How to choose an analytical model for the stress state of sandwich beams under bending

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SUMMARY. The difficulty in describing the stress state of sandwich beams under bending (and shear) can been seen as due to the lack of the Saint-Venant principle for such structures, mostly when extreme design leads a core several orders of magnitude softer than the skins. Each of the analytical models available in the literature turns out to be appropriate for a specific range of relative stiffness of core and skins. In this work, after extensively comparing the results obtained from analytical models and finite element simulations, we shall provide some abaci which can help in selecting the suitable model for each case.

1 INTRODUCTION

We wish to get an insight on the relation between the heterogeneity peculiar of sandwich beams and the analytical models to be employed in order to describe the sandwich stress state under bending and shear. We limit our attention to sandwiches whose cross-section is symmetric with respect to the neutral axis, say z, x being the beam axis. In other words, we consider sandwiches whose top and bottom layers, i.e., the skins, are identical, having *relevant* Young's modulus E and *arbitrary* thickness t. The heterogeneity is expressed in terms of the ratios between the longitudinal elastic moduli of core (i.e., the intermediate layer) and skins, E_c/E, and between the skin and core thicknesses, t/c, $h = c + 2t$ being the cross section height. Since we leave E_c/E and t/c unrestricted, finding an accurate and simple analytical model for any situation is almost impossible [1, 2].

The main point of concern is that the behaviour of a sandwich beam may be strongly affected by the exact way the loads are applied and the constraints are realised, details usually neglected in *beam models*. For instance, should the core be much softer than the skins (E_c/E of about 0.01 or less is feasible in sandwich cores made up of foam) the stress state in a simply supported sandwich beam may be largely different whether a uniformly distributed transversal load (i.e., acting along y) is applied on the top skin or on the bottom skin; moreover, the resulting stress state may also be strongly dependent, *in any section*, on the exact way the reaction forces are developed by the supports: are they distributed along the whole sandwich height (that is for instance the case of fictitious constraints used to impose some symmetry condition) or are they localised on one skin? In other words, the problem may become two-dimensional. Because of this, let us indicate this feature left out of consideration by standard beam models as the "dependence of the boundary conditions on y" or "*lack of the Saint-Venant principle*".

To our purpose, we compare the results of plane stress finite element simulations with the predictions of (i) First-Order Shear Deformation (FOSD) models (see [3, 2] and references therein) and (ii) the theory of Frostig et al. [4], whose peculiar feature is the inclusion of the contribution of the core deformability along the thickness direction, y.

FOSD models are the simplest possible models accounting for the shear deformation. They are developed within the context of the single-layer theory for laminate structures and are based on the Timoshenko model for homogeneous beams, in which the heterogeneity enters the model only through appropriate choices of the bending stiffness D and shearing rigidity S. In passing, we note that the choice of S is usually considered the main difficulty for a successful application of FOSD models, though, recently, it has been pointed out that also the choice of D may deserve a more sophisticated analysis than the standard one usually exploited [2]. By pretending that there is no warping, the FOSD models predict a stress state which, in statically determinate beams, leaves out of consideration the deformability and just depends on the bending moment and shear force acting on the section considered, so that, in statically determinate beams, all FOSD models predict the same stress state, estimated by straightforward extensions of the classical theories of Bernoulli-Navier and Jourawsky [5]. FOSD models cannot account for the "dependence of the boundary conditions on y" and, hence, we indicate the stress state they predict "à la de Saint-Venant".

Contrariwise, the model of Frostig et al. [4] aims at describing the effects of concentrated loads when the core is very soft with respect to the skins, so that, it somehow accounts for the "dependence of the boundary conditions on y". This is done by including, through the Total Potential Energy functional, the contribution due to the normal deformation ε_y along the core thickness. The sandwich is modelled as two Euler-Bernoulli beams (i.e., the skins) connected by a two-dimensional plane stress continuum (i.e., the core), where the normal stress σ_x along the longitudinal axis x is neglected, or, in words typical of sandwich structures [1], the core is assumed to be "antiplane". The resulting model is quite complicated in spite of the fact that it neglects the shear deformation in the skins and assumes an antiplane core. This last approximation, easily avoided in FOSD models, requires the shear stress τ_{xy} in the core to be independent upon y (i.e., uniform along the thickness) and makes the Frostig model unable to well represent the stress state in sandwiches with a not-too-soft core.

In particular, our comparison with the results of finite element simulations will show that for each choice of the relative thicknesses t/c between core and skins there is a range of the ratio E_c/E in which both the FOSD and Frostig descriptions inaccurately represent the stress field over a large part of the sandwich. More precisely, for each choice of t/c, so far, we have found a quite well defined range $\mathcal{E} \subset (0,1)$ of values E_c/E above which the stress state is effectively estimated by the FOSD modelling in any sandwich section at a distance larger than the sandwich height h from concentrated loads, while for values of E_c/E falling below the range $\mathcal{E}$ the Frostig theory becomes accurate in computing the stress state, still at a distance larger than h from concentrated loads. We wish to provide some formulæ for this "switch of modelling", also accounting for the beam slenderness. At least, we aim at providing an abacus that be a guide for choosing the most appropriate model for sandwiches of given materials and geometry, possibly reliable for a wide range of boundary conditions.

2 THE BENCHMARK

In order to avoid the relevant complications concerning statistically indeterminate structures (see, e.g., [2]), we focus our attention on a simply supported sandwich beam subjected to a uniform transversal load (i.e., acting along the y direction).

2.1 *The finite element model*

In the finite element simulations, run with the code ABAQUS [6], the sandwiches are discretised by means of eight-noded plane stress continuum elements with reduced integration. The symmetry of the problem allows the modelling of half beam by imposing zero displacements along x at the midspan ($x = L$). An example is given in Figure 1, where the mesh is represented on the deformed shape of a sandwich with extreme relative stiffness between core and skins ($E_c/E = 10^{-5}$ and

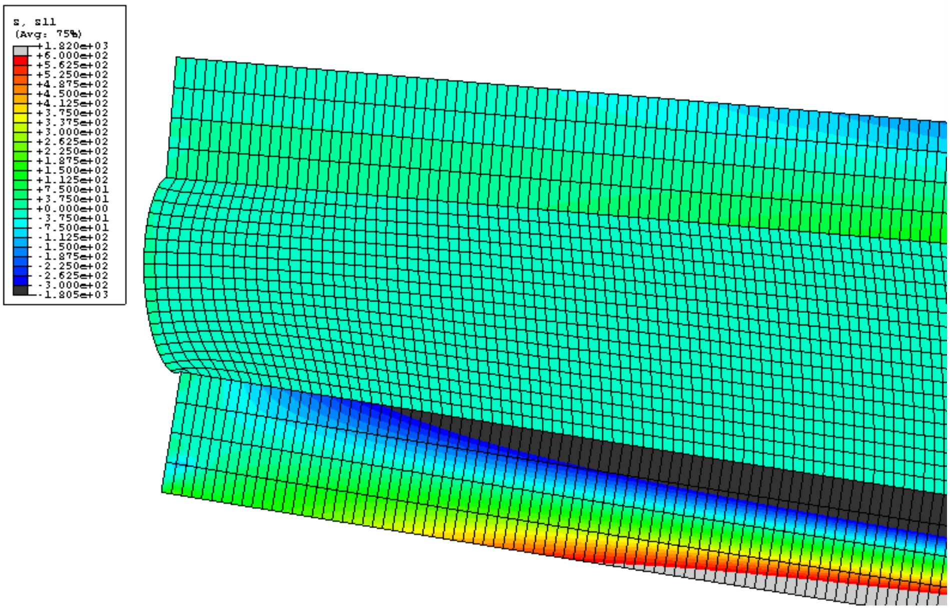

Figure 1: The deformed mesh with the contour of the longitudinal normal stress in the case $t/c = 1/3$ and $E_c/E = 10^{-5}$.

$t/c = 1/3$); coarser meshes have been employed for milder relative stiffnesses.

How the support is modelled is extremely important when E_c/E becomes very small. Since we are here interested in the load diffusion accompanied with beam deflection, we consider the most severe case, in which the support is modelled in such a way as its reaction force is concentrated in the bottom corner node ($y = c/2+t$, $x = 0$), while the distributed load consists of a pressure applied to the top surface ($y = -c/2 - t$). Such boundary conditions can be directly imposed in the Frostig model.

2.2 *The comparison with the analytical models*

Here, we define how we compare the results obtained from the analytical models considered (described in section 1) with those of the finite element simulations.

The outcome of the comparison is the distance d from the support, represented in Figure 2, at which the stress state predicted by the analytical models becomes almost coincident with that of the finite element analysis. This is considered to be the case when the relative error between the normal longitudinal stresses in the Gauss points farther from the neutral axis z is less than 3%. This choice has been guided by numerical tests which have shown that it usually requires a shorter distance for the shear stresses to converge.

For what concerns the FOSD theory, the distance d may then be called the *stress diffusion distance* and it would be approximately equal to the cross section height, $h = c+2t$, if the Saint Venant principle held. The expectation is that as E_c/E decreases d/h increases, because softer the core with respect to the skins, more difficult for the stresses to propagate from one skin to the other.

Instead, one of the main purposes of the higher-order model of Frostig et al. [4] is the description

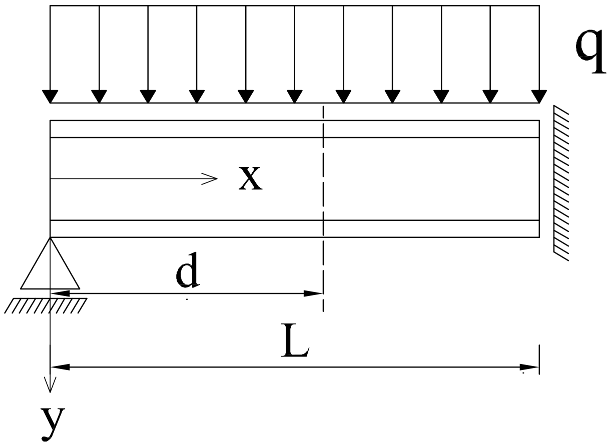

Figure 2: Definition of the distance d after which the models analysed accurately predict the stress state.

of the stress state also *within* the diffusion zone. This notwithstanding, we still expect an open range of values of E_c/E in which the Frostig model is unable to accurately represent the stresses, because of the model assumption that the longitudinal normal stresses in the core be negligible. Hence, contrary to the FOSD model, we shall evaluate an increase in d as E_c/E augments.

The qualitative considerations above lead to the question: is there a range $0 < E_c/E < 1$ in which both models can satisfactorily represent the stress field? This should be checked for many parameters; here, we shall restrict out attention to the most important one, i.e., ratio t/c. Notice that since the Frostig model also neglects the shear strain within the skins, it is expected to better work for small values of t/c.

3 NUMERICAL RESULTS

All the results are obtained for the case of isotropic materials with Poisson ratio $\nu = 0.3$ for both skins and core. We have kept the beam slenderness under control by imposing the ratio $h/(2L)$ to be always equal to $3/40$, where $h = c + 2t = 30\ mm$ is the cross section height and $2L$ is the total beam length.

The accuracy of the results has been checked by properly refining the mesh, case by case. Figure 1 reports one of the most refined mesh, employed in the extreme case in which $E_c/E = 10^{-5}$ and $t/c = 1/3$; Figure 1 also includes the contour for the normal longitudinal stress (related to the choice of a section width $b = 5h/3$ and a uniform load $q = 50\ N/mm$, so that the top skin is subjected to a pressure equal to $1\ MPa$).

The results obtained so far are collected in Figure 3 for three different values of the ratio t/c. The continuous curves indicate the distance from the support after which the solution predicted by the model of Frostig et al. [4] gets very close to that of the finite element simulation (by meeting the criterium described in subsection 2.2), whereas the dashed curves are referred to the analogous comparison for the FOSD model.

The main result consists in the fact that there is a range of the ratio E_c/E dependent on t/c in which both models inaccurately predict the stress state for a large sandwich region.

The peculiar behaviour of the continuous plots, related to the Frostig model, can be explained

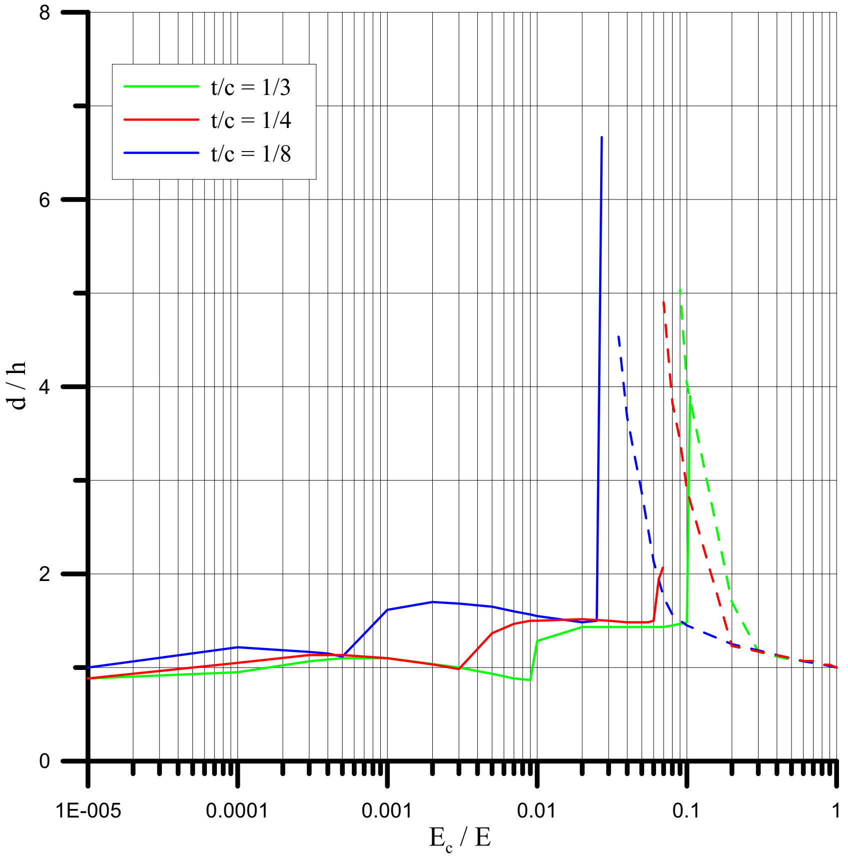

Figure 3: Beam region d where both the FOSD and Frostig models inaccurately predict the sandwich stress state.

by the competition between the facts that, when E_c/E decreases, on the one hand, the connection of the two skins is more difficult but, on the other hand, the Frostig model gets closer to the real behaviour as the normal longitudinal stress σ_x within the core becomes really negligible.

In Figure 4, for the case $t/c = 1/8$, we have highlighted the range of E_c/E where the stress field is badly represented by the models considered.

4 CONCLUDING REMARKS

For statically determinate sandwich beams, we have compared the stress fields obtained from finite element simulations with those predicted by two analytical models available in the literature: the First-Order Shear Deformation (FOSD) model (e.g., [3, 2]) and the higher-order model of Frostig et al. [4].

We have found that such analytical models may be inadeguate to accurately describe the stress behaviour for certain values of the relative stiffness between core and skins, expressed in terms of their relative thicknesses, t/c, and longitudinal moduli, E_c/E. We have also connected the failure of the prediction of the FOSD model within a large sandwich region from concentrated loads with

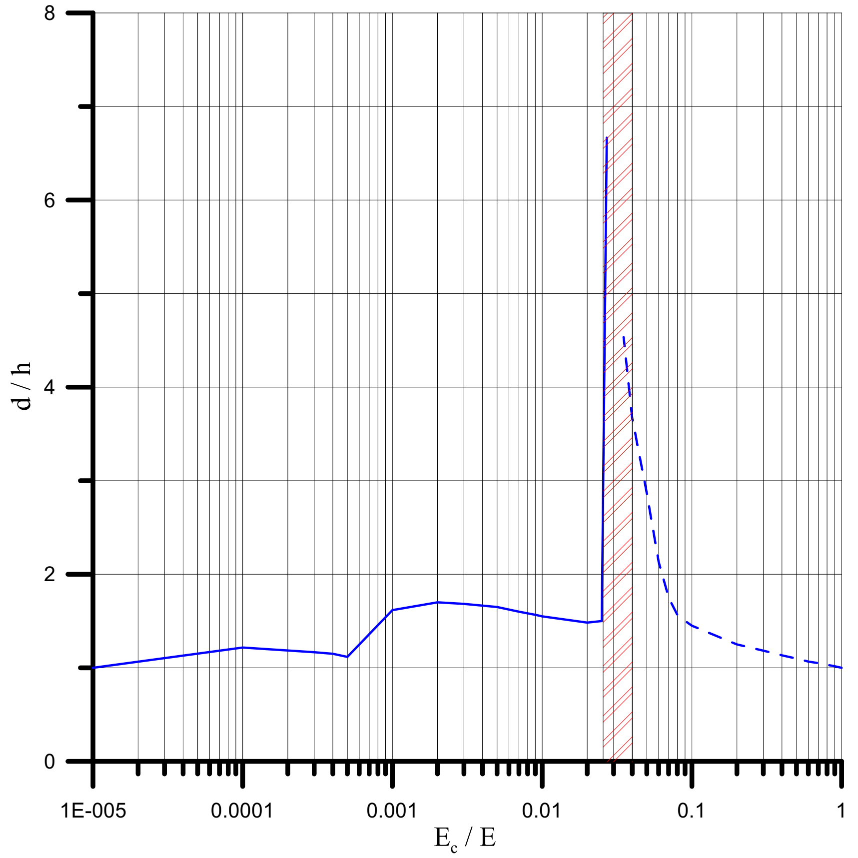

Figure 4: Range of E_c/E in which the models analysed are inaccurate for the case $t/c = 1/8$.

the lack of the Saint Venant principle in those sandwich beams where it happens.

A model that should be able to represent the stress field of any sandwich with E_c/E greater than a very low value is that developed by Krajcinovic [7]. In fact, this model both allows a zig-zag warping of the cross section and accounts, within the core, for both the longitudinal normal stress σ_x and the deformation along the direction y normal to the neutral plane, even though the description of the latter is poorer than that allowed by the Frostig model. In particular, in the Krajcinovic model, the displacement along y within the core is constrained to be linear in y, while Frostig et al. [4] describe the core as a plane stress continuum under the further hypothesis $\sigma_x = 0$. Hence, for extremely low values of E_c/E, the model of Frostig is expected to better describe the sandwich behaviour than the Krajcinovic model. By the way, the main problem with the comprehensive and purely structural theory of Krajcinovic is that it requires a cumbersome numerical implementation, nowadays probably more expensive than modelling the problem into finite elements. This notwithstanding, it would be interesting to verify whether the Krajcinovic model can fill the gap left by the FOSD and Frostig models in describing the sandwich stress state under bending and shear.

In the near future, we aim at extending and extrapolating the results obtained in such a way as to be able to provide a simple analytical rule which should indicate, for any sandwich beam, which model available in the literature is the most appropriate. In particular, we shall analyse the case of statically indeterminate sandwich beams, for which each different FOSD model, characterised by a different shearing rigidity S, provides different results [2]. Most of all, we think it would be extremely useful to know under which circumstances (expressed in terms of relative stiffnesses between core and skins) the very simple FOSD modelling may be accurate enough in all the cross sections at distances larger than the sandwich height from concentrated loads.

Moreover, we shall try to link our results to those of Serpilli and Lenci [8], whose purpose was the validation of structural models by means of a proper limit process involving asymptotic expansions of the solutions of two-dimensional continuum models. In particular, we are interested in the fact that they found a lack of convergence in the case of a three-layered elastic strip under bending and shear where the inner layer (i.e., the core) becomes extremely soft with respect to the outer layers (i.e., the skins), so that $E_c/E \to 0$. Of course, this is the extreme case of a sandwich in which the Saint Venant principle does not hold.

5 ACKNOWLEDGEMENTS

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