STABILITY AND VIBRATION OF ROADS UNDER FOLLOWER FORCES; the Boundary Characteristic Orthogonal Polynomials (BCOP) method.

N.M. Auciello

Department of Structural Engineering,
University of Basilicata,Via dell'Ateneo Lucano,10 - 85100 – Potenza – ITALY
E-mail: auciello@unibas.it

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SUMMARY. A procedure is illustrated in order to study beams with variable cross-sections subjected to sub-follower forces. The analysis is based on a variational approach with Boundary Characteristic Orthogonal Polynomials (BCOP) are chosen as trial functions polynomial trial functions, employed by the same authors for beams with constant cross sections [9]. The instability regions by divergence and flutter are examined in some detail, so allowing to determine the range of applicability of the static instability criterion. Some numerical examples are compared with other classical results from the bibliography, so confirming the goodness of the proposed approach.

1. INTRODUCTION

As well known, the instability phenomena of various engineering structures depend on the boundary conditions. Moreover, it is important to consider if the applied forces admit or not a potential function, because some kind of nonconservative follower forces lead to structural behaviour which cannot be examined by using the classical static criterion. Therefore, in the presence of follower forces it is mandatory to use the dynamic criterion in order to predict the correct critical load multiplier. [1-3].

Beams have been used for various purposed for many structures and hence the vibration behaviour of beams has a great importance in many engineering applications such as in the design of machines and structures.

For nonconservative systems, the frequencies can be either real or complex. Therefore, when instability occurs, the lowest frequency can pass trough the origin, as for the conservative system, or two frequencies can approach each other, coincide, and the become complex conjugate. The latter situation is defined as flutter instability, and the load at which the two frequencies coincide is defined as the flutter load.

Recently, Chen [3] studied the instability behaviour of beams with variable cross section subjected to sub-tangential nonconservative follower forces, and the solution is numerically attained by using a Runge-Kutta routine.

An approximed – yet general – solution is proposed by Glabisz [2] for a generic structural element subjected to distributed follower forces (Leipholz) or to a concentrated tip force (Beck). The problem is approximated using a power series approach, and the dynamic loss-of-stability is deduced for beams on elastic soil and for various boundary conditions.

In this paper the author gives a variational Rayleigh-Ritz solution to the instability problem of beams with variable cross-section subjected to subtangential follower forces. If the rotations at the ends of the beam are denoted by $\varphi_{L,}$ and φ_R , then the nonconservative applied forces will considered to be functions of $\eta\cdot\varphi_L$ and $\eta\cdot\varphi_R$ respectively. The η parameter completely defines the system of sub-tangential concentrated follower forces at the ends, and, consequently, completely defines the dynamic behaviour of the system. The classical conservative Euler case is recovered at η=0, whereas if η=1 the beam is subjected to purely tangential forces (follower forces). As η varies in the range [0,1] critical loads are reached by means of divergence or flutter, so permitting to deduce the range of applicability of the static criterion. Usually, a precise parameter value $\eta=\eta_c$ exists, which separate the divergence region from the flutter region, and this parameter strongly influences the frequency-force relationship. As well known, in the case of a cantilever beam with constant cross section it is η_c = 0.5. If $\eta< \eta_c$ then the problem is governed by a selfadjoint boundary value problem and, hence, the instability is of the divergence type. Conversely, for $\eta> \eta_c$ the problem is governed by the flutter.

The paper will be devoted to this problem, for various boundary conditions, for beams with varying cross section subjected to sub-tangential follower forces [2]. The whole procedure is confirmed by numerical examples and comparisons with other classical results.

2. FORMULATION OF PROBLEM

Let us consider the general extended Hamilton principle, in the presence of nonconservative applied loads:

$$\int_{t_1}^{t_2} \delta(T-V)\,dt + \int_{t_1}^{t_2} \delta W\,dt = 0 \tag{1}$$

where T is the kinetic energy of the system, V is the potential energy, and dW is the virtual work of the applied nonconservative loads.

For this kind of problems, the functions u $\in$ [0,L] are square integrable, so that it is defined the following inner product *(u,u)* [4]:

$$(u,u) = \int_0^L u\,u\,dx\ . \tag{2}$$

If the oscillations are supposed to be small, the elastic energy of the system can be written as:

$$V = \frac{1}{2}(EI\,u'',u'') + \frac{1}{2}(P\,u',u'), \tag{3}$$

where P is the horizontal projection of the applied nonconservative forces at the ends.

The other, nonconservative part of the applied loads gives rise to the following virtual work:

$$\delta W = P\sin\eta\,[u']_0^L \cong P\eta[u']_0^L = P\eta\,[u'\delta u]_0 - P\eta\,[u'\delta u]_L\ . \tag{4}$$

Finally, the kinetic energy can be written as:

$$T = \frac{1}{2}\rho\,\omega^2\,(A\,w,w) \tag{5}$$

where ρ is the mass density, ω the free vibration frequency, and A is the cross sectional area.

According to the Rayleigh-Ritz approximation method, the solution will be expressed as a linear combination of independent functions, and it is convenient to express the displacements of the beams as a sequence of orthogonal polynomials.

It will be:

$$u_n = q_i\, f_i = \mathbf{f}^{\mathbf{T}}\,\mathbf{q} \qquad i = 1,..n \tag{6}$$

where f is the eigenfunction vector, and q contains the lagrangian coordinates. The eigenfunctions fi should only satisfy the essential boundary conditions, and they can be deduced following an iterative method based on the ortho-normalization Schmidt method [4].

If the approximation (6) is introduced into the Hamilton principle (1), the following homogeneous system is obtained:

$$[\mathbf{K} + p\,(\mathbf{B} - \eta\,\mathbf{W}) - \Omega^2 \mathbf{M}]\mathbf{q} = \mathbf{0}, \tag{7}$$

where some nondimensional quantities have been defined:

$$\Omega_i^2 = \omega_i^2 \frac{\rho\, A_0\, L^4}{E\, I_0}, \qquad p = \frac{P\, L^2}{E\, I_0} \tag{8}$$

and the following matrices can be easily built up:

$$\begin{aligned} [\mathbf{K}]_{ij} &= (f_i\,'', f_j\,''), \\ [\mathbf{B}]_{ij} &= (f_i\,'\ , f_j\,'), \\ [\mathbf{W}]_{ij} &= \left[f_i\, f_j \right]_0 + \left[f_i\, f_j \right]_L , \\ [\mathbf{M}]_{ij} &= (f_i, f_j)\ . \end{aligned} \tag{9}$$

The free vibration frequencies can be calculated by imposing :

$$\det[\mathbf{K} + p\,(\mathbf{B} - \eta\,\mathbf{W}) - \Omega^{\,2}\mathbf{M}] = 0. \tag{10}$$

From a computational point of view, the presence of unsymmetrical matrices leads to complex conjugate solutions, and an iterative approach seems to be the simplest solution algorithm.

Two different cases can be faced, according to the η value.

If $\eta<\eta c$ the normalized critical load p_c corresponds to $\Omega 1=0$, and it can be deduced using the static criterion. The condition:

$$\det\left[\mathbf{K}+p\left(\mathbf{B}-\eta\,\mathbf{W}\right)\right]=0, \tag{11}$$

gives the solutions pi and the critical load is $p_c=p_1$. As η increases, a threshold value ηc is reached, beyond which the structure loses stability by flutter, and the static criterion is no longer applicable. At $\eta>\eta_c$ the solutions pi of equation (11) turn out to be complex, and the critical load must be calculated using eqn.(10), corresponding to the coalescence of the first two free vibration frequencies.

3. NUMERICAL RESULTS AND DISCUSSION

In order to illustrate some numerical examples and comparisons with other known results, let us consider now a beam with varying cross section, in which area and moment of inertia of the cross section obey to the following laws [3]:

$$A(x)=A_0\left(1+\frac{\alpha x}{L}\right)^2,$$

$$I(x)=I_0\left(1+\frac{\alpha x}{L}\right)^4; \tag{12}$$

where A_0 and I_0 area the cross sectional area and moment of inertia at *x=0,* respectively. For a circular cross section, it will be:

$$I_0=a^4\frac{\pi}{4}, \qquad A_0=a^2\pi, \tag{13}$$

where *a* is the radius of the section at x=0, and therefore:

$$\Omega_i^2=\omega_i^2\frac{\rho A_0 L^4}{\pi^2 E I_0}, \qquad p=\frac{P L^2}{\pi^2 E I_0} \tag{14}$$

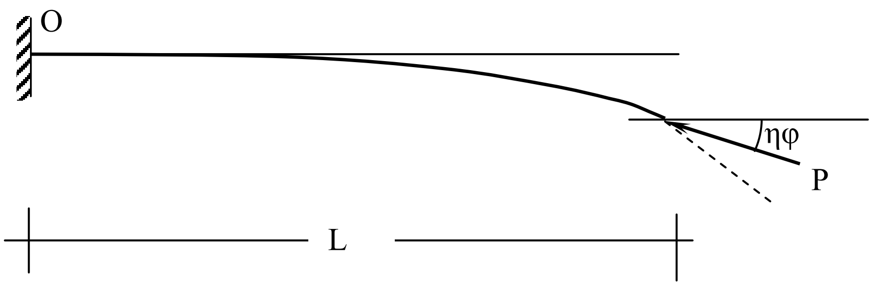

Figure 1: Cantilever beam with sub-tangential force.

As a first example, the column with constant cross section has been studied, subjected to a sub-tangential load, α=0; (Fig. 1).

Using eqn. (11) the critical load is given, for different parameter value η.

At η=0 we have two different values p_1 and p_2 , whereas the difference $(p_1 - p_2)$ diminishes with increasing η, and at $\eta = \eta_{c\,r}$ the two values coalesce. At this parameter value η_c we have the passage from divergence to flutter [1,2, 9]. The results are given in the first columns of table 1, where a comparison with the Chen [3]. For the sake of completeness, the p_2 values are also given. In table 2 the results can be compared with the critical loads given by Glabisz [2]. It can be the correlation of the result are excellent.

α=0	Chen [3]	Present		α=0,5	Chen [3]	Present	
η	p_1	p_1	p_2	η	p_1	p_1	p_2
0	0,24	0,2499	2,24999	0	0,4	0,3997	4,9096
0,2	0,337	0,3369	2,0151	0,3	0,625	0,6252	4,3078
0,3	------	0,4109	1,8469	0,59	1,673	1,6725	2,5503
0,45	------	0,6519	1,4279	0,6	------	1,9355	2,2499
0,48	------	0,7644	1,2671	0,601	------	2,0036	2,1780
0,49	0,829	0,8291	1,1868	0,6014	------	2,0630	2,1171
0,5	1	1	1	0,60144	2,09	2,0860	2,0860

α=–0,5	Present	
η	p_1	p_2
0,0000	0,1042	0,6114
0,2000	0,1509	0,4960
0,3000	0,2058	0,4007
0,3200	0,2274	0,3703
0,3300	0,2431	0,3501
0,3400	0,2702	0,3185
0,3425	0,2938	0,2938

Table 1 :The dependence of critical load vs subtangential parameter; $\eta<\eta_{cr}$.

A geometrical sketch of the functional relationship between critical loads and subtangential parameter is given in Figure 2 for various α values. As can be seen, $p_1 \to p_2$ for $\eta \to \eta_c$.

If η is higher than its critical value η_c the static criterion is no longer applicable, and the critical flutter load must be sought by applying the dynamic criterion and the complete equation (11). The critical value η_c is the threshold value between the divergence region and the flutter region.

$\alpha=0$	Chen [3]		Present		$\alpha=0,5$	Chen [3]		Present	
η	p_C	$\Omega_1=\Omega_2$	p_C	$\Omega_1=\Omega_2$	η	p_C	$\Omega_1=\Omega_2$	p_C	$\Omega_1=\Omega_2$
0,51	1,627	0,732	1,6267	0,7315	0,61	3,854	0,799	3,8542	0,7982
0,55	1,632	0,788	1,6321	0,7876	0,80	4,020	0,964	4,0203	0,9631
0,80	1,782	1,009	1,7815	1,0085	1,0	-----	-----	4,3731	1,0733
1,00	2,032	1,118	2,0316	1,1161	1,2	4,932	1,145	4,9325	1,1438
1,5	-----	-----	3,1033	1,2215	1,5	-----	-----	6,2413	1,1697
2	-----	-----	3,8272	1,1855	1,6	6,765	1,155	8,4900	1,0313

$\alpha=-0,5$	Glabisz [2]		Present	
η	p_C	$\Omega_1=\Omega_2$	p_C	$\Omega_1=\Omega_2$
0,3425	0.3771	0.5490	0,3771	0,5490
0,5	0,3808	0,8735	0,3808	0,8735
0,80	0.4968	1.1433	0,4969	1,1434
1,0	0.6560	1.2650	0,6592	1,2709
1,5	0,9486	1,3211	0,9487	1,3212
2	1.0251	1.2780	1,0252	1,2782

Table 2: The dependence of critical load vs subtangential parameter; $\eta>\eta_{cr}$.

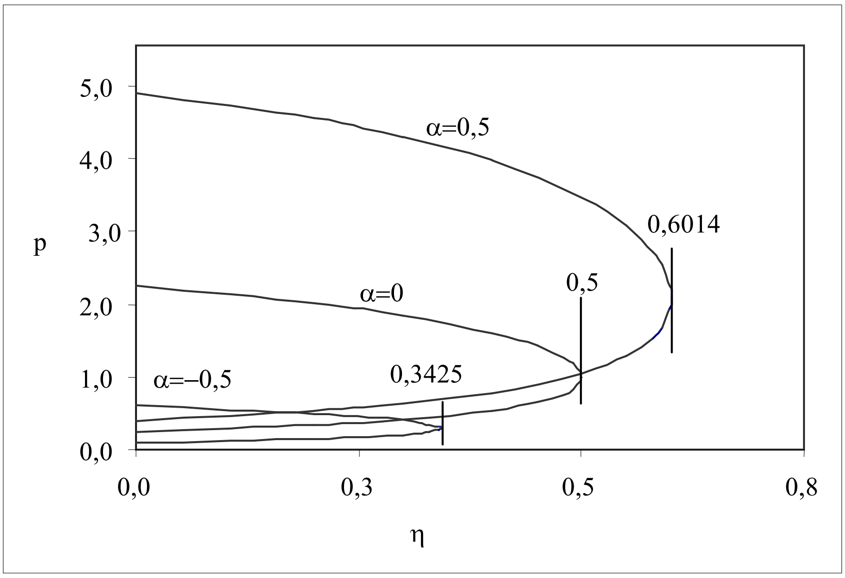

Figure 2: The dependence of critical load vs subtangential parameter for various tapered coefficients (α).

In the flutter region $\eta \geq \eta c$, the dynamic criterion is followed, and the axial load is found, such that the first two frequencies coalesce, $\Omega_1 = \Omega_2$. The corresponding data for different taper ratio, are given in the Figure 3. For $\eta = \eta_c = 0.5$ the first two frequencies become zero at p=1 the two force-frequency curves are merged into one point Q, and the force-frequency curves become AQ and BQ. In fact, $\eta_c = 0.5$ is the limitation for which the static criterion of the loading by following; for $\eta < \eta_c = 0{,}5$, the lowest root of *p* from equation (11) is the bucking loading of the cantilever. In the case $\eta > \eta_c$, the relevant curves are plotted in Figure 3, we see that the force-frequency curve is significantly changed, if one compare to the case of $\eta = \eta_c$. For example, for $\eta = 0.51$ case, we see that there is no solution for *p* from the equation (11). That is to say, the static formulation of the buckling problem cannot give a solution of *p*.

For $\eta > \eta_c$ the force-frequency curves are gradually changed from *AB* for $\eta = 0.51$ to *ATB* for $\eta = 2$. It is natural to define the critical force by the following dynamic criterion.

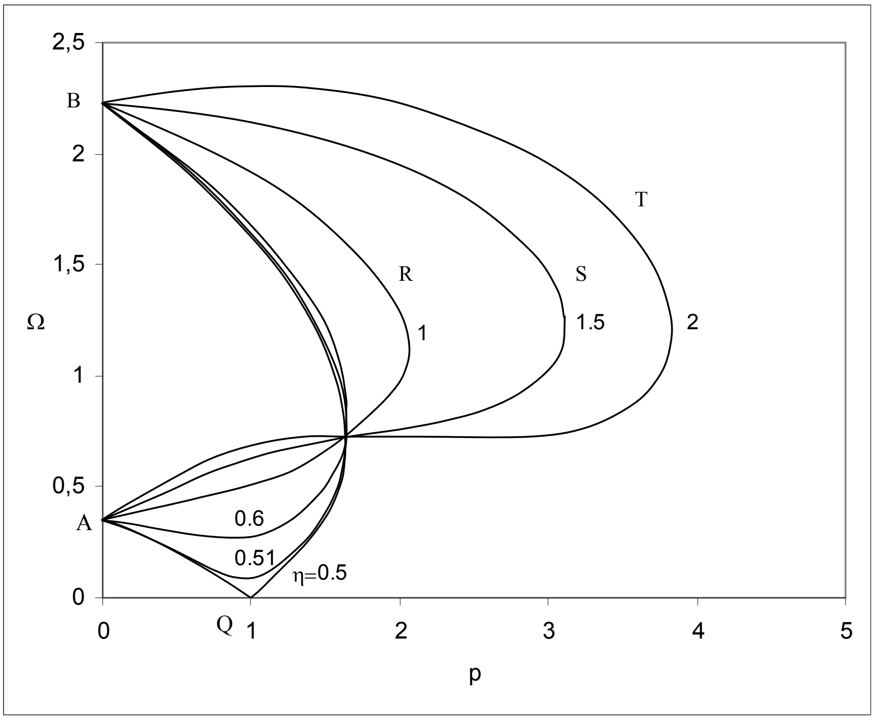

Figure 3: The dependence of the free vibration frequency vs subtangential force for various η; uniform section $\alpha = 0$.

Analogous results are shown in Figures 4 for $\alpha = 0.5$ with $\eta > \eta_c = 0.60144$, the force-frequency curves are gradually changed from *AB* for $\eta = 0.61$ to *ATB* for $\eta = 2$.

Similarly, for $\alpha = -0.5$ with $\eta > \eta_c = 0.3425$, the results are shown in Figures 5.

4. CONCLUSIONS

A general approach is discussed for the analysis of tapered beams subjected to non-conservative sub-tangential loads. The analysis does not depend on the boundary conditions, and allows the determination of the critical parameters signalling the passage from divergence to flutter. The whole procedure is extremely stable, even in critical conditions.

The use of the dynamic loss-of-stability criterion made it possible to find the critical loads of the flutter type for different boundary condition.

Using one computer package the natural frequencies are calculated and the effects of the incorporated parameters are examined. Moreover, numerical examples are solved to make comparisons with the existing results in open literature and it is observed that the agreement between the results is very good.

The numerical examples have been completely carried through by means of the powerful software symbolic program.

The basic concept to form this set of admissible function is reasonable and simple and requises no complicated mathematical knowledge. Moreover, from the above analysis, it can be seen that the admissible functions presented in the paper are closely related to the variation of flexural rigidity of the beam, but near unrelated to the variation of cross-sectional area of the beam which shows that the effect of flexural rigidity on the model shape function of the tapered beams is much greater than that of cross-sectional area.

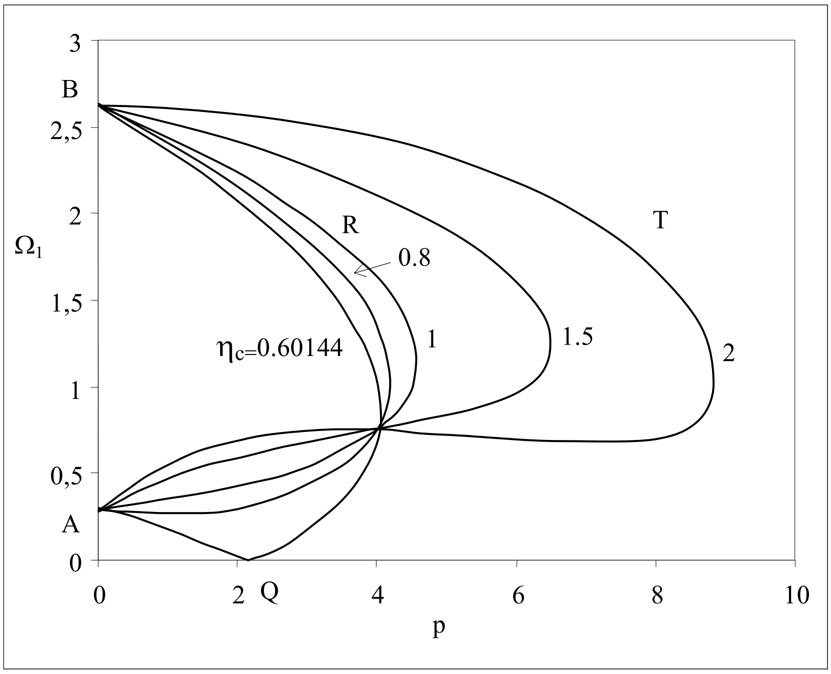

Figure 4: The dependence of the free vibration frequency vs subtangential force for various η; tapered beam α= 0,5.

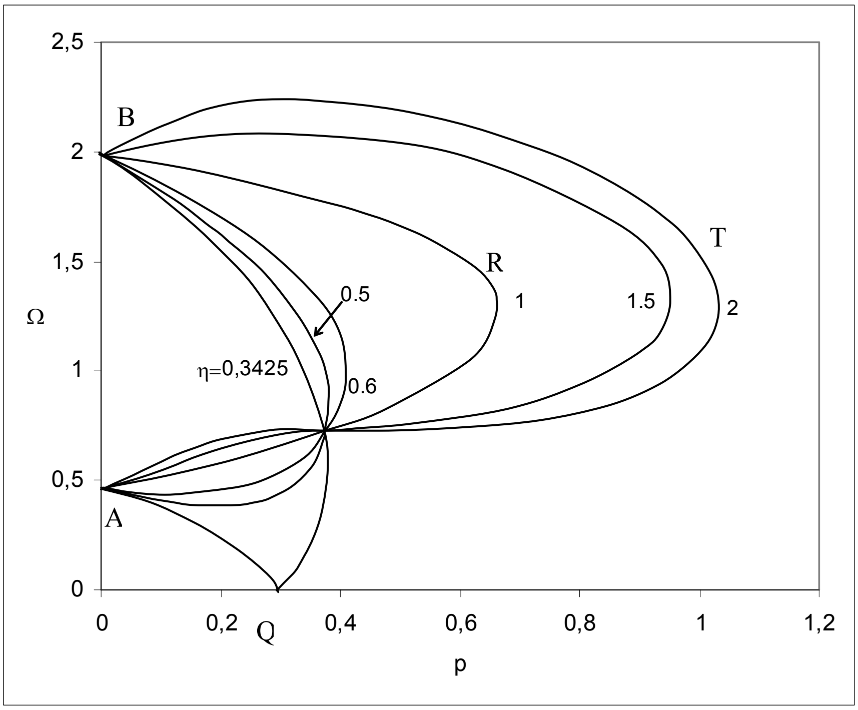

Figure 5: The dependence of the free vibration frequency vs subtangential force for various η; tapered beam α= -0,5.

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Nomenclature

a	radius of section in x=0
δW	virtual work
A; A_o	cross-sectional area of beam; cross-sectional area of beam in *x=0*
B; **W**	matrix in eq. (11)
E	Young's modulus of beam material
f	vector eigenfunction vector
I; I_o	area moment of inertia; area moment of inertia in x=0
K; **M**	stiffness matrix; mass matrix
L	length of the beam
$\bar{p}$;p	dimensionless partially tangential load, eq. (7); eq. (14)
P	partially tangential load
p_c	critical buckling load parameter
q	vector coefficients of trial function
T	kinetic energy
u	amplitude of the transverse deflection
V	potential energy
$\bar{\Omega}_i ; \Omega_i$	*i*th non-dimensional eigenfrequency of beam, eq. (8); eq. (14)
α	thickness ratio
η	tangential coefficient
η_c	critical tangential coefficient
ρ	mass density
$\omega, \bar{\omega}$	natural frequency, eq. (5); eq. (14)