Nonhomogeneous nonlocal elasticity: a finite element approach

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SUMMARY. A variationally consistent finite element approach for solving nonhomogeneous nonlocal elastic 2D boundary value problems is presented and the related computational issues are discussed. The method is developed in the framework of nonlocal integral elasticity grounding on a numerical procedure known, in the relevant literature, as *Nonlocal Finite Element Method*. The behaviour of the considered nonhomogeneous nonlocal elastic material is described assuming a recently proposed phenomenological *strain-difference-based* constitutive law. Numerical results, concerning a simple 2D example, are presented and discussed.

1 INTRODUCTION

The classical (local) elasticity theory has been widely employed in the context of continuum theories to solve with success a large number of engineering problems. However, as known, such theory is unable to capture phenomena which indeed can be reasonably explained only at microstructural level. Well-known examples are: the occurrence of size effects; the dispersion of elastic waves; the singularity of the stress at crack tip. Several remedies have been proposed since the late fifties to get around the failure of classical continuum theories. Approaches dealing with the detailed description of material microstructure have been developed in the framework of atomic and lattice mechanics (see e.g. [1]). However, it has been soon recognized that the application of such theories to realistic engineering problems is computationally unfeasible. Therefore, it appeared more sensible to account for the effects of material microstructure within a continuum formulation. This is the key idea behind the nonlocal elasticity theories which include, besides the contact forces between particles (occurring in a local elastic approach), long-range cohesive forces [2,3] able to catch the capacity of an elastic material to transmit information to neighbouring points within a certain distance. This distance, herein named *influence distance*, is strictly related to an *internal length material scale* which enters the constitutive material model in different ways i.e. by considering body couples as in polar elasticity; or by gradient operators or, moreover, by integral operators or, very recently, by fractional quantities (see e.g. [4-8]). Several applications of nonlocal elastic continuum approaches to nanomaterials, where size effects often become prominent, are also traceable in the recent literature from the work of Peddieson *et al.* [9], who developed a nonlocal Euler-Bernoulli beam model, till the remarkable contribution of Aifantis [10]. The latter study shows that continuum elasticity can indeed be properly extended to address a variety of problems at micro/nano regime by including long-range or nonlocal material point interactions and surface effects in the form of higher-order stress/strain gradients.

The above list of contributions, far to be exhaustive, testifies the increasing interest of researchers towards nonlocal approaches. In the authors' opinion, besides some theoretical aspects still open to discussion, the issues that need further investigations are those related to the solution of nonlocal elastic boundary value problems. However, the complexity of analytical solutions, even for simple 1D problems, implies that only numerical approaches can be efficiently applied in a more general 2D or 3D context. Recently, the authors have implemented a finite element method for analyzing 2D Eringen-type nonlocal elastic problems [11]. The procedure, named NL-FEM - where NL stands for "NonLocal"- has been first theorized by Polizzotto [12] on the base of a *nonlocal total potential energy principle* conceivable as an extension of the analogous principle of classical (local) theory. The obtained numerical results, though showing the potentialities and the effectiveness of the NL-FEM, are often distressed by some numerical oscillations or incoherencies such as undesired boundary effects. A typical example is the increasing trend of the strain profile close to the end sections of an Eringen-type nonlocal bar under uniform tension [13].

In order to overcome the above drawbacks, recently the authors have implemented an enhanced version of the NL-FEM [14] by assuming a phenomenological nonlocal constitutive model proposed by Polizzotto *et al.* [15] for (macroscopically) *nonhomogeneous* linear elastic materials in isothermal conditions. The latter, based on a firm thermodynamic formulation, turns out to be a two components local/nonlocal model since the stress is expressed as the sum of two contributions: one is the local stress governed by the standard elastic moduli tensor assumed variable in space; the other is of nonlocal nature and is expressed in terms of an *averaged strain-difference field*. A notable feature of the assumed constitutive relationship is that in the presence of a uniform strain field the nonlocal aliquot of the stress vanishes and the local behavior is recovered both in terms of stress and energy, while the symmetry of the nonlocal operators is preserved. This circumstance enables to avoid numerical instabilities and incoherencies on the strain distribution close to the domain boundaries. The present paper aims to explore the validity of the strain-difference-based NL-FEM when dealing with nonhomogeneous nonlocal elastic 2D boundary value problems. In particular, the method is formulated assuming nonhomogeneous elastic moduli, constant internal length and an attenuation function depending on the Euclidean distance. The numerical implementation of the NL-FEM is described with reference to a simple 2D example focusing the attention on the nonlocal operators, their physical meaning as well as on the main differences they exhibit with respect to the operators pertaining to the standard FEM.

2 CONSTITUTIVE ASSUMPTIONS

Let us consider a nonlocal linear elastic nonhomogeneous continuum in its undeformed state, occupying the volume V of a three-dimensional Euclidean domain referred to orthogonal axes with Cartesian coordinates $\boldsymbol{x}=(x_1,x_2,x_3)$. The constitutive behavior of the considered material is herein described assuming the phenomenological nonlocal model proposed by Polizzotto *et al.* [15], that is:

$$\boldsymbol{\sigma}(\boldsymbol{x})=\boldsymbol{D}(\boldsymbol{x}):\varepsilon(\boldsymbol{x})-\alpha\int_V \boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}'):\left[\varepsilon(\boldsymbol{x}')-\varepsilon(\boldsymbol{x})\right]\mathrm{d}\boldsymbol{x}' \quad \forall(\boldsymbol{x},\boldsymbol{x}')\in V. \tag{1}$$

The constitutive relation (1) expresses the stress response, $\boldsymbol{\sigma}(\boldsymbol{x})$, to a given strain field, $\boldsymbol{\varepsilon}(\boldsymbol{x})$, as sum of two contributions: the first one is the local stress governed by the standard symmetric and positive definite elastic moduli tensor $\boldsymbol{D}(\boldsymbol{x})$ assumed variable in space; the second one is of nonlocal nature and depends on the strain difference field $\left[\varepsilon(\boldsymbol{x}')-\varepsilon(\boldsymbol{x})\right]$ through the symmetric nonlocal tensor $\boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}')$ defined as:

$$\boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}'):=\left[\gamma(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x})+\gamma(\boldsymbol{x}')\boldsymbol{D}(\boldsymbol{x}')\right]g(\boldsymbol{x},\boldsymbol{x}')-\boldsymbol{q}(\boldsymbol{x},\boldsymbol{x}') \quad \forall\ (\boldsymbol{x},\ \boldsymbol{x}')\in V, \tag{2}$$

where:

$$\gamma(\boldsymbol{x}) := \int_V g(\boldsymbol{x},\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'; \qquad (3a,b)$$
$$\boldsymbol{q}(\boldsymbol{x},\boldsymbol{x}') := \int_V g(\boldsymbol{x},z)g(\boldsymbol{x}',z)\,\boldsymbol{D}(z)\,\mathrm{d}z.$$

In the above operators, $g(\boldsymbol{x},\boldsymbol{x}')$ denotes a positive, symmetric scalar *attenuation function* which assigns a weight to the nonlocal effects induced at the field point $\boldsymbol{x}$ by a phenomenon acting at the source point $\boldsymbol{x}'$; it contains the internal length material scale, say ℓ, and it rapidly decreases with increasing (Euclidean) distance between $\boldsymbol{x}$ and $\boldsymbol{x}'$, vanishing beyond the so-called *influence distance* L_R. The latter is a multiple of the internal length ℓ and both L_R and ℓ are much smaller than the smallest dimension of the body. Moreover, in Eq. (1) α is a positive scalar coefficient driving the "degree" of nonlocality (the positiveness of the latter material parameter being required to avoid numerical instabilities; refer to the above quoted paper for details).

An attractive feature of the assumed nonlocal constitutive model (1) is that for any uniform strain field the nonlocal contribution vanishes and the stress recovers the local value, in agreement with some experimental findings on thin wires in tension executed by Fleck *et al.* [16] dealing with strain gradient plasticity. To this concern, the averaged strain-difference in Eq. (1) plays the same role of the strain gradient for strain gradient-dependent materials. By inspection of Eq. (1), it is also observed that material inhomogenities affect both the local and nonlocal part of the stress related to a given strain field through the spatially variable elastic moduli tensor $\boldsymbol{D}(\boldsymbol{x})$.

It can be verified that the stress-strain relationship (1) can be expressed in the following notable alternative form:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) + \alpha\int_V \boldsymbol{S}(\boldsymbol{x},\boldsymbol{x}') : \boldsymbol{\varepsilon}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}' \qquad \forall(\boldsymbol{x},\boldsymbol{x}')\in V \qquad (4)$$

where $\boldsymbol{S}(\boldsymbol{x},\boldsymbol{x}')$, related to the additional nonlocal stress contribution, can be interpreted as a *nonlocal stiffness tensor*. Specifically, the (singular) symmetric and positive definite tensor $\boldsymbol{S}(\boldsymbol{x},\boldsymbol{x}')$ is given by:

$$\boldsymbol{S}(\boldsymbol{x},\boldsymbol{x}') := \frac{1}{2}\left[\gamma^2(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) + \gamma^2(\boldsymbol{x}')\boldsymbol{D}(\boldsymbol{x}')\right]\delta(\boldsymbol{x}'-\boldsymbol{x}) - \boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}') \quad \forall\ (\boldsymbol{x},\boldsymbol{x}')\in V, \qquad (5)$$

where $\delta(\boldsymbol{x}'-\boldsymbol{x})$ denotes the Dirac delta function.

Notice that both $\boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}')$ and $\boldsymbol{S}(\boldsymbol{x},\boldsymbol{x}')$ vanish outside the influence zone defined by L_R being $g(\boldsymbol{x},\boldsymbol{x}')\approx 0$. This implies that the nonlocal contribution to the stress in the assumed constitutive model (Eq. (1) or (5)) depends only on the strains within the influence region.

3 NONLOCAL FINITE ELEMENTS FOR NONHOMOGENEOUS MATERIALS

Introducing the further hypotheses of small displacements and loads acting in a quasi-static manner, the boundary value problem for the above considered nonlocal linear elastic nonhomogeneous continuum is governed by the standard equilibrium and compatibility equations besides the assumed constitutive relation (Eq. (1) or (5)). In Ref. [15], the extension of the total potential energy principle to nonlocal continua obeying to the strain-difference based constitutive model has been presented. Taking into account Eq. (4) and the definition (5) of the nonlocal stiffness tensor $\boldsymbol{S}(\boldsymbol{x},\boldsymbol{x}')$, the pertinent functional can be expressed as follows:

$$\begin{aligned}\pi[\boldsymbol{u}(\boldsymbol{x})] := \frac{1}{2}\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{D}(\boldsymbol{x}) : \nabla\boldsymbol{u}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \gamma^2(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) : \nabla\boldsymbol{u}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \\ -\frac{\alpha}{2}\int_V\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}') : \nabla\boldsymbol{u}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x} - \int_V \boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{S_t} \boldsymbol{t}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\end{aligned} \tag{6}$$

where: ∇ denotes the gradient operator; $\boldsymbol{u}(\boldsymbol{x})$ is the unknown displacement field subject to the boundary conditions $\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x})$ on the surface portion S_u; $\boldsymbol{b}(\boldsymbol{x})$ are assigned body forces; finally, $\boldsymbol{t}(\boldsymbol{x})$ denotes tractions prescribed on $S_t = S - S_u$.

A finite element procedure for analyzing nonlocal nonhomogeneous elastic problems can be formulated grounding on the discretized form of the nonlocal total potential energy functional given in Eq. (6). To this aim, the domain V is subdivided into N_e finite elements (FEs) of volume V_n and, adopting the standard formalism of FEM, the displacement and strain fields within the n-th FE are given the following shapes

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{N}_n(\boldsymbol{x})\,\boldsymbol{d}_n; \qquad \boldsymbol{\varepsilon}(\boldsymbol{x}) = \nabla\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{B}_n(\boldsymbol{x})\,\boldsymbol{d}_n, \qquad \forall\boldsymbol{x}\in V_n, \tag{7a,b}$$

where $\boldsymbol{N}_n(\boldsymbol{x})$ and $\boldsymbol{B}_n(\boldsymbol{x})$ are the matrices of shape functions and their derivatives, respectively; $\boldsymbol{d}_n$ denotes the nodal displacement vector. Substituting Eqs. (7a,b) into Eq. (6), the functional $\pi[\boldsymbol{u}(\boldsymbol{x})]$ takes the following discretized form:

$$\pi[\boldsymbol{d}_n] = \frac{1}{2}\sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_n^{loc}\,\boldsymbol{d}_n + \frac{\alpha}{2}\sum_{n=1}^{N_e}\left[\boldsymbol{d}_n^T\,\boldsymbol{k}_n^{nonloc}\,\boldsymbol{d}_n - \sum_{m=1}^{N_e}\boldsymbol{d}_n^T\,\boldsymbol{k}_{nm}^{nonloc}\,\boldsymbol{d}_m\right] - \sum_{n=1}^{N_e}\boldsymbol{d}_n^T\,\boldsymbol{f}_n, \tag{8}$$

where $\boldsymbol{k}_n^{loc}$ and $\boldsymbol{f}_n$ are the standard element (local) stiffness matrix and equivalent nodal forces vector, given by:

$$\begin{aligned}\boldsymbol{k}_n^{loc} &:= \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x})\,\boldsymbol{D}(\boldsymbol{x})\,\boldsymbol{B}_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}; \\ \boldsymbol{f}_n &:= \int_{V_n} \boldsymbol{N}_n^T(\boldsymbol{x})\,\boldsymbol{b}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{S_{t(n)}} \boldsymbol{N}_n^T(\boldsymbol{x})\,\boldsymbol{t}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{9a,b}$$

The novelties in Eq (8), with respect to the standard local FEM formulation, are represented by the *element nonlocal stiffness matrices* $\boldsymbol{k}_n^{nonloc}$ and $\boldsymbol{k}_{nm}^{nonloc}$, defined as:

$$\begin{aligned}\boldsymbol{k}_n^{nonloc} &:= \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x})\,\gamma^2(\boldsymbol{x})\,\boldsymbol{D}(\boldsymbol{x})\,\boldsymbol{B}_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}; \\ \boldsymbol{k}_{nm}^{nonloc} &:= \int_{V_n}\int_{V_m} \boldsymbol{B}_n^T(\boldsymbol{x})\,\boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}')\,\boldsymbol{B}_m(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{10a,b}$$

Notice that the matrix $\boldsymbol{k}_n^{nonloc}$ differs from the standard local stiffness matrix in Eq. (9a) just for the presence of the nonlocal operator $\gamma^2(\boldsymbol{x})$ (see Eq. (3a)) which accounts for the influence exerted on the n-th element by the nonlocality diffusion processes over the whole domain. A quite different meaning may be given to $\boldsymbol{k}_{nm}^{nonloc}$ which, as can be inferred by its mathematical structure, takes into account the nonlocality effects of the m-th element on the n-th current one through the nonlocal operator $\boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}')$ and is therefore called *element nonlocal cross-stiffness matrix*. Notice that $\boldsymbol{k}_{nm}^{nonloc}$ vanishes when the elements $\# m$ and $\# n$ are far from each other with respect to the influence distance L_R since, as already mentioned, beyond this distance the attenuation function $g(\boldsymbol{x},\boldsymbol{x}')$ and consequently $\boldsymbol{J}(\boldsymbol{x},\boldsymbol{x}')$ take on almost negligible values. Following a similar reasoning, it is also observed that $\boldsymbol{k}_{n}^{nonloc}$ actually accounts just for the nonlocality diffusion processes taking place within the influence zone of the n-th element, namely within the portion of the whole domain in which $g(\boldsymbol{x},\boldsymbol{x}')$ and therefore $\gamma^2(\boldsymbol{x})$ are different from zero.

Equation (8) can be obviously rewritten in terms of global DOFs, collected in the vector $\boldsymbol{U}$, by using the standard correspondence $\boldsymbol{d}_n = \boldsymbol{C}_n\boldsymbol{U}$ where $\boldsymbol{C}_n$ denotes the usual connectivity matrix. The following global solving linear equation system, formally similar to the one pertaining to the local FEM, is then recovered in the shape:

$$\hat{\boldsymbol{K}}\boldsymbol{U} = \boldsymbol{F}, \tag{11}$$

where:

$$\begin{aligned} \hat{\boldsymbol{K}} &:= \sum_{n=1}^{N_e} \boldsymbol{C}_n^T \boldsymbol{k}_n^{loc} \boldsymbol{C}_n + \alpha \sum_{n=1}^{N_e} \left[\boldsymbol{C}_n^T \boldsymbol{k}_n^{nonloc} \boldsymbol{C}_n - \sum_{m=1}^{N_e} \boldsymbol{C}_n^T \boldsymbol{k}_{nm}^{nonloc} \boldsymbol{C}_m \right]; \\ \boldsymbol{F} &:= \sum_{n=1}^{N_e} \boldsymbol{C}_n^T \boldsymbol{f}_n . \end{aligned} \tag{12a,b}$$

$\hat{\boldsymbol{K}}$ and $\boldsymbol{F}$ denote the *nonlocal global stiffness matrix* and the standard global force vector, respectively. It is worth mentioning that $\hat{\boldsymbol{K}}$ turns out to be symmetric, positive semi-definite and banded but with a bandwidth larger than in the local FEM. Moreover, it can be observed that the nonlocal global stiffness matrix in Eq. (12a) clearly reflects the two/components local/nonlocal nature of the assumed constitutive model since it is given by the sum of two contributions: the first one coincides with the standard local global stiffness matrix; the second one represents the nonlocal addition controlled by the coefficient α. Finally, it is noted that, contrary to $\boldsymbol{k}_{n}^{nonloc}$, the matrices $\boldsymbol{k}_{nm}^{nonloc}$ do not follow the standard assemblage procedure thus leading to a bandwidth of $\hat{\boldsymbol{K}}$ larger than in the local case.

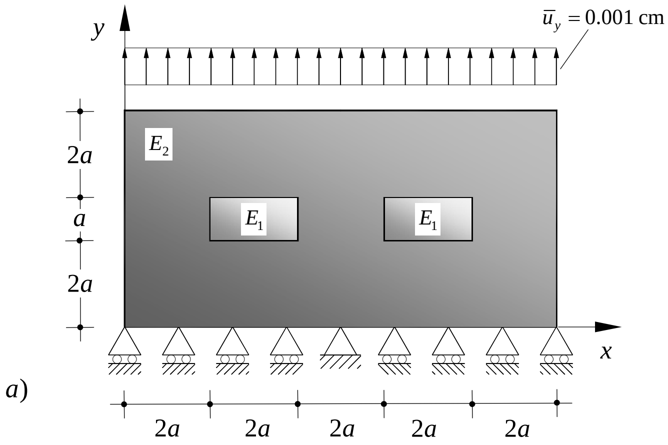
From a numerical point of view, special attention should be devoted to the evaluation of the nonlocal operators $\gamma(\boldsymbol{x})$ and $\boldsymbol{q}(\boldsymbol{x},\boldsymbol{x}')$ as well as of the cross-stiffness matrices $\boldsymbol{k}_{nm}^{nonloc}$. In particular, both $\gamma(\boldsymbol{x})$ and $\boldsymbol{q}(\boldsymbol{x},\boldsymbol{x}')$ (see Eqs. (3a,b)) require integrations over the whole domain which, following a standard FE treatment, can be performed resorting to a Gauss-Legendre quadrature involving all the Gauss points in the mesh. The computation of $\boldsymbol{k}_{nm}^{nonloc}$ can be carried out by standard quadrature as well, considering in this case only the Gauss points belonging to elements $\# n$ and $\# m$ since cross integrations between elements are involved (see Eq. (10b)).

To reduce the relatively prohibitive computational efforts implied by such integrations, the nonlocal operators $\gamma(\boldsymbol{x})$ and $\boldsymbol{q}(\boldsymbol{x},\boldsymbol{x}')$ can be evaluated considering not all the Gauss points in the mesh but only those falling within an influence region determined by L_R. In a similar way, for the generic element $\# n$, the computation of the cross-stiffness matrices $\boldsymbol{k}_{nm}^{nonloc}$ can be confined to the elements $\# m$, say M_e, falling within the influence region.

As regards the numerical integrations, it has also to be mentioned that an advantage of the adopted quadrature rule is that it allows the implementation of the expounded NL-FEM just by enriching standard (local elastic) FE codes with apposite subroutines. Appropriate comparisons with different standard procedures, available in many commercial codes (see e.g. [17]), have shown that a standard Gauss quadrature provides reliable approximations of the nonlocal operators and cross-stiffness matrices, at least for the run cases.

4 NUMERICAL APPLICATION

The presented NL-FEM has been implemented and applied to the nonhomogeneous plate under tension depicted in Fig. 1a.



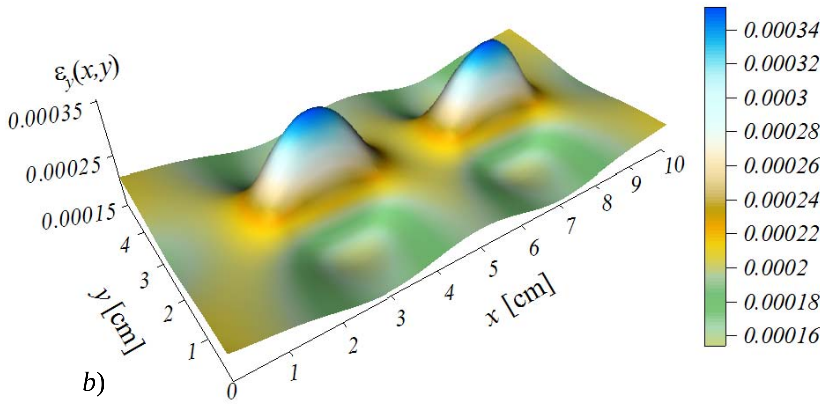


Figure 1: a) A nonhomogeneous plate under tension with piecewise constant Young modulus; b) 3D plot of the strain distribution $\varepsilon_y(x, y)$.

The plate, with $a = 1\text{cm}$ and thickness $t = 0.5\text{cm}$, is constrained along the edge at $y = 0$ and it is subjected to a uniform displacement $\bar{u}_y = 0.001\text{cm}$ at the opposite edge $y = 5a$. The Young modulus is piecewise constant with $E_2 = E_0$ over the whole structure except in two rectangular regions where it takes a smaller value $E_1 = E_0/3$. The strain-difference-based nonlocal constitutive model (1) with an attenuation function of the form $g(\boldsymbol{x}, \boldsymbol{x}') = \lambda_0 e^{-|\boldsymbol{x}-\boldsymbol{x}'|/\ell}$ (with $\lambda_0 = 1/(2\pi\ell^2 t)$ denoting the normalization factor, t the thickness of the structure and $L_R = 6\ell$)

has been adopted. The material properties have been selected as follows: Young modulus $E_0 = 2.1\times10^6\,\text{daN/cm}^2$, Poisson ratio $\nu = 0.2$, internal length $\ell = 0.1\text{cm}$ and $\alpha = 50$. Finally, the analysis has been carried out using 8-node Serendipity elements with 3×3 Gauss points per element and assuming a uniform mesh of 800 FEs with 40 subdivisions along x and 20 along y.

Figure 1b displays a 3D plot of the NL-FEM solution in terms of strain distribution $\varepsilon_y(x,y)$ over the plate.

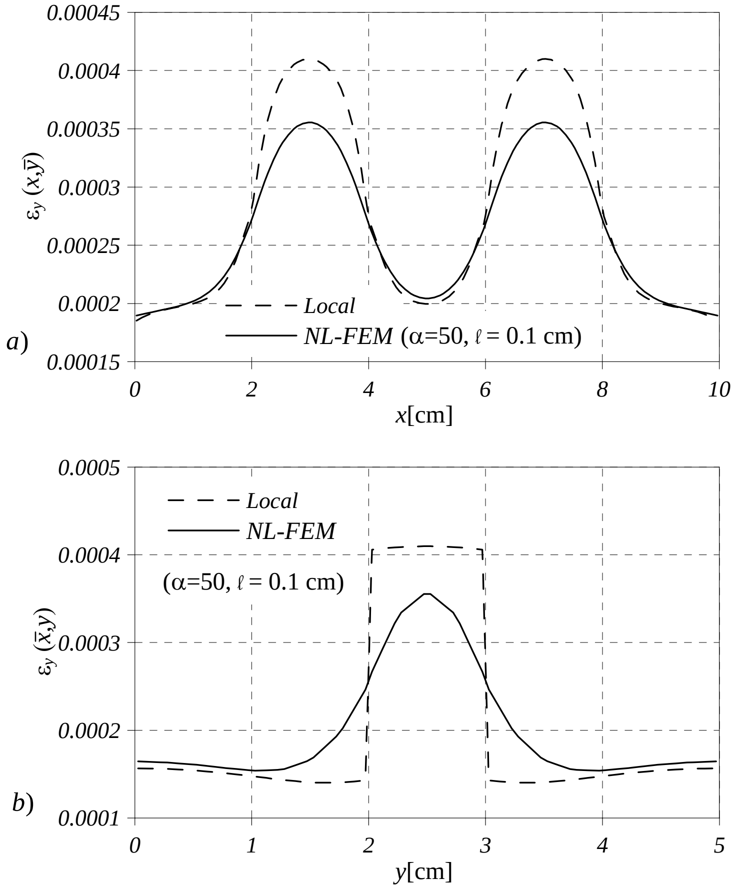


Figure 2: Strain profile of the local (dashed lines) and nonlocal (solid lines) solutions: a) ε_y versus x at $\bar{y} = 2.528\text{cm}$; b) ε_y versus y at $\bar{x} = 3.028\text{cm}$.

Plane sections of the 3D plot given in Fig. 1b at $\bar{y} = 2.528\text{cm}$ and $\bar{x} = 3.028\text{cm}$, respectively, along with the corresponding local FEM solutions are reported in Fig. 2(a,b). These plots clearly show that, both along the x and y directions, the nonlocal diffusion processes have a notable influence on the strain distribution arising around the transition sections between E_1 and E_2. In particular, by inspection of Fig. 2a it is observed that around the sections at $y = 2a$ and $y = 3a$, where the value of the elastic modulus abruptly changes, the strain profile along the direction of the prescribed displacement $\bar{u}_y$ (see Fig. 1a), is smoother than the one given by the local approach. Furthermore, it is noted that near the boundaries, the NL-FEM solution is very close to the corresponding local one thus confirming that the enhanced strain-difference-based constitutive model does not give rise to undesired boundary effects.

The nonlocal plate herein considered has also been analyzed in the case of homogeneous elastic modulus, i.e. $E = E_0$ over the whole structure. In agreement with the essential feature of the strain-difference-based nonlocal model, it has been found that the NL-FEM yields a uniform strain distribution, exactly coincident with the local one, for any choice of ℓ and α. Finally, it is worth mentioning that several numerical analyses have been carried out considering different FE meshes and the obtained results, herein omitted for conciseness, do not exhibit mesh dependence. Nevertheless, a rigorous proof to this concern is still missing and further investigations would certainly be necessary.

5 CONCLUSIONS

The paper has focused on the implementation of a nonlocal finite element approach which makes use of a strain-difference-based constitutive model. A nonlocal elastic nonhomogeneous material as well as a 2D boundary value problem have been considered. Issues related to the definition of the nonlocal operators have been discussed highlighting their physical meaning and giving details on the related integration procedures. The obtained results, though confined to a simple 2D academic example, have proved the capability of the method to overcome some drawbacks of nonlocal integral elasticity typically arising at the domain boundaries as well as to avoid numerical instabilities. The strain-difference-based NL-FEM thus seems to be a useful tool to handle a wide class of engineering problems.

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