Application of the Linear Matching Method to materials that exhibit softening.

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SUMMARY: The paper considers the problem of evaluating the maximum load that an elastic-plastic frame structure can withstand when material or element softening is present. Here we propose an extension of the Linear Matching Method to take into account material softening. A three steps procedure is described which systematically evaluates the structural response for different levels of softening. Stable solutions are obtained for high and low levels of softening, but numerical stabilities in the procedure can occur for intermediate degrees of softening.

1 INTRODUCTION:

It is well known that localized strain softening behavior causes important consequences for the overall structural response. There are a number of circumstances where the softening issue is relevant: local buckling of beams in portal frames; local buckling in sandwich shell structures; and degradation in strength of composite structures due to internal cracking and fiber failure. In all such cases classical limit analysis is not appropriate and the prime motivation is the development of appropriate mathematical programming methods for the direct evaluation of a maximum load. It is well known that localized strain softening behavior causes important consequences for the overall structural response. Recently, the application of mathematical programming methods to the limit analysis of portal frames has been consolidated and summarized by Cocchetti and Maier [1]. These methods have also been extended to the behavior of elastic-softening plastic portal frames, emphasizing the importance of Mathematical Programming with Equilibrium Constraints (MPEC) methods by Ferris and Tin-Loi [2] and Tangaramvong and Tin-Loi [3, 4]. Here we propose an extension of the Linear Matching Method to take into account material softening. Linear Matching Methods are a class of programming methods where, at each iteration, equilibrium and compatibility are satisfied and convergence is imposed by ensuring material consistency. Convergent methods have been derived for classical limit analysis by Ponter, Fuschi and Engelhardt [5] and shakedown by Ponter and Engelhardt [6]. Recently, a detailed study of convergence of both upper and lower bounds for portal frames has been carried out by Barrera, Cocks and Ponter [7]. In this paper the work is extended to evaluate the maximum load that portal frames with a softening moment/curvature relationship can support.

2 PROBLEM SPECIFICATION:

In common with all structural systems, portal frames can be analyzed within a kinematic framework defined by a set of displacements, in this case the deflections Δ_i^c at the intersection of beams at x_i that are compatible with a set of plastic hinge rotations Φ_j^c at positions h_j . Hence the deformation of the structure is subjected to a severe subclass of all the possible modes of behavior, those defined by Δ_i^c and Φ_j^c. The equilibrium of bending moments M_j^* with loads F_i^* is then defined by a Galerkin criterion, such that equilibrium is satisfied if the following virtual work relationship holds for *all* possible sets of Δ_i^c and Φ_j^c;

$$\sum_i F_i^* \Delta_i^c = \sum_j M_j^* \Phi_j^c \tag{1}$$

Consider a structure composed of an elastic-plastic material that exhibits softening (see Figure.1a). The structure is subjected to a set of proportional loads λF_i and the objective is to find the values of Δ_i^c $\Phi_j{}^c$ that will yield the largest value of the load factor

$$\lambda \sum F_i \Delta_i^c = \sum M_{pj} \left| \Phi_j^c \right| \tag{2}$$

Where M_{pj} is the plastic moment corresponding with the hinge rotation $\Phi_j{}^c$ at hinge position h_j. We consider the moment-rotation relationship indicated in Figure 1a, where three different regions can be distinguished: elastic region (*e*), plateau region (δ), and softening region (*s*). The value of M_{pj} for each region is given below:

$$\begin{aligned} M_{pj} &= R\Phi_e & \Phi_e &< \Phi_c \\ M_{pj} &= M_c & \Phi_c &\le \Phi_\delta < \Phi_c^I \\ M_{pj} &= M_c - P\left(\Phi_s - \Phi_c{}^I\right) & \Phi_s &\ge \Phi_c^I \end{aligned} \tag{3a-c}$$

where *P* is the slope of the softening branch as shown in Figure 1.

3 LINEAR MATCHING METHOD .

The Linear Matching Method (LMM) attempts to construct, as the limit of an iterative procedure, linear solutions, $\tilde{\Phi}_j$ and $\tilde{M}_j$, for the load λF_i by varying the set of linear moduli R_j :

$$\tilde{\Phi}_j = \frac{1}{R_j} \tilde{M}_j \tag{4}$$

by a particular scaling factor, which is different for each region in Fig.1, as discussed below. Eq. (4) describes an arbitrary sign consistent description of the relationship between the moments, in equilibrium, and compatible rotations which is capable of describing any type of holonomic constitutive assumption. The procedure described below provides an iterative procedure which seeks a sequence of values of R_j denoted by R_j^k, so that each solution more closely approaches the correct solution.

We start the procedure with a linear solution for $R_j^0 = R$, a constant and arbitrary λ, producing an initial solution $\tilde{\Phi}_j^0$ and $\tilde{M}_j^0$. In the subsequent iterative procedure the moduli R_j^k are adjusted according to a specified design criterion so that a distribution of R_j^{k+1} can be found, which for a prescribed rotation $\mu\tilde{\Phi}_j^k$, where μ is a scaling factor that is determined by the design criterion, the moment can be brought onto the moment-rotation curve (see Figure 1b). This procedure takes into account the three different regions of the constitutive response as given by (3 a-c). Hence the new R distribution is:

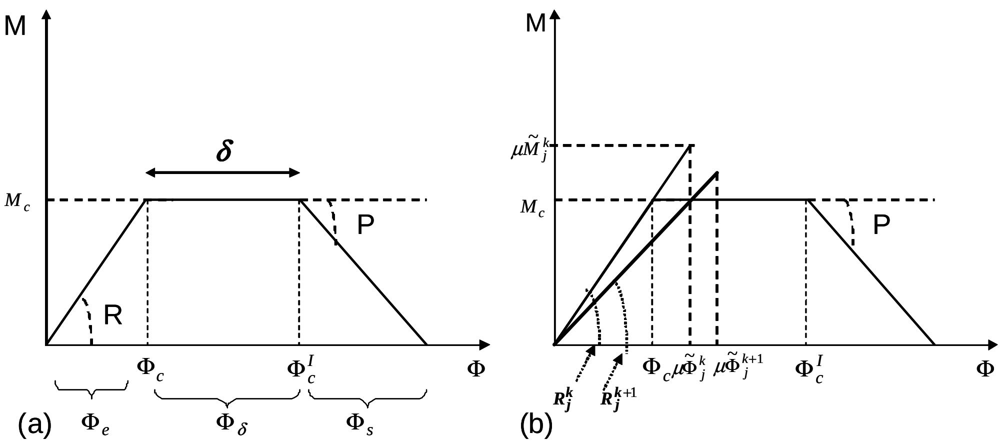


Figure 1a-b. a) Material behavior; b) Matching procedure at increment *k+1* in the plateau region

$$R_j^{k+1} = \frac{M_{pj}}{\left|\mu\tilde{M}_j^k\right|} R_j^k \tag{5}$$

where the value of M_{pj} associated with the prescribed rotation $\mu\tilde{\Phi}_j^k$ is given by (3a-c).

A new linear solution is now constructed for $R_j = R_j^{k+1}$. The load for this *(k+1)*th solution is chosen by computing the loading parameter λ_{KIN}^k corresponding to the previous solution $\tilde{\Delta}_j^k$ and $\tilde{\Phi}_j^k$.

At each iteration a corresponding static approximation to the maximum load can be found by scaling the moment distribution $\tilde{M}_j^k$, which is in equilibrium with $\lambda_{KIN}^{k-1} F_i$ so that, for the largest possible value of $\lambda = \lambda_{ST}^k$ the scaled moments lie on or below the moment-curvature curve of Fig 1:

$$\lambda_{ST}^k = \lambda_{KIN}^{k-1} \min \frac{M_{pj}}{\left|\mu\tilde{M}_j^k\right|} \tag{6}$$

In this process the solution converges to the exact if the kinematic and static bounds are equal. For P=0 convergence certainly occurs [7] but, in order to proceed to the more general case, we need to identify a suitable criterion for scaling the rotations. This is the subject of the following section.

4 EXTENSION OF THE LINEAR MATCHING METHOD FOR DETERMINATION OF THE MAXIMUM LOAD

The process of evaluating the maximum load a structure can support for a softening moment/curvature relationship consists of the following three major steps based on the LMM (see Figure 2):

1) The Linear Matching Method is employed to determine the maximum load at which $\Phi \le \Phi_c^I$ throughout the structure.
2) The range of values of the slope *P* for which the load can be increased beyond that determined in (1) is identified.
3) For values of *P* which satisfy the criteria established in (2) the maximum load that the structure can withstand is determined through a two stage iterative procedure based on the Linear Matching Method.

4.1 Step 1 – rotation limit

A linear analysis with an initial arbitrary value of λ and an initial moduli distribution R_j is performed. The solution is then scaled considering the constraints on the rotations $\Phi \le \Phi_c^I$ (see Figure 2) so that a scalar parameter μ is given by:

$$\mu = \frac{\Phi_c^I}{\max \Phi_j} \tag{7}$$

A kinematic bound, λ_{KIN}^k, to the maximum load λ_δ^{MAX} at which $\Phi \le \Phi_c^I$ is determined by the following virtual work statement:

$$\lambda_{KIN}^k F_0 \{\mu\Delta_i\} = \sum_j M_{pj}\left(\mu\Phi_j\right)\mu\Phi_j \tag{8}$$

where the expressions for M_{pj} are given by (3a-c). At each iteration the rotational stiffnesses are updated using the LMM and a new linear solution with the new moduli and load given by (8) to calculate an improved kinematic bound. At each iteration a static bound can be determined using (6). The iterative process is stopped when the kinematic and static bounds are within 0.00001% of each other.

4.2 Step 2 – Values of P for which the load can be increased beyond that of step 1

We now evaluate the values of the softening slope *P* for which the load can be increased beyond that given by eqn (8). Step 1 provides a set of hinge rotations $\mu\Phi_j$. From the moment curvature relationship of (3) in Fig 1 we can determine the tangent stiffness *K* (see Figure 2). Note that for hinges where $\mu\Phi_j = \Phi_c^I$ the tangent stiffness is *K*=-*P*. It can be shown [8] that a sufficient and necessary condition for the load parameter to be able to be increased beyond λ_δ^{MAX} is

$$\det\left(K^{TOT}\right) > 0 \tag{9}$$

where K^{TOT} is the global tangent stiffness matrix for the structure, which, for a prescribed elastic stiffness *R* , is a function of the softening modulus *P*. Inequality (9) is satisfied for $P < P_{crit}$, where P_{crit} is the value of *P* at which the determinate is zero. For larger values of *P* the load cannot be increased beyond that determined in step 1 and the maximum value of the load parameter is $\lambda^{MAX} = \lambda_\delta^{MAX}$.

4.3 Step 3- Maximum load for $P < P_{crit}$

Barrera et al [8] demonstrate that a two stage min-max iterative procedure can be employed to determine λ^{MAX} for values of the softening slope $P \leq Pcrit$. The first stage involves using the LMM to provide a set of rotations for an arbitrary load factor λ (in practice it is generally most appropriate to start with the set of rotations determined from the calculation of λ_δ^{MAX}). We now scale the distribution $\mu\phi_j$. As μ is increased the rotations translate along the moment relation curve of Fig 1 as illustrated in Fig 2. For a given value of μ we can calculate the local tangent stiffness *K* and global tangent stiffness for the structure K^{TOT} . We now wish to maximise the value of μ subject to the constraint that the solution is stable. This again requires that the determinate of K^{TOT} is positive, i.e. the maximum value of μ is obtained by equating the determinate to zero. This maximisation project mirrors that for step 2, but now *P* is prescribed and μ is the variable, whereas before μ was prescribed from step 1 and *P* was the variable.

Having determined a compatible set of rotations from stage 1, eqn (8) is used to provide a kinematic estimate of the maximum load and a corresponding static estimate can be obtained using (6). The kinematic and static estimates are no longer formal bounds to the exact value of λ^{MAX} , but it can be shown [8] that the exact result is obtained when the two solutions are equal to each other.

The rotations $\mu\Phi_j$ are now employed to determine a new set of moduli using the LMM procedure described in section 3. These provide the moduli for the new linear problem which is solved by minimising the total potential energy of the system. This procedure results in a new compatible displacement field which can be used as input into stage 1 of the min-max process.

For the situation where *P=0*, the constitutive model of 3(a-c) reduces to an elastic perfectly plastic material. In this limit the determinant of K^{TOT} is equal to zero when μ is increased to a value such that a mechanism of collapse is activated by the hinges corresponding to the set of curvatures that lie along the plateau of Figure 1a. The LMM sequentially evolves the mechanism until the exact limit load for the structure, λ_L , is obtained. In this limit the procedure is equivalent to the method described by Barrera et al [7] for determining the collapse load of a perfectly plastic material.

For values of P in the range 0 to P_{crit} λ^{MAX} is bounded from above by λ_L and from below by λ_δ^{MAX} . The min-max procedure of step 3 interpolates between these extreme values. We describe an application of this three steps procedure in the following section.

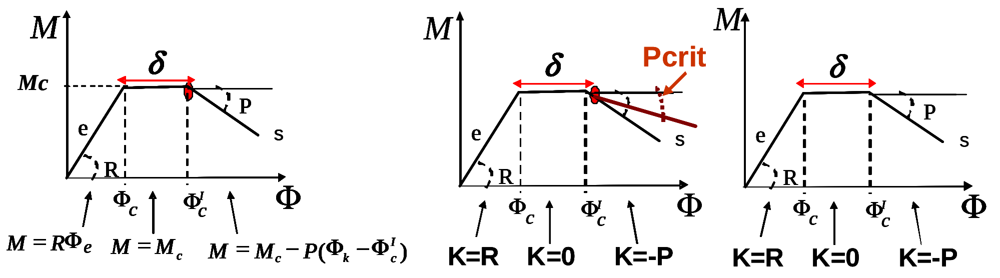
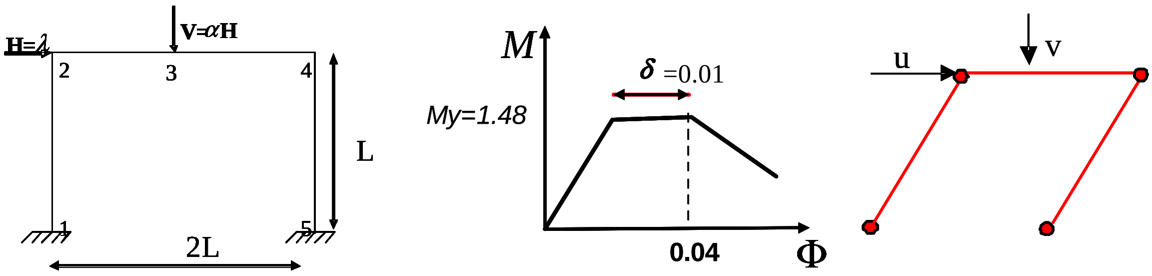


Figure 2. Three steps procedure for determining the maximum load

5. APPLICATION OF THE THREE STEPS PROCEDURE TO A SIMPLE PORTAL FRAME

In this section we apply the procedures described in section 4 to the single story portal frame of Figure 3. The frame is fixed at its base nodes 1 and 5. The vertical load $V = \alpha H$ and horizontal load $H = \lambda$ remain proportional, with the magnitude represented by the load factor λ . We present results for the situation α =0.25. For problems where concentrated loads are applied, equilibrium requires that the maximum and minimum bending moments occur at the intersection of uniform beam sections, or the points of application of the loads, i.e. at nodes 1 to 5 of Figure 3. Hence plastic hinges may only occur at these nodes and the rotation of local plastic hinges are given by Φ_j, $j=1$ to $j=5$ as shown in Figure 4. For this combination of loads the exact limit load solution consists of a sway mechanism [7] (see Figure 3c).

We consider the situation where the elastic modulus R=45 kNm and $\phi_c = 0.033$, so that the plastic moment $M_c = 1.48$ kNm, and define the extent of the plateau region as $\delta = \phi_c^I - \phi_c$. Figure 4a shows the variation the λ_δ^{MAX} as a function of the size of plateau region δ . For values of δ greater than 0.018, step 1 of the above procedure gives $\lambda_\delta^{MAX} = \lambda_L = 0.594$. Thus steps 2 and 3 are redundant. For smaller values of δ, $\lambda_\delta^{MAX} < \lambda_L$ and we need to employ steps 2 and 3 to determine the peak load. For $\delta = 0.01$ $\lambda_\delta^{MAX} = 0.576$ and the value of the critical slope *Pcrit* determined from step 2 of the procedure described in section 4.2 is 22 kNm. Results for the third step of the procedure (see section 4.3) is shown in Figure 4b where the evolution of the maximum load is reported as a function of the slope *P*, where λ^{MAX} decreases monotonically from λ_L to λ_δ^{MAX} as *P* is increased from 0 to 22 kNm.



(a) (b) (c)

Figure 3. a-c. a) A single story frame; b) constitutive behaviour; c) collapse mechanism for α=0.25.

5.1 *Graphical representation of the solution process*

Convergence of step 3 of the maximum load procedure may be shown through the following graphical representation of the iterative process for the example shown in Figure 3a-c for $\delta = 0$. The problem of Fig 3 is essentially a two degree of freedom problem. It proves convenient to illustrate the solution process in terms of the displacements u and v defined in Figure 3. A graphical representation of the solution process is plotted in u-v space in Figure5. In the first part of step 3 of the procedure outlined in section 4.3, the method simply scales the mechanism determined from a linear calculation. Thus the solution lies along a radial line radiating from the origin in u-v space as illustrated in Figure 5. The process of maximising the scaling factor μ locates a solution along this radial path (point1′, in the Figure) and this is used to calculate the load factor λ_{KIN}^1 using eqn (8). In the second part of the procedure described in section 4.3 the effective stiffnesses are updated by applying the LMM and a linear problem is solved for a prescribed load λ_{KIN}^k . A surface of constant potential energy for this load is plotted in Figure 5 that passes through point 1′. The radial solution path used in the first part is tangential to the surface at this point. This is a general feature of the solution process [8]. The combination of u and v that minimises the total potential energy corresponds to point 2 of Figure 5.

The above process is repeated with the displacement pattern obtained from the minimising process scaled to determine a new value of the load factor λ_{KIN}^2 . The new radial path plotted on the figure passes through point 2, with the new solution represented by point 2′. A surface of constant total potential energy is again tangential to the radial line. Minimising the total potential energy produces a new solution, point 3. This process is repeated until the kinematic and static results agree within a small tolerance. It is evident from Figure 5 that as this iterative process proceeds the elliptic surfaces of constant potential energy get smaller and successive solutions become closer together.

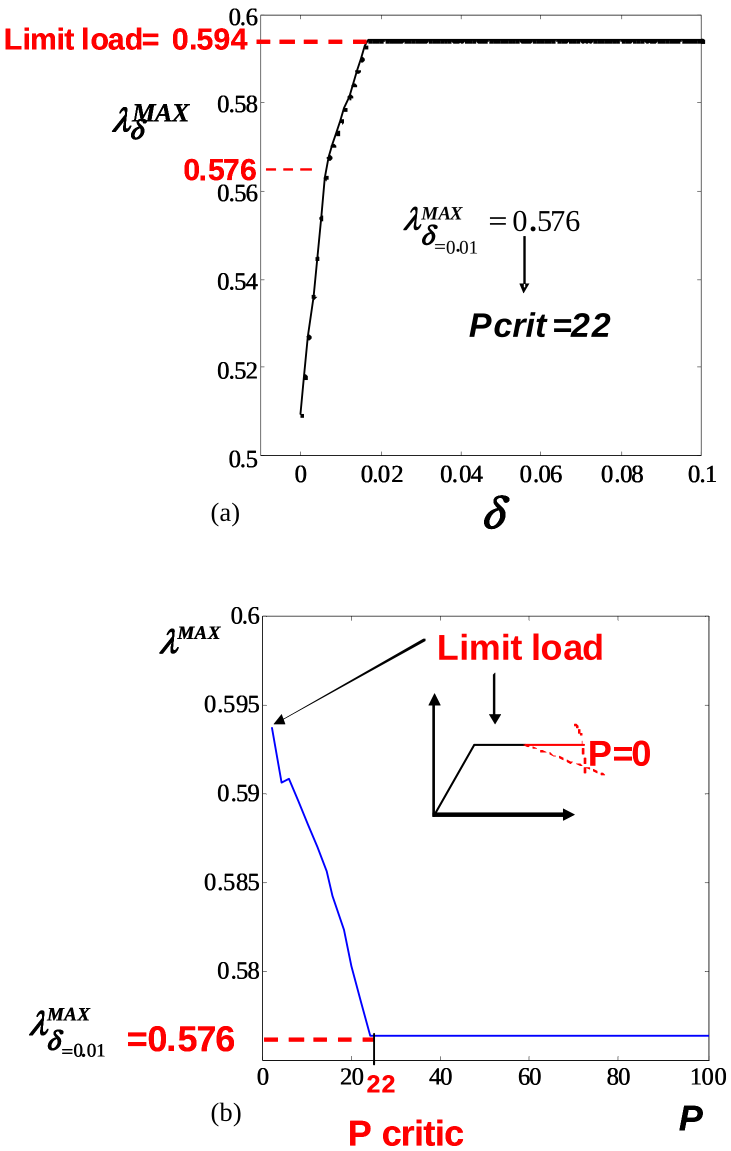


Figure 4 a-b. a) steps 1-2 max load procedure; b) step 3 max load procedure.

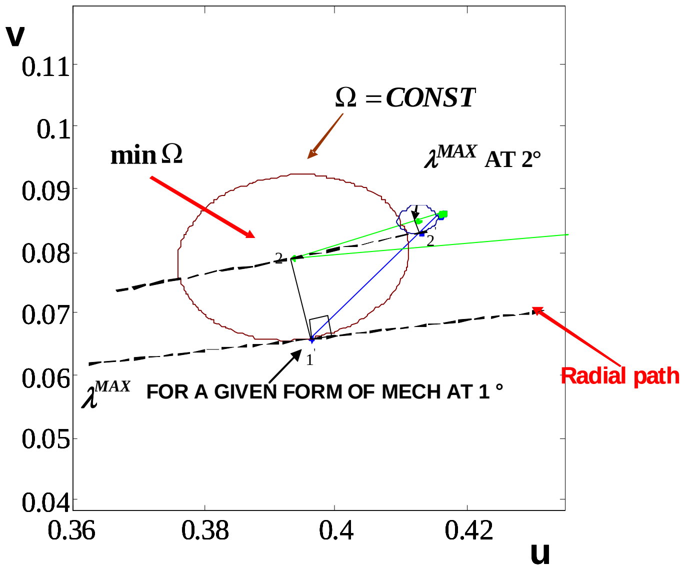


Figure 5. Graphical representation of the min-max process for step 3 of the iterative procedure

6. CONCLUSIONS

The Linear Matching Method provides a programming method for the evaluation of limits in classical plasticity that differs significantly from other programming methods. As demonstrated in [7] the procedure exhibits strong convergence properties. This paper has concentrated on the extension of the LMM to materials which exhibit softening. A three step process has been described which systematically maps out how the maximum load that a structure can support depends on parameters within the model. As with the classical method of Ponter et al [5-7], step 1 of the process exhibits strong convergent properties. Also a unique solution for P_{crit} is obtained directly through implementation of step2, but there are currently no uniqueness and convergence proofs for the min-max procedure of step 3. For more complex problems than considered here the solution process can become unstable for values of P close to P_{crit}. Solutions for frameworks with more degrees of freedom than considered here and an evaluation of these instabilities is presented elsewhere [8].

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