Non-local finite element method for the analysis of elastic continuum with long-range central interactions.

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Keywords: Non-local elasticity, long-range interactions, finite element method.

SUMMARY. In this paper the Finite Element Method (FEM) for the mechanically-based non-local elastic continuum model is proposed. In such a model non-adjacent elements are considered mutually interacting by means of central body forces that are monotonically decreasing with their interdistance and proportional to the product of the interacting volume elements. The resulting governing equation is an integro-differential one and for such a model both kinematical and mechanical boundary conditions are exactly coincident with the classical boundary conditions of the continuum mechanics. The solution of the integro-differential problem is framed in the paper by the finite element method. Finally, the solution obtained in the context of FEM is compared with finite difference method (FDM).

1 INTRODUCTION

In the last decades experimental results which cannot be explained by classical local continuum mechanics have forced scientists and engineers to formulate alternative theories which may better fit to the observed phenomena. These include shear bands in tensile specimens, acoustic wave dispersion in granular materials, softening phenomena and smoothing effects of concentrated stress at crack tips. It is generally believed that most of these phenomena may be provided with an exhaustive explanation at a micro-structural level and, since the end of the fifties, a first approach to this problem has been then framed in the context of atomic theory and lattice mechanics. Anyway, the strong computational efforts involved by the use of such theories have soon motivated an increasing interest towards continuum formulations where microstructural effects are accounted for by properly modified constitutive relations, including non-local terms as weighted integrals ([1],[2]) or gradients [3]. These approaches enjoy the beauty of an unique constitutive relation that does not involve additional state variables of the elastic problem. On the other hand, such a constitutive relation yields a non-convex potential energy functional. Recently, a nonlocal continuum model has been proposed for a 1D bar using a physically-based approach [4] resorting to a discrete grid of interacting volumes. In this model adjacent elements exert mutual classical contact forces while non-adjacent elements exert mutual, distance-decaying, central forces. The latter are taken to be proportional to the interacting volumes and to their relative displacement. As the size of the volume elements reduces to zero, the resulting 1D continuum is found to be governed by an integro-differential equation, for which either mechanical or kinematic boundary conditions may be consistently enforced. It may be shown that the physically-based approach enjoys all the features of the elasticity theory since

the associated elastic potential is convex and positive definite. Moreover since the long-range interactions are assumed to be central body forces, then they do not affect the mechanical boundary conditions. The physical model used has been set upon a physical description of the long-range interactions in the form of a point-spring network in which each point of the model is connected with all the others by linear springs with distance-decaying stiffness. Solution of the integro-differential problem may not be performed in closed-form and some analytical solution strategies, as extensions of the well-known Galerkin and Rayleigh-Ritz method have been introduced in [5]. Extension to three dimensional elastic problem has been also recently provided in [6]. In this paper a different, finite element strategy will be proposed to yield an approximate solution of the elastic problem. The finite element solution will be obtained introducing a proper discretization mesh of the bar domain and, introducing the locally-based shape function a discrete, algebraic system of equations will be obtained, similar to well-known basic FEM [7]. Numerical applications providing the capabilities of the proposed finite element method to handle non-local elastic problem will be provided and the results have been contrasted with finite difference method.

2 A PHYSICALLY-BASED APPROACH FOR 1D NON-LOCAL MODEL

Let us consider a simple 1D solid as depicted in fig.(1) under axial stress. Be L the length, $\tilde{E} = (1-\beta)\, E\ (0 \leq \beta \leq 1)$ the elastic Young modulus, $A(x)$ the cross-sectional area; let $u(x)$ the axial displacement and $b(x)$ the external body force. Moreover let us assume that in a generic volume element $V_j = A_j \Delta x \ \ \big(A_j = A(x_j)\big)$ whose centroid has coordinate x_j, the non-local contribution is provided by a central body force due to the other volume element $V_h\ (h = 1, 2, ..., n)$. Each of these body forces will be denoted as $Q^{(h,j)}$, then the overall resultant force in the volume element V_j denoted as Q_j is given as

$$Q_j = \sum_{\substack{h=1 \\ h \neq j}}^{n} Q^{(h,j)} \tag{1}$$

where n is the total number of elements, $n\Delta x = L$. Moreover let us assume that the central body forces are proportional to the product of interacting volume elements as $Q^{(h,j)} = q^{(h,j)} V_j V_h$ and, in the physically-based approach, $q^{(h,j)}$ are expressed as

$$q^{(h,j)} = \left[u\left(x_h\right) - u\left(x_j\right)\right] g\left(x_h, x_j\right) \tag{2}$$

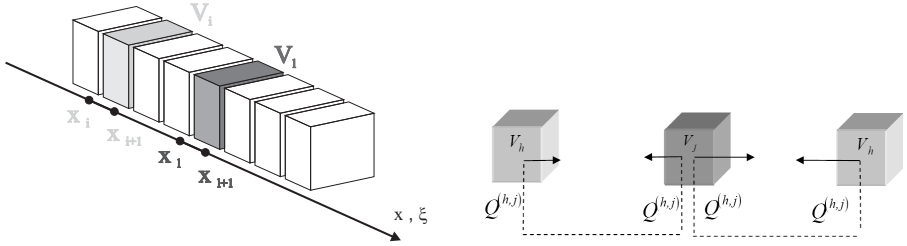
where $g(x_h, x_j) = g\left(|x_h - x_j|\right)$ is an appropriate distance decaying function wich accounts that in nature any long-range force decays with the interdistance of the material particles. Thermodynamic consistence as well as material stability of the proposed model [6] has been verified as $g\left(x_h, x_j\right)$ is a symmetric strictly positive distance decaying function.

With the physically-based model it is possible to consider the overall stress $\sigma\left(x\right)$ as a superposition $\sigma\left(x\right) = \sigma^{(l)}(x) + \sigma^{(nl)}(x)$ where $\sigma^{(l)}$ is the local (Cauchy) stress $\sigma^{(l)}(x) = N(x)/A(x)$ related to the strain $\varepsilon(x) = d\, u(x)/d\, x$ by the Hooke law $\sigma^{(l)}(x) = \tilde{E}\varepsilon(x)$. The non-local stress $\sigma^{(nl)}$ is given as a double integral involving long-range interactions

$$\sigma^{(nl)}(x) = \int_{\xi=x}^{L} \int_{\zeta=0}^{x} q(\zeta, \xi) A(\zeta) d\zeta d\xi = \int_{\xi=x}^{L} \int_{\zeta=0}^{x} g(\xi, \zeta) A(\zeta) \eta(\xi, \zeta) d\xi d\zeta \tag{3}$$

where we introduced the relative displacement function $\eta\left(x, \xi\right)$ defined as

$$\eta(x, \xi) = u(\xi) - u(x) \tag{4}$$



(a) (b)

Figure 1: (1) Mesh of a 1D bar. (b) Volume elements and long-range interactions in a 1D bar.

It is to be remarked that the overall stress $\sigma(x)$ for unbounded domain coalesces with that proposed in the Eringen model [4]. The equilibrium, compatibility and constitutive law for 1D solid are:

$$\begin{cases} \dfrac{d\sigma\left(x\right)}{dx} = \dfrac{d\sigma^{(l)}\left(x\right)}{dx} + \displaystyle\int_0^L q\left(x,\xi\right) A\left(\xi\right) d\xi \\ \dfrac{du\left(x\right)}{dx} = \varepsilon\left(x\right) \\ \sigma^{(l)}\left(x\right) = \widetilde{E}\varepsilon\left(x\right) \end{cases} \tag{5}$$

that, in conjunction with eq.(2) yields the complete formulation of the elastic equilibrium problem of the physically-based model of non-local elasticity. The goverining field equation for the simple 1D case is obtained by proper combination of eqs.(5) yielding the field equation in the form

$$\frac{d}{dx}\left[\tilde{E}A\left(x\right)\frac{du\left(x\right)}{dx}\right] + A\left(x\right)\int_0^L A\left(\xi\right) g(x,\xi)\left[u(\xi) - u(x)\right] d\xi = -f\left(x\right) A\left(x\right) \tag{6}$$

The kinematic and the static boundary conditions associated to the proposed model of non-local elastic mechanics have been reported in the form [6]

$$u(0) = u_0; \qquad u(L) = u_L \tag{7}$$

$$\sigma^{(l)}(0)A(0) = -F_0; \qquad \sigma^{(l)}(L)A(L) = F_L \tag{8}$$

where F_0 and F_L are the external loads at $x = 0$ and $x = L$ respectively. Eq. (8) may be explained in two different ways: *i)* Close inspection of eq. (3) reveals that for $x = 0$ and $x = L$ the integral in the constitutive equation disappear then only the local stress gives contribution at the the ends of the bar. *ii)* in the proposed model the non-local actions are volume body forces and then at the two elementary volumes located at $x = 0$ and $x = L$ they are infinitesimal of third order and then only the contact forces equilibrate the external loads.

3 FINITE ELEMENTS IN PRESENCE OF LONG-RANGE INTERACTIONS

The elastic equilibrium problem reported in the previous section may be efficiently formulated in terms of the total elastic potential energy. This is a convenient formulation to provide the element equations of the FEM algorithm. Similar considerations may be provided resorting to a weak form of the governing integro-differential eq. (6). The total potential energy $\Pi\left(u,\varepsilon,\eta\right)$ of

the elastic equilibrium problem is defined as $\Pi(u,\varepsilon,\eta) = \Phi(\varepsilon,\eta) + P(u)$ where we defined $P(u) = P^{(r)}(u) + P^{(s)}(u)$ the potential function of the external load being $\bar{b}(x) = -\nabla\left[P^{(r)}(u)\right]$, $p_n(x) = -\nabla\left[P^{(s)}(u)\right]$ where ∇ is the gradeient operator that in the 1D case is reverted to the total derivative $\nabla[\cdot] = {}^{\partial[\cdot]}/_{\partial u}$ and $\Phi(\varepsilon,\eta)$ the elastic potential energy of the solid that is provided in the form

$$\Phi(\varepsilon,\eta) = \frac{1}{2}\tilde{E}A\int_0^L \varepsilon^2(x)\,dx + \frac{1}{4}A^2\int_0^L\int_0^L g(x,\xi)\,\eta^2(x,\xi)\,d\xi dx \tag{9}$$

so that the total potential energy of the elastic problem fully equivalent restoring, at the stationarity, the governing equations of sec. (2) reads

$$\begin{aligned}\Pi(u,\varepsilon,\eta) =& \frac{1}{2}\tilde{E}A\int_0^L \varepsilon^2(x)\,dx + \\ &+\frac{1}{4}A^2\int_0^L\int_0^L g(x,\xi)\,\eta^2(x,\xi)\,d\xi dx - A\int_0^L P^{(r)}(x)\,dx - \left(P^{(s)}(L) + P^{(s)}(0)\right)\end{aligned} \tag{10}$$

It has been prooved in previous papers that the solution of the elastic problem corrisponds to a minimum of the eq.(10) and all the basic theorems of the linear elasticity are fullfilled. In this context approximate solutions obtained using the same projection base and obtained from the first variation of eq.(10) or via weak formulation of the governing eq. (6) yields the same solution [6]. Such problems are frequently encountered also in the use of continuum field theories and finite elements solutions are an efficient tool to provide an approximate numerical solution of the elastic problem. This is the crucial point that is considered in the paper that aims to introduce the finite element method for the solution of non-local elastic problems with long-range central interactions. Generally speaking the finite element method aims to introduce proper discrete mesh subdividing the volume V of the solid into disjoint volume elements $V_j (j = 1, 2, \ldots, n)$ with common vertices shared by adjacent volumes defined as mesh nodes. The discrete mesh is such that $V = V_1 \cup V_2 \cup \ldots \cup V_n$. So that the following realtion holds true

$$\Pi(u,\varepsilon,\eta) = \sum_{i=1}^{n}\Pi_i(u,\varepsilon,\eta) = \sum_{i=1}^{n}\left(\Phi_i(\varepsilon,\eta) + {P_i}^{(r)}(u) + {P_i}^{(s)}(u)\right) \tag{11}$$

The approximation involved in the finite element scheme, beside the spatial discretization approximation, is related to the interpolation of the displacement field in the volume element $V_i(x)$, namely $u_i(x)$ between the nodal displacement in the form

$$u_i(x) = \boldsymbol{N}_i(x)\cdot\boldsymbol{a}_i \qquad i = 1, 2, \ldots, n \tag{12}$$

where $[\cdot]\cdot[\cdot]$ means the tensor contraction operator and $\boldsymbol{N}_i(x)$ is a vector collecting the interpolation functions and $\boldsymbol{a}_i$ is the vector of the unknown nodal displacement. The element equations of the proposed model of non-local elasticity may be obtained introducing eq.(12) into the generic contribution in eq.(11), yielding

$$\Pi_i(u_i,\varepsilon_i,\eta) \cong \Phi_i(\varepsilon_i,\eta) + {P_i}^{(r)}(u_i) + {P_i}^{(s)}(u_i) \tag{13}$$

where the elastic potential energy of the i^{th} volume element is expressed in the form

$$\begin{aligned}\Phi_i\left(\varepsilon_i,\eta\right) =& \frac{1}{2}\tilde{E}A\,\boldsymbol{a}_i^T\left\{\int_{x_i}^{x_{i+1}}\boldsymbol{B}_i^T\left(x\right)\boldsymbol{B}_i\left(x\right)dx\right\}\boldsymbol{a}_i+\\ &+\frac{1}{4}A^2\sum_{l=1}^{n}\int_{x_i}^{x_{i+1}}\int_{x_l}^{x_{l+1}}g\left(x,\xi\right)\eta^2\left(x,\xi\right)d\xi dx\end{aligned} \tag{14}$$

where

$$\boldsymbol{B}_i\left(x\right) = \frac{\partial\left[\boldsymbol{N}_i\left(x\right)\right]}{\partial x} \tag{15a}$$

$$\boldsymbol{\eta}\left(x,\xi\right) = u\left(\xi\right) - u\left(x\right) = \boldsymbol{N}_l\left(\xi\right)\boldsymbol{a}_l - \boldsymbol{N}_i\left(x\right)\boldsymbol{a}_i \tag{15b}$$

The main consideration that may be withdrawn from the observation of eq.(14) is related to the sum of n double integral at the right-hand side that is extended to each couple of dV_l and $dV_i \quad l = 1,2,\ldots,n$ in the solid body domain. The presence of this additional term is due to the long-range interactions as it represents the amount of elastic energy stored in the long-range bonds. Moreover it may be observed that the elastic potential of element V_i do involve the displacement field in the whole body due to the sum in eq.(14). To this aim let us define the block vector $\boldsymbol{a}^T = \left[\begin{array}{cccc}\boldsymbol{a}_1^T & \boldsymbol{a}_2^T & \cdots & \boldsymbol{a}_n^T\end{array}\right]$ gathering the nodal displacements of all the nodes of the solid body elements so that nodal displacement $\boldsymbol{a}_i$ may be obtained by means of the boolean connectivity matrix $\boldsymbol{C}_i$ of the i^{th} element as

$$\boldsymbol{a}_i = \boldsymbol{C}_i\cdot\boldsymbol{a} \qquad i = 1,2,\ldots,n \tag{16}$$

Substitution of eqs.(15b,16) into eq.(14) yields the compact expression

$$\Phi_i\left(\varepsilon_i,\eta\right) = \frac{1}{2}\left(\boldsymbol{a}^T\left(\boldsymbol{K}_i^{(l)} + \boldsymbol{K}_i^{(nl)}\right)\boldsymbol{a}\right) \tag{17}$$

where we denoted $\boldsymbol{K}_i^{(l)}$ the local and non-local stifness matrices of the i^{th} volume element expressed in the form

$$\boldsymbol{K}_i^{(l)} = \tilde{E}A\int_{x_i}^{x_{i+1}}\boldsymbol{B}_i^T\left(x\right)\boldsymbol{B}_i\left(x\right)dx \tag{18a}$$

$$\begin{aligned}\boldsymbol{K}_i^{(nl)} =& \sum_{l=1}^{n}\frac{1}{2}A^2\int_{x_i}^{x_{i+1}}\int_{x_l}^{x_{l+1}}g\left(x,\xi\right)\left\{\boldsymbol{C}_l^T\boldsymbol{N}_l^T\left(\xi\right) - \boldsymbol{C}_i^T\boldsymbol{N}_i^T\left(x\right)\right\}\\ &\left\{\boldsymbol{N}_l\left(\xi\right)\boldsymbol{C}_l - \boldsymbol{N}_i\left(x\right)\boldsymbol{C}_i\right\}d\xi dx\end{aligned} \tag{18b}$$

So that the total potential energy associated to the volume element V_i reads

$$\Pi_i\left(\boldsymbol{a}\right) = \frac{1}{2}\left(\boldsymbol{a}^T\left(\boldsymbol{K}_i^{(l)} + \boldsymbol{K}_i^{(nl)}\right)\boldsymbol{a}\right) - \boldsymbol{a}^T\boldsymbol{f}_i \tag{19}$$

with the load vector expressed as

$$\boldsymbol{f}_i = A \int_{x_i}^{x_{i+1}} \boldsymbol{C}_i^T \boldsymbol{N}_i^T (x) \, \overline{b}(x) \, dx + A \left(\boldsymbol{C}_i^T \boldsymbol{N}_i^T (x_i) \, p_n (x_i) + \boldsymbol{C}_i^T \boldsymbol{N}_i^T (x_{i+1}) \, p_n (x_{i+1}) \right) \quad (20)$$

and accounting for $\boldsymbol{N}_i^T (x_i) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\boldsymbol{N}_i^T (x_{i+1}) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $p_n (x_i) = -\sigma^{(l)} (x_i)$ and $p_n (x_{i+1}) = \sigma^{(l)} (x_{i+1})$ eq.(20) then becomes

$$\boldsymbol{f}_i = A \int_{x_i}^{x_{i+1}} \boldsymbol{C}_i^T \boldsymbol{N}_i^T (x) \, \overline{b}(x) \, dx + A \boldsymbol{C}_i^T \boldsymbol{\sigma}_i^{(l)T} \quad (21)$$

where $\boldsymbol{\sigma}_i^{(l)} = \begin{bmatrix} -\sigma^{(l)} (x_i) & \sigma^{(l)} (x_{i+1}) \end{bmatrix}$. The total potential energy of the whole body, introducing the finite element discretization, is obtained introducing eq.(19) into eq.(11) to yield

$$\Pi = \Pi (\boldsymbol{a}) = \frac{1}{2} \left(\boldsymbol{a}^T \left(\boldsymbol{K}^{(l)} + \boldsymbol{K}^{(nl)} \right) \boldsymbol{a} \right) - \boldsymbol{a}^T \boldsymbol{f} \quad (22)$$

and

$$\boldsymbol{K}^{(l)} = \sum_{i=1}^{n} \boldsymbol{K}_i^{(l)} \quad (23a)$$

$$\boldsymbol{K}^{(nl)} = \sum_{i=1}^{n} \sum_{l=1}^{n} \boldsymbol{K}_{il}^{(nl)} \quad (23b)$$

where

$$\begin{aligned} \boldsymbol{K}_{il}^{(nl)} = & \frac{1}{2} A^2 \int_{x_i}^{x_{i+1}} \int_{x_l}^{x_{l+1}} g(x, \xi) \left\{ \boldsymbol{C}_l^T \boldsymbol{N}_l^T (\xi) - \boldsymbol{C}_i^T \boldsymbol{N}_i^T (x) \right\} \\ & \{ \boldsymbol{N}_l (\xi) \, \boldsymbol{C}_l - \boldsymbol{N}_i (x) \, \boldsymbol{C}_i \} \, d\xi dx \end{aligned} \quad (24)$$

that is the non-local stiffness matrix $\boldsymbol{K}^{(nl)}$ is a real, fully populated, symmetric and positive definite matrix accounting for the long-range interactions in the whole solid domain. The character of positive definition of the stifness matrix $\boldsymbol{K}^{(nl)}$ is fully accomplished as long as the strong formulation of the Drucker stability principle is fulfilled in the whole domain [6]. The elastic equilibrium problem involving the solution of the nodal displacement vector $\boldsymbol{a}$ of the mesh grid is obtained at stationarity of the total potential energy in eq.(22) as

$$\delta \Pi = \frac{\partial \Pi}{\partial \boldsymbol{a}^T} \delta \boldsymbol{a}^T = \boldsymbol{K} \boldsymbol{a} - \boldsymbol{F} = 0 \quad (25)$$

where we introduced the total stiffness matrix of the considered elastic solid in the form

$$\boldsymbol{K} = \boldsymbol{K}^{(l)} + \boldsymbol{K}^{(nl)} \quad (26)$$

At this stage some comments involving the non-local stiffness matrix represented in eqs.(23) may be reported in the perspective of the elastic continuum with long-range central interactions. As in fact the double sum in eq.(23b) involves either terms $K_{ii}^{(nl)}$ $i = 1, 2, \ldots, n$ representing the self-contributions of the long-range forces due to the elementary $dV_l\left(\boldsymbol{\xi}\right)$ volume elements involved in the i^{th} finite element as well as cross-terms $K_{il}^{(nl)} = K_{li}^{(nl)}$. These latter terms involve the contribution to the equilibrium of the i^{th} (or l^{th}) finite element due to the long-range forces exerthed by the l^{th} (or i^{th}) finite elements. Cross-contributions $K_{il}^{(nl)}$ are nearly vanishing as finite elements V_i and V_l are well-separated so that the matrix $K^{(nl)}$ is a block matrix that is banded with a band width larger than in standard FEM (with only local-type stiffness matrices). In this context a FE library may be built accounting only for self-stiffness contributions $K_{ii}^{(nl)}$ as the cross-terms $K_{il}^{(nl)}$ are related to the mesh geometry and they can be neglected in many practical cases.
A different approach to the solution of non-local elastic problems consists in setting up an iterative algorithm which, at the i^{th} step, evalutes the long-range contributions by means of the displacements calculated at step $i-1$. Considering in fact eq.(25), the following equation may be written

$$\boldsymbol{K}^{(l)}\boldsymbol{a}_i \cong \boldsymbol{f}^{(l)} + \boldsymbol{f}_{i-1}^{(nl)} \Longrightarrow \boldsymbol{a}_i \cong \left[\boldsymbol{K}^{(l)}\right]^{-1}\left(\boldsymbol{f}^{(l)} + \boldsymbol{f}_{i-1}^{(nl)}\right) \tag{27}$$

where $\boldsymbol{f}^{(l)}$ is the load vector defined in eq.(21) and $\boldsymbol{f}_{i-1}^{(nl)} = -\boldsymbol{K}^{(nl)}\boldsymbol{a}_{i-1}$. In this way, each solution step will require the inversion of a sparse stiffness matrix $\boldsymbol{K}^{l}$ instead of a full one becouse the non-local effect will be considered as a known load vector in eq.(27), reducing therefore the solution time as the number of elements rises up. Moreover it is fair to assume that the number of iterations needed to obtain an acceptable solution will be as lower as the effect of long-range interactions is small as in many engineering applications.

4 NUMERICAL APPLICATION

A Matlab code has been developed to solve the problem faced in the previous sections, and its results have been compared to those obtained by a FDM code devoloped previously [6]. A bar of length $L = 20\,cm$, Young modulus $E = 2.1\,10^6\,daN{\cdot}cm^{-2}$,constant cross-section area $A = 1\,cm^2$, loaded with a self-equilibrated couple of normal forces $F = 100daN$ is subdivided along direction x in n elements of constant length as depicted in fig.(1a); nodes of the i^{th} element have coordinate x_i and x_{i+1}. The shape function used are linear so that

$$\boldsymbol{N}_i = \begin{pmatrix} \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \dfrac{x - x_i}{x_{i+1} - x_i} \end{pmatrix} \tag{28}$$

and, considering eq.(15a)

$$\boldsymbol{B}_i = \begin{pmatrix} -\dfrac{1}{x_{i+1} - x_i} & \dfrac{1}{x_{i+1} - x_i} \end{pmatrix} \tag{29}$$

The attenuation function, accordingly to [6], has been choosen as follows

$$g\left(x, \xi\right) = \frac{\beta E}{2\lambda A^2} e^{-\frac{1}{\lambda}|x-\xi|} \tag{30}$$

In figs.(2 - 4) some results in term of axial displacement, axial strain,local axial stress and non-local axial stress have been reported. In fig.(5) a comparison between the dimensionless total elastic energy for different values of β and λ is reported; Φ_∞ has been defined as the total elastic energy

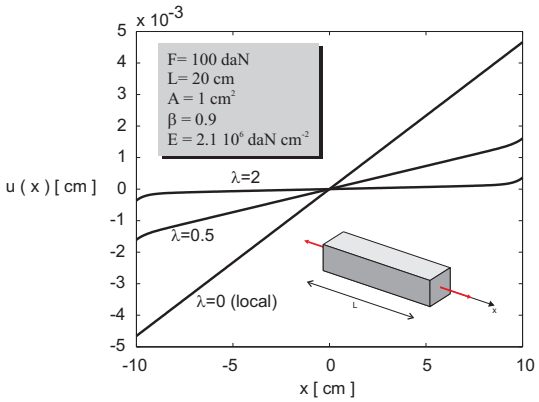


Figure 2: Axial displacement for different λ ($\beta = 0.9$)

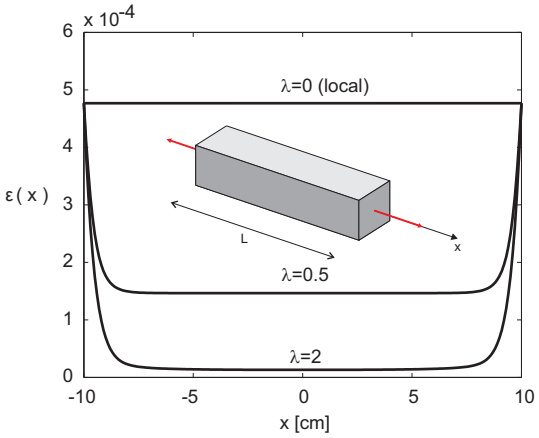
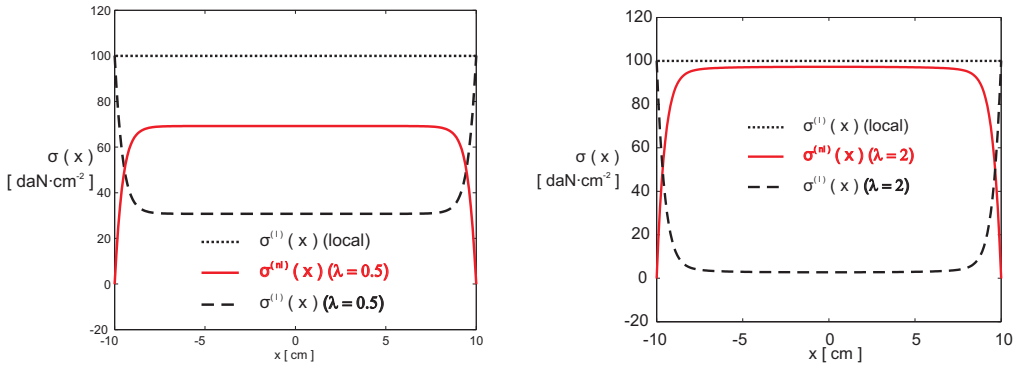


Figure 3: Axial strain for different λ ($\beta = 0.9$)

calculated with a FEM solution of 2000 elements. Figure (6a) shows that total CPU time behaves differently for FDM and FEM; for higher number of elements FEM becomes faster and, together with this advantage, it has to be accounted also the faster convergence of it respect to finite difference solution as showed in fig.(5). Looking at fig.(6b) it is clear that for a non-local FEM the assembling time is larger respect to the solution time; in the non-local FEM for each element there are n non-local element matrix $\boldsymbol{K}_{il}^{(nl)}$ and a local one such that the number of the significant element matrices is $(n+1)n$ while in the classic local FEM they are just n.

5 CONCLUSIONS

In this paper a first attempt to the formulation of a consistent finite element method for the physically-based approach to non-local elasticity has been investigated. The need for such a methods, based upon discretization of the original integro-differential problem is even more pregnant in the solution of two or three dimensional problems with complex boundaries. The results shown in the paper yields to conclude that FEM solution reaches convergence with fewer elements compared



(a) (b)

Figure 4: Axial stress for different λ ($\beta = 0.9$)

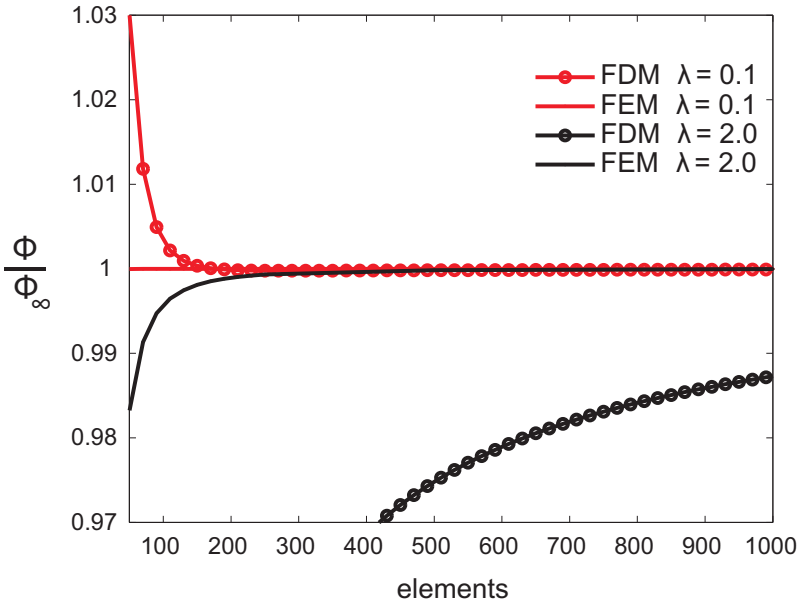
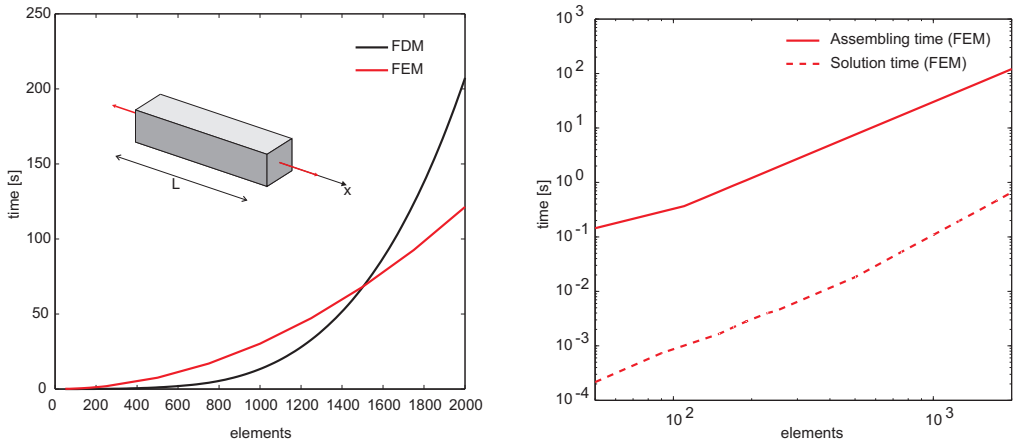


Figure 5: Elastic potential energy convergence for different λ ($\beta = 0.9$)



(a) (b)

Figure 6: *(a)* Total CPU time versus number of elements; *(b)* assmbling time and solution time of FEM . ($\beta = 0.9$, $\lambda = 0.1$)

to finite difference one, when the number of elements grows, FEM speeds up. As infact the FE non-local matrices $\boldsymbol{K}_{il}^{(nl)}$ can be calculated solving the integral in eq.(24) in an exact form, while FDM approximates that integral [6]. Summing up FEM applied to non-local mechanic is the smartest choice twice; firstly becouse the native FEM capability to manage complex boundaries respect to other methods and secondly becouse it supplies more accurate solutions on equal time required.

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