An analytical model for elastic stability of full 2-D plates.

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SUMMARY. An analytical approach for the elastic stability of thin rectangular plates under arbitrary boundary conditions is here presented. Because a closed-form solution is developed, the proposed method can be described as 'exact' under the Kirchhoff-Love assumption. The analytical nature of the proposed procedure returns exact buckling load and modal displacements using a very coarse mesh. In the present paper an enriched longitudinal variation of displacements is adopted, in order to remove the restriction about one-dimensional nature of analytical model usually represented in literature and obtain a full two-dimensional model. Several cases of plate buckling are studied to show the generality of the method. The improvement of the method is demonstrated through comparison with finite element analysis and existing analytical solution available in literature.

1 INTRODUCTION

The understanding of the buckling behavior of plates subjected to in-plane forces has been an important area of investigation for many researchers, due to its wide use in many engineering applications. For the elastic buckling analysis numerical methods have been employed over the last years, most of them dealing with rectangular plates. Many of the useful results have been summarized in several texts and handbooks [1-3]. Finite Element Method (FEM) has represented a powerful tool, able to define buckling load and buckling mode to any structural member, loading and boundary condition, and a large number of FEM software packages is available. However, to provide accurate prediction of buckling response a large number of elements is required, and numerical instabilities as well as inefficient results can occur, especially when close buckling modes are present. When the geometry of structure becomes regular (i.e. open ruled surface), more efficient techniques can be successfully adopted. The Finite Strip Method (FSM), based on the discretization of the structure along the transverse direction only, is systematically employed in buckling analysis and it is often found to be more efficient to determine the critical loads for thin plates because of its reduced both computation times and numerical instabilities. The price of the reduced computational effort is that the method can be only applied on structures with specific geometry and boundary conditions: it works only on prismatic structures, and require simply supported edges. Semi-analytical, or *exact*, models can be considered variation of FSM. In the usual FSM the lengthwise variation of displacements is represented by harmonic functions, and a polynomial shape function is retained for the transverse variation of displacements. In a semi-analytical approach, under the same kinematical assumption about lengthwise displacements, represented by harmonic functions, it is possible to reduce the partial differential equilibrium equations (PDEE), that doesn't allow closed-form solutions, into a set of one-dimensional ordinary differential equilibrium equations (ODEE), suitable for analytical solution. Using closed form solutions as shape function in numerical approach can define *exact* buckling load and modal displacements throughout very coarse mesh. Semi-analytical method for buckling analysis has been extensively studied by Williams et al. works [4,5]. They proposed a FEM-like procedure for critical and post-critical behavior of isotropic and homogeneous plates rigidly connected, solving equilibrium equations obtained via perturbation technique [6-7]. Closed form solutions of buckling analysis of rectangular plates with different in-plane loads and improved mechanical models, removing Von-Karman or Kirchhoff hypothesis, can be found in literature: Hosseini-Hashemi et al. [8] presented an analytical closed form solutions of rectangular Mindlin plates, in order to deal with thicker and laminated composite plates. Leissa and Kang found out the solutions for free vibration and buckling of plates loaded by linearly varying in-plane distributed forces and moments, simply supported edges in *y* direction, clamped [9] or free [10] in *x* direction, using the classical power series method. Reddy and Pan [11] in a pioneer work defined a model capable to obtain critical load and critical mode of isotropic/orthotropic laminated plate. Minutolo et al. [12] removed the Von-Karman assumption about deformation-displacement relationship, in order to couple the second-order in-plane and out-of plane displacements in stiffened rectangular plates. Iuspa e Ruocco [13] adopted a closed form approach in an optimization procedure for the optimal design of isotropic/orthotropic thin structures involving weight limitation and/or buckling load, using genetic algorithm. For kinematical assumption in any analytical models proposed in literature, two opposite edges simply supported are required. In the present paper an enriched longitudinal variation of displacements is adopted, in order to remove the restriction of simply supported edges. It has been obtained coupling two different 1D models to get a 2D model capable to describe the boundary conditions completely. The improvement of the method is shown through comparison with results obtained using commercial FEM package ANSYS and existing analytical solution available in literature.

2 THE MODEL

We can consider an isotropic thin rectangular plate having lengths of *a* e *b* and constant thickness *h*, subject to in-plane compression N_L (fig. 1).

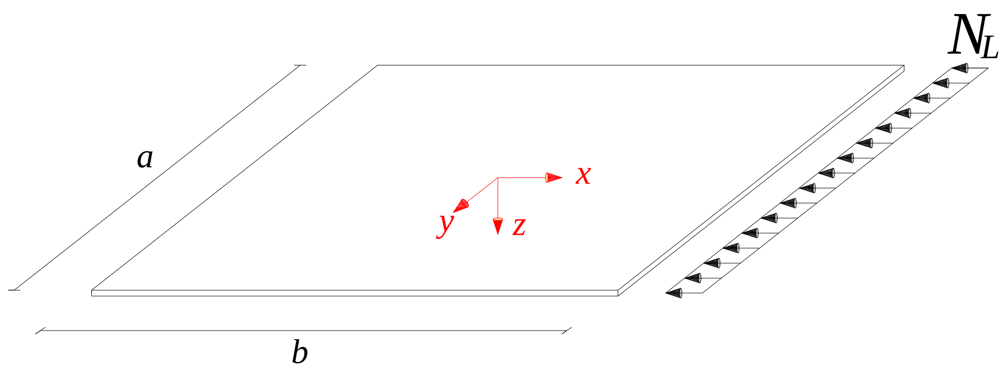

Fig. 1. Thin rectangular plate under in-plane load: dimension and reference frame

Under the classical thin plate hypothesis (Kirchhoff-Love's theory and von Kàrman strain-displacement relationship), the partial differential equation of equilibrium, in terms of out-of-plane displacement, is:

$$D\nabla^4 w + N_L w,_{xx} = 0 \qquad (1)$$

Where ∇^4 is the biharmonic differential operator ($i.e., w,_{xxxx} + 2w,_{xxyy} + w,_{yyyy}$ in rectangular coordinates), $D = \frac{Et^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate. Here $(\bullet)_{,i}$ represents $\frac{\partial(\bullet)}{\partial x_i}$.

Eq. (1) does not allow a closed form solution but, under more restrictive hypothesis on the displacement field, it is possible to reduce the PDEE to a set of one-dimensional ordinary differential equations of equilibrium, suitable for analytical solution. Classical approaches based on closed-form solution are *Semi-analytical approach* (SAA), where displacements are represented by harmonic function in *y* direction and *beam-like theory* (BLT), where displacements are represented by constant function in *x* direction.

The buckling solution defined using analytical solutions as shape function in a FEM-like approach doesn't depend on discretization adopted, and the number of representative elements is the minimum required for a complete representation of the geometry. In fig. 2 classical discretization required for a full description of plates in a FEM, FSM, and analytical approach is represented.

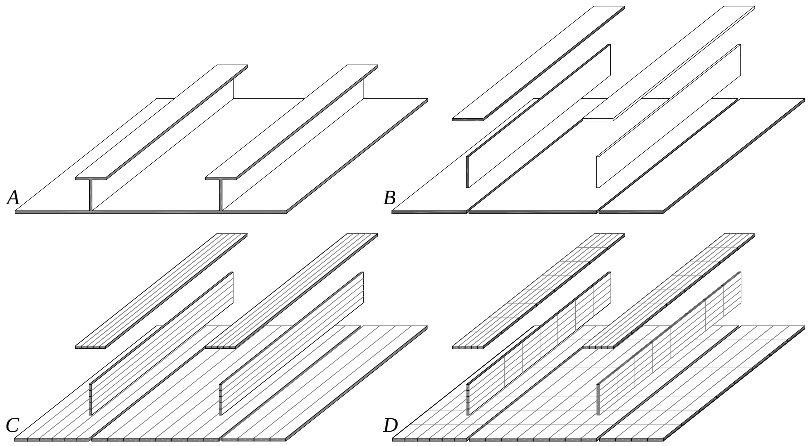

Fig. 2 A-B-C-D. (A) Typical stiffened plate and comparison between analytical (B), FSM (C) and FEM (D) required discretization

Counterpart of numerical and computational advantages are some geometrical restrictions, due to intrinsic one-dimensional nature of the analytical models adopted in literature. Both SAA and BLT models are briefly reported to show similitude and restrictions.

Let's consider the out-of-plane displacement $w(x, y)$ as:

$$w(x, y) = w_1(x) w_2(y) \qquad (2)$$

substituting Eq. (2) into eq. (1) we obtain:

$$D\left(w_1,_{xxxx} + 2w_1,_{xx}\, w_2,_{yy} + w_2,_{yyyy}\right) + N_L w_1,_{xx} = 0 \qquad (3)$$

Two analytical models, characterized by closed form solutions of Eq. (3), can be derived using in eq. (2) suitable displacements field. In a *Semi-analytical approach* we put $w_1(x)$ as

$$w_1(x) = \cos\frac{m\pi x}{b} \quad (m = 1,2,3,....) \qquad (4)$$

With m number of half-wave characterizing the buckling mode. Substituting Eq. (4), the eq. (3) becomes

$$\cos\frac{m\pi x}{a}\left[w_2,_{yyyy} - 2\frac{m^2\pi^2}{a^2} w_2,_{yy} + \left(\frac{m^4\pi^4}{a^4} - \frac{N_L}{D}\frac{m^2\pi^2}{a^2}\right) w_2\right] = 0 \qquad (5)$$

Eq. (5) can be solved analytically:

$$w_2(y) = c_1 e^{-\alpha y} + c_2 e^{\alpha y} + c_3 \cos\beta y + c_4 \sin\beta y \qquad (6)$$

Depending by N_L via α and β:

$$\alpha = \sqrt{\frac{m^2\pi^2}{b^2} + \sqrt{\frac{N_L}{D}\frac{m^2\pi^2}{b^2}}} \qquad \beta = \sqrt{-\frac{m^2\pi^2}{b^2} + \sqrt{\frac{N_L}{D}\frac{m^2\pi^2}{b^2}}} \qquad (7)$$

In (6) c_i represent coefficients defined imposing boundary conditions along $y = \pm\frac{b}{2}$.

In a *Beam-like theory* a solution for the displacement $w_2(y)$ may be taken as a constant function

$$w_2(y) = k \qquad (8)$$

Substituting (8), the (3) assuming the form

$$k\left[w_1(x),_{xxxx} + \frac{N_L}{D} w_1(x),_{xx}\right] = 0 \qquad (9)$$

Eq. (2.9) is suitable to following closed form solution

$$w_1(x) = d_1 \cos\gamma x + d_2 \sin\gamma x + d_3 x + d_4 \qquad (10)$$

Depending by N_L via γ :

$$\gamma = \sqrt{\frac{N_L}{D}} \tag{11}$$

In eq. (10) the constants d_i are defined by imposing boundary conditions along $x = \pm\frac{a}{2}$.

The SAA and BLT models have the same philosophy, and they are both well posed. Using the (6) or the (10) as shape function in a fem-like procedure, it is possible to define critical load and critical mode of structures with complex geometry and general boundary conditions along *y-edges* (SAA procedure) or *x-edges* (BLT procedure). The required *input* is one-dimensional, according to hypothesis (4) or (8) on displacement field, therefore buckling response of structures represented in figure 3 can be easily defined.

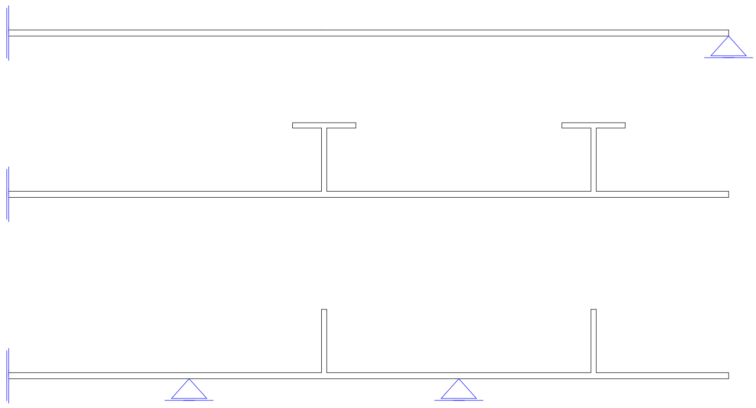

Fig. 3. Examples of typical one-dimensional input required by SAA and BLT method

Due to different displacement functions, the one-dimensional input represented in figure 3 recalls different two-dimensional structures (see fig. 4), with different boundary conditions imposed: the sinusoidal behaviour in y-direction for the SAA model requires simply supported edges in $x = \pm\frac{a}{2}$, where constant behaviour in x-direction for BLT model requires free ends in $y = \pm\frac{b}{2}$.

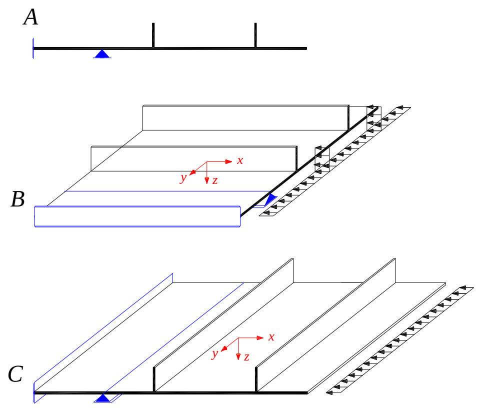

Fig. 4 A-B-C: two-dimensional representation of a typical one-dimensional input (A), in SAA approach (B) and BLT approach (C)

If we consider condition of plate clamped/free represented in fig. 5, for example, using a SAA approach we get:

$$\begin{aligned} & w^{(A)} = 0; \\ & \varphi_x^{(A)} = -w,_x^{(A)} = 0 \\ & M_x^{(B)} = -D\left(w,_{xx}^{(B)} + \nu\, w,_{yy}^{(B)}\right) = 0 \\ & T_y^{(B)} = -D\left(w,_{xxx}^{(B)} + (2-\nu) w,_{xyy}^{(B)}\right) = 0 \end{aligned} \Rightarrow k \begin{bmatrix} \cos\frac{\gamma a}{2} & -\sin\frac{\gamma a}{2} & -\frac{1}{2} & 1 \\ -\alpha\sin\frac{\gamma a}{2} & -\alpha\cos\frac{\gamma a}{2} & -\frac{1}{a} & 0 \\ D^2\gamma\cos\frac{\gamma a}{2} & D\gamma^2\sin\frac{\gamma a}{2} & 0 & 0 \\ 0 & 0 & -D\frac{\gamma^2}{2} & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (12)$$

whereas for BLT model the same conditions are defined as:

$$\begin{aligned} & w^{(A)} = 0; \\ & \varphi_y^{(A)} = -w,_y^{(A)} = 0 \\ & M_y^{(B)} = -D\left(w,_{yy}^{(B)} + \nu\, w,_{xx}^{(B)}\right) = 0 \\ & T_y^{(B)} = -D\left(w,_{yyy}^{(B)} + (2-\nu) w,_{yxx}^{(B)}\right) = 0 \end{aligned} \qquad (13)$$

it considers the expression (2.6), representative of $w_2(y)$, yields:

$$\cos\frac{m\pi x}{b} \begin{bmatrix} -\sinh\frac{\alpha b}{2} & \cosh\frac{\alpha b}{2} & -\sin\frac{\beta b}{2} & \cos\frac{\beta b}{2} \\ \alpha\cosh\frac{\alpha b}{2} & -\alpha\sinh\frac{\alpha b}{2} & \beta\cos\frac{\beta b}{2} & \beta\sin\frac{\beta b}{2} \\ D\left(\alpha^2 - \frac{\nu\pi^2}{b^2}\right)\sinh\frac{\alpha b}{2} & D\left(\alpha^2 - \frac{\nu\pi^2}{b^2}\right)\cosh\frac{\alpha b}{2} & -D\left(\beta^2 + \frac{\nu\pi^2}{b^2}\right)\sin\frac{\beta b}{2} & -D\left(\beta^2 + \frac{\nu\pi^2}{b^2}\right)\cos\frac{\beta b}{2} \\ D\left(\alpha^3 - \frac{(2-\nu)\pi^2\alpha}{b^2}\right)\cosh\frac{\alpha b}{2} & D\left(\alpha^3\cosh\frac{\alpha b}{2} - (2-\nu)\frac{\pi^2\alpha}{b^2}\sinh\frac{\alpha b}{2}\right) & D\left(\alpha^3\cosh\frac{\alpha b}{2} - \frac{(2-\nu)\pi^2\beta}{b^2}\cos\frac{\beta b}{2}\right) & D\left(\alpha^3\cosh\frac{\alpha b}{2} + \frac{(2-\nu)\pi^2\beta}{b^2}\sin\frac{\beta b}{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (14)$$

The (12) and (14) assume the form $k\mathbf{K}(\gamma)\mathbf{x} = \mathbf{0}$ and $\cos\frac{m\pi x}{b}\mathbf{K}(\alpha,\beta)\mathbf{x} = \mathbf{0}$, respectively.

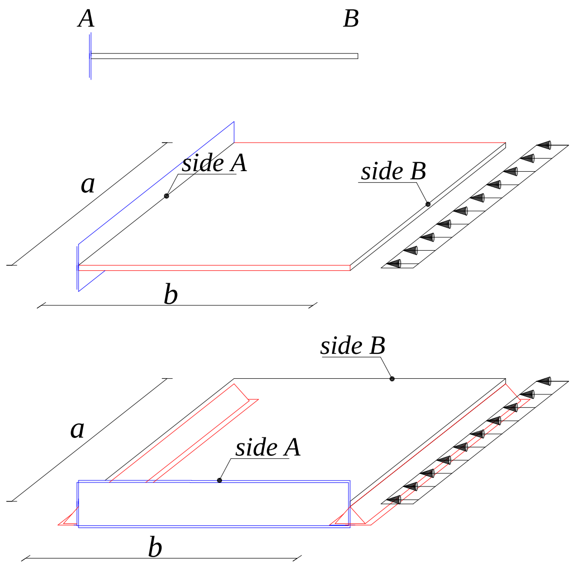

Fig. 5. Imposed boundary condition of a free plate (A) in BLT (B) and SAA (C) approach

Critical load can be defined imposing $\det(\mathbf{K}) = 0$. Taking into account the expressions (7) and (11) of $\alpha\left(N_{cr}^{BLT}\right), \beta\left(N_{cr}^{SAA}\right), \gamma\left(N_{cr}^{SAA}\right)$, it gets the critical loads represented in fig. (6) for different *a* values.

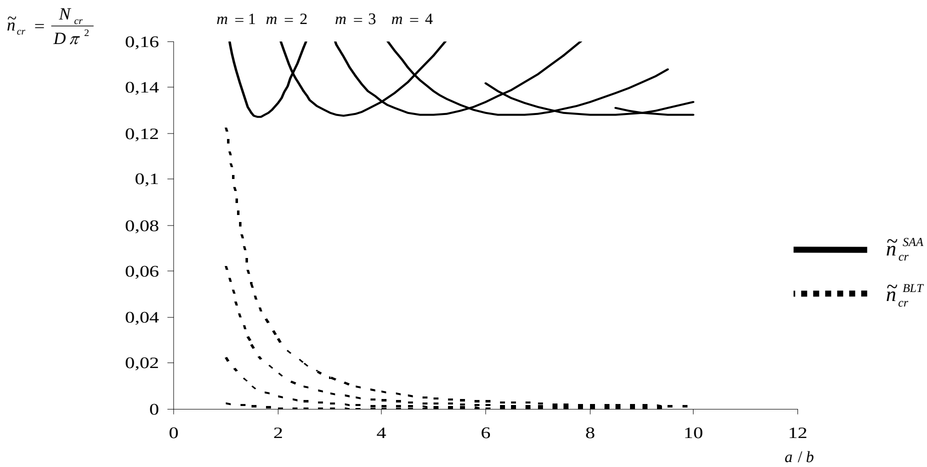

Fig. 6: Critical load of clamped-free plate via SAA approach and BLT approach

The correlate *critical mode*:

$$w_{cr}^{BLT} = \delta\left(1+\sin\frac{n\pi x}{a}\right)$$
$$w_{cr}^{SAA} = \delta\left(1+\sin\frac{m\pi x}{a}\right)\left(\cos\frac{\pi y}{b}\right) \qquad (15)$$

are representative of different normalized displacements field, as shown in fig. (7) and fig. (8).

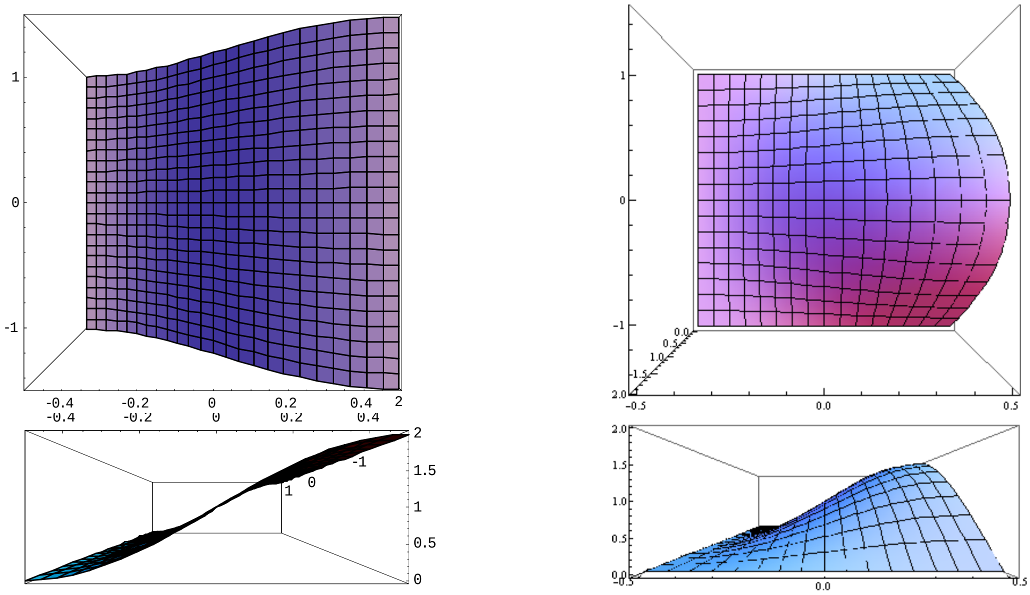

Fig 7: first mode of clamped-free plate of dimensions L and H, with BLT approach

The basic idea of the proposed model is a coupling of one-dimensional SAA and BLT models, in order to remove the restriction contained in both and to obtain a full two-dimensional model. Let us assume that the displacement function (2.2) is:

$$w(x,y) = \mathbf{n}(x)\mathbf{w}(y) \qquad (16)$$

Where

$$\mathbf{n}(x) = \begin{bmatrix} \sin\dfrac{m\pi}{xa} & \cos\dfrac{m\pi}{xa} & \dfrac{x}{a} & 1 \end{bmatrix}^T \qquad (17)$$

and $\mathbf{w}(y)$ a vector containing the four unknown functions

$$\mathbf{w}(y) = \left[w_A(y) \quad w_B(y) \quad w_C(y) \quad w_D(y)\right]^T \qquad (18)$$

Substituting the eq. (16), the eq. (3) assuming the form

$$\sin\frac{m\pi x}{a}\left[w_{A,yyyy}-2\frac{m^2\pi^2}{a^2}w_{A,yy}+\left(\frac{m^4\pi^4}{a^4}-\frac{N_L}{D}\frac{m^2\pi^2}{a^2}\right)w_A\right]+$$
$$\cos\frac{m\pi x}{a}\left[w_{B,yyyy}-2\frac{m^2\pi^2}{a^2}w_{B,yy}+\left(\frac{m^4\pi^4}{a^4}-\frac{N_L}{D}\frac{m^2\pi^2}{a^2}\right)w_B\right]+ \tag{19}$$
$$\frac{x}{a}\left[w_{C,yyyy}\right]+\left[w_{D,yyyy}\right]=0$$

The (19) admits solution for each x if each expression in square bracket is equal to zero. It is then possible to obtain four uncoupled ODEE , suitable for analytical solution:

$$\mathbf{w}(y)=\mathbf{F}(y)\cdot\mathbf{a} \tag{20}$$

Where

$$\mathbf{F}(y)=\begin{bmatrix} e^{-\alpha y} & e^{\alpha y} & \cos\beta y & \sin\beta y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\alpha y} & e^{\alpha y} & \cos\beta y & \sin\beta y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^3 & y^2 & y & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^3 & y^2 & y & 1 \end{bmatrix} \tag{21}$$

Depending on $\alpha(N_L)$ and $\beta(N_L)$, defined in (7). In (20) $\mathbf{a}$ is a vector containing 16 constants defined imposing boundary conditions in four points of each element of the structure

$$\mathbf{a}=\begin{bmatrix}a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 & c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 & d_4\end{bmatrix}^T \tag{22}$$

Substituting (3.5) in (3.3) we obtain

$$w(x,y)=\mathbf{n}(x)\cdot\mathbf{F}(y)\cdot\mathbf{a} \tag{23}$$

Using (23) as a shape function in a Fem procedure, it is possible to obtain critical load and critical mode of full 2D structures solving the associated eigenvector and eigenvalue problem. It does not depend on discretization adopted, as (21) are closed form solutions of equilibrium equation, and solutions are obtained with very coarse mesh. Since the uncoupling of equation of equilibrium (3) in four independent ordinary differential equations, it is not possible to obtain kinematical congruence everywhere: the classical fem approach is based on a *kinematic formulation:* it means that chosen displacement functions satisfy the displacement continuity between each point of adjacent elements but the equilibrium equation only in some representative nodes [14]. Differently, the proposed approach can be seen as a *static formulation* which means that chosen displacement functions satisfy the equilibrium equations between each point of adjacent elements but the displacement continuity only in some representative nodes.

3 CONCLUSIONS

In the presented study an exact solution procedure for buckling analysis of plates having all possible combinations of boundary conditions was performed. The proposed work unifies two one-dimensional models usually adopted in literature, obtaining a full 2-dimensional model capable to remove geometrical constrains on boundary conditions, in 1D models applied continuously along the *x* and *y* edges. Comparison with numerical results and analytical solutions available in literature has shown the performance of the model, capable to obtain closed form solutions and comparable results with very coarse mesh. The study can be extended defining a FEM procedure, coupling in-plane and out-of-plane displacements, in order to define critical response of stiffened plates with general geometry.

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