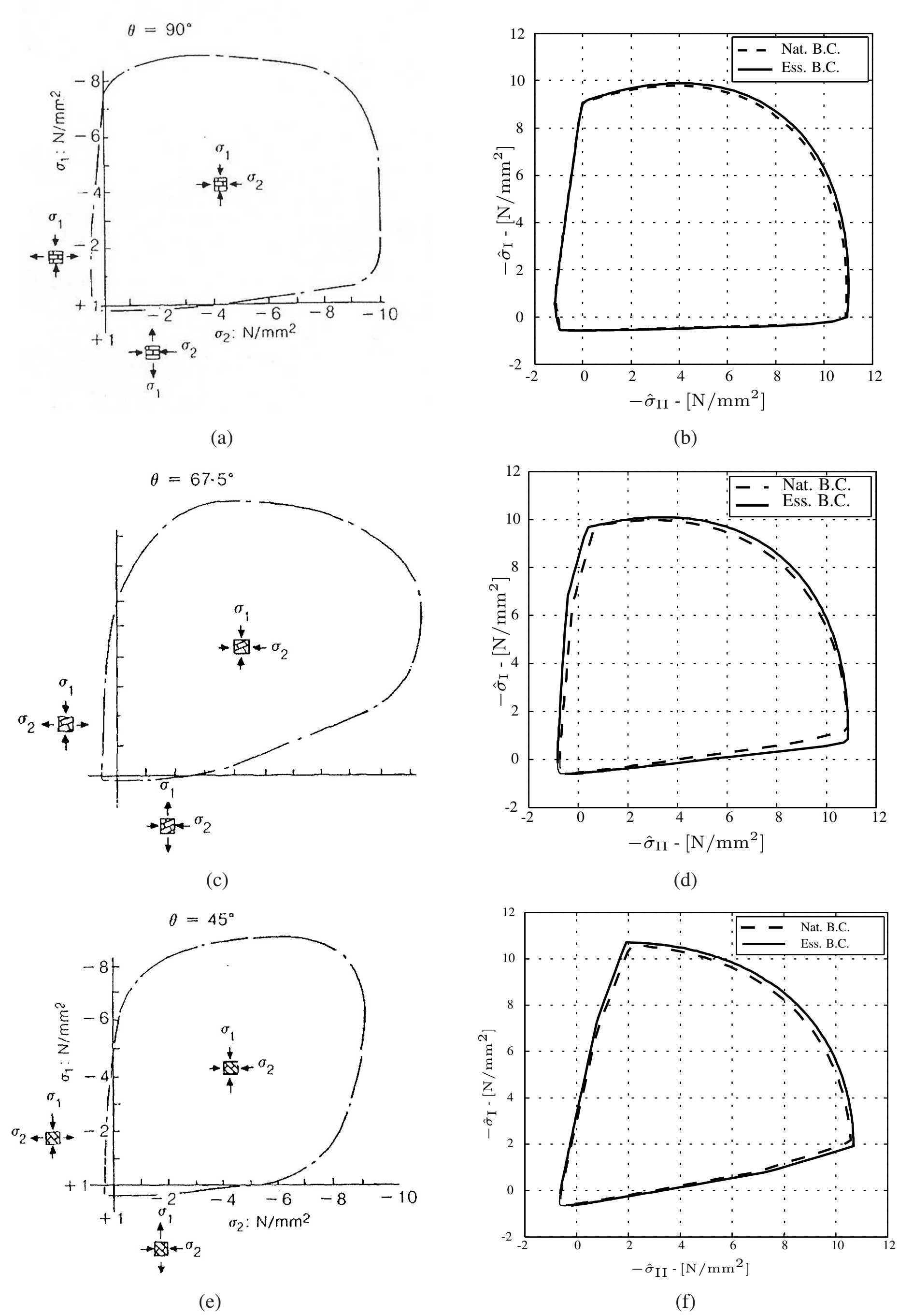
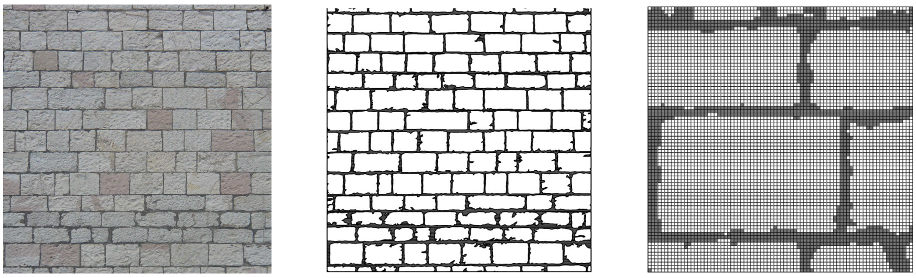
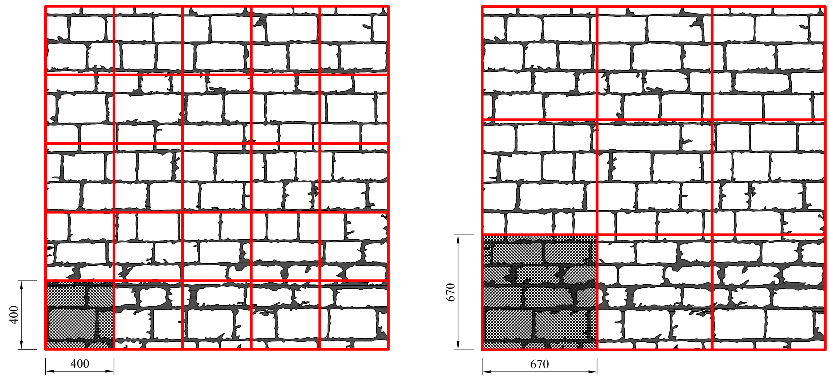


Figure 2: Strength domains of periodic masonry obtained by experimental tests [4] (a)-(c)-(e) and numerical procedures (b)-(d)-(f).



(a) (b) (c)

Figure 3: Photographic image of the non-periodic masonry wall analysed (a); processed digital image (b) and a portion after the discretization process.



(a) (b)

Figure 4: Examples of partitions of the masonry wall: P_1 with $n_1 = 25$ (a) and P_3 with $n_3 = 9$ (b).

$\boldsymbol{\varepsilon}_1^0, \boldsymbol{\varepsilon}_2^0, \boldsymbol{\varepsilon}_3^0$ have been considered to obtain three-dimensional surfaces in the σ_n, σ_p, τ space, where n and p indicate the direction normal and parallel to the bed joints. This choice of reference frame identifies an intrinsic orthotropic property of masonry that directly arises from the pattern and the arrangement of inclusions.

For each portion, the superposition principle has been applied by varying a and b coefficients at fixed values of c (eq. 12) for natural and essential boundary conditions, to obtain several closed curves which belong to the spatial strength domain [20].

The strength domains $\mathscr{D}_{\sigma\delta}^{app}$ and $\mathscr{D}_{\varepsilon\delta}^{app}$ are obtained from the relations (7).

Figure 5 graphically shows the convergence of the surfaces with the increase in size of portions. In particular, the surfaces of partitions P_1 (red), P_2 (blue) and P_4 (black) evaluated in natural (dashed line) and essential (continuous line) boundary conditions have been sectioned by planes with constant values of τ.

Figure 6 shows the overall strength domain obtained for the homogenized continuum, represented by iso-tangential curves projected in the σ_n, σ_p plane (Fig. 6(a)), and by curves in σ_n, σ_p, τ 3D space (Fig. 6(b)).

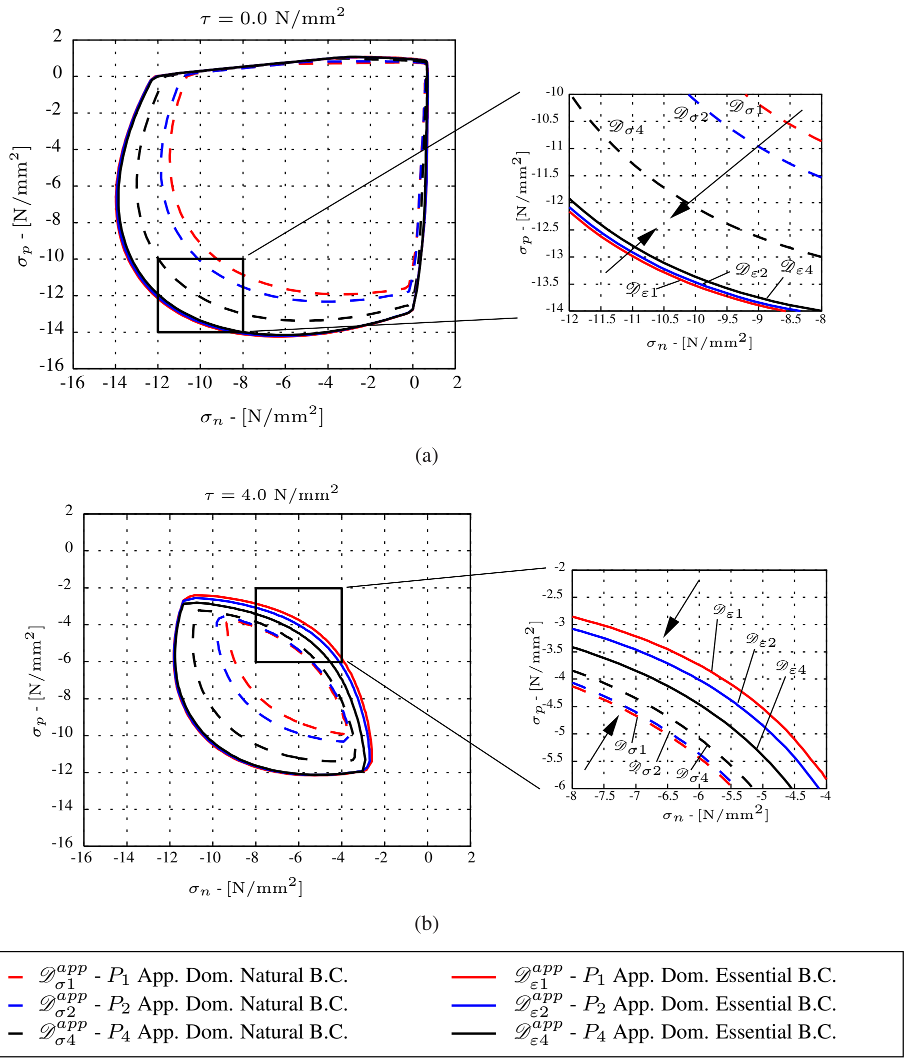


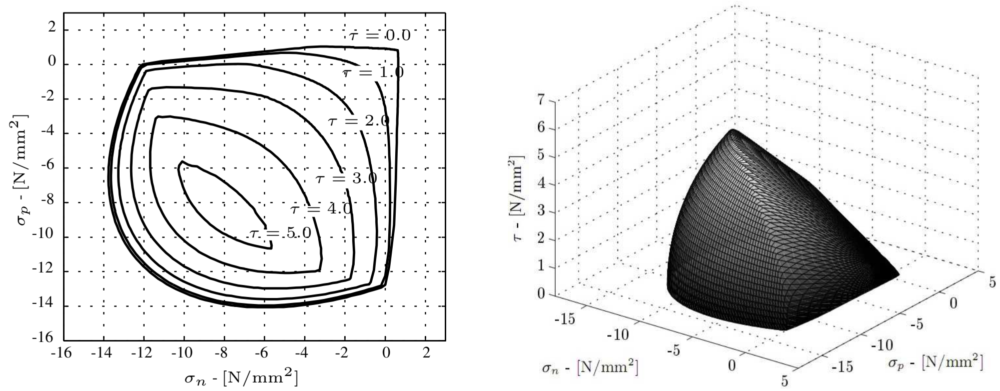
Figure 5: Strength domains: convergence of the homogenized failure surfaces of the partitions P_1, P_2 and P_4 on $\tau = 0.0$ (a) and $\tau = 4.0$ (b) planes.

6 CONCLUSIONS

Starting from a critical study of the experimental results given in literature and considering the masonry material as a composite, in this paper a method for the evaluation of strength domain of non-periodic masonry has been proposed.

The adopted procedure uses a generalized plane state of axial strain formulation, which allows to keep into account the presence of out-of-plane stresses with a low computational load; the failure criterion adopted for brick and mortar is concrete-type, while for the whole specimen it is defined through average stresses in the phases.

The failure surfaces are obtained through the application of natural and essential boundary con-



(a) (b)

Figure 6: Strength domain: (a) Projections of failure surface sections at $\tau = \mathrm{const}$ planes on σ_n, σ_p plane. (b) three-dimensional representation in σ_n, σ_p, τ space.

ditions. The homogenization technique is based on the Hill's definition of the RVE; using partitions with increase in size of the portions, the hierarchy relations that can be established among the strength domains allow to single out the failure criterion for the homogenized material.

One of the best advantages of the procedure is the quickness of calculus. In fact the general state of stress of a test portion is obtained by the effect superposition of three basic cases for each type of boundary condition analyzed in the elastic field.

The reliability of the method is proved, for periodic masonry, by the comparison of the numerical results with the experimental ones given in literature and by the recognition of some basic failure modes. Then, the method is applied to a non-periodic masonry wall: the results obtained for portions of different size show a good convergence.

In this way for the first time, as much as the authors are acquainted with, a method for the estimation of the strength domain of non-periodic masonries which takes into account the actual material heterogeneity and its pattern is proposed.

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