Probabilistic Buckling Analysis of Frame Structures with uncertain parameters

Umberto Alibrandi[1], Nicola Impollonia[2], Giuseppe Ricciardi[1]

[1]*Department of Civil Engineering, University of Messina, Italy*
E-mail: *umbertoalibrandi@ingegneria.unime.it, riccioi@ingegneria.unime.it*

[2]*Department ASTRA, University of Catania, Siracusa, Italy*
E-mail: *nimpo@unict.it*

Keywords: reliability analysis, buckling analysis, response surface method.

SUMMARY. The paper describes a procedure for the reliability analysis of frame structures with respect to the buckling limit state under the assumption that the structural parameters and imperfections are uncertain and modeled as random variables. The procedure relies on a Response Surface Method adopting simple ratio of polynomials without cross-terms as performance function. Such a relationship approximates analytically the dependence between the buckling load and the basic variables furnishing a limit state function which is very close to the exact one when a proper experimental design is adopted. In this way a Monte Carlo Simulation applied to the response surface leads to a good approximation with low computational effort. Some numerical examples show the accuracy and effectiveness of the method, for different values of probability failure.

1 INTRODUCTION

The sensitivity of buckling load to structural parameters and imperfections has long been investigated in structural design, assessing that probabilistic considerations are unavoidable when stability problems are of concern. In this context, the reliability theory [1] is a powerful tool for the rational treatment of uncertainties and for the evaluation of structural safety with respect to the buckling limit state. Many investigations on the role of uncertainties on structural stability have addressed the effects of random initial geometric imperfections [2-4], whereas only a few studies considered the effects of uncertain material properties on the buckling load [5-7]. Perturbation methods or crude Monte Carlo simulations are usually employed to estimate probabilistic information on the uncertain buckling load. However, perturbation-based methods furnish unsatisfactory approximation of the Probability Density Function (PDF) of the buckling load, in particular a lack of accuracy is encountered at the tails of the distribution, whereas crude Monte Carlo simulations are computationally expensive for evaluation of small probabilities of failure. Consequently, both methods give good approximations for the mean value and the coefficient of variation of the random buckling load, but they are not very effective for a reliability analysis.

Aim of the Probabilistic Buckling Analysis (PBA) is the evaluation of the Conditional Probability of Buckling (CPB), that is the probability of buckling of the structural system for assigned value of the acting loads, and it is equivalent to the Cumulative Distribution Function of the random buckling load [8]. The focus of the PBA is mainly to very small failure probabilities, which are the most crucial for this kind of analysis, in view of real world engineering applications.

In this paper the PBA is solved within the classical framework of the structural reliability analysis. The evaluation of the failure probability with respect to the buckling ultimate limit state of frame structures is of concern. The reliability analysis has been performed by a Response Surface Method (RSM), which adopts as a performance function a Ratio of Polynomials Surface (RPS), previously proposed for linear and non-linear [9, 10] static analysis of structures with uncertain parameters. The RPS has been successfully applied also in the analysis of dynamical systems [11] and for the evaluation of the explicit inverse of stiffness matrices [12].

Recently it has been shown that the RPS is effective also in the context of the PBA of perfect structures [13-15]. In fact it furnishes a good fit of the actual implicit limit state function by performing a saturated design (where the number of experimental designs coincides with the number of unknown coefficients). The proposed RPS with $2n+1$ unknown coefficients, n being the number of basic variables, performs better than a classical quadratic polynomial without cross-terms, which requires the same number of experiments. The favorable property may be explained by the capability of the proposed ratio of polynomials to fit the analytical structure of the actual implicit buckling limit state, as in detail described in [15].

However, practical engineering structures have inevitable small structural imperfections which are inherent in their manufacturing. Generally these imperfections have only a mild effect on the structural response, however there are certain cases in which the structural response is strongly modified. This is the case of the so called imperfection-sensitive structures, namely thin shells, but also some types of rings, frames and arches. In these kind of structures, the randomness of the elastic parameters and of the imperfections may bear different behaviors of the structural systems, and the distinction between symmetric stable, symmetric unstable and asymmetric post-buckling response is lost. As a consequence, also the buckling load may be strongly affected by the uncertainties of the structural parameters and of the imperfections. It is known that in this kind of structural systems the buckling load, for assigned realization of the basic random variables, cannot be evaluated as an eigenvalue problem, but a full nonlinear analysis must be performed. However, it is possible to see that the RPS and the proposed experimental plan described in [15] for perfect structures fits well also the limit state equation relative to imperfection-sensitive structures, as it will be described in the following in this paper and validated by the presented numerical experimentations.

The proposed Limit State Function (LSF) can be employed for PBA in conjunction with First and Second Order Reliability Method (FORM and SORM) or Monte Carlo Simulation (MCS) to estimate failure probability. In the first case, the RPS is employed for detecting the design points (FORM) and for evaluating the curvature of the limit state at these points (SORM). If, as usual, one is mainly interested in the evaluation of the very small probabilities, several design points can be detected. Numerical experiments have shown that the estimated design points are quite close to the exact ones. The probability of failure is evaluated by standard FORM/SORM if a single design point is present otherwise it may be necessary to apply a Multi-Point FORM/SORM [16]. The union of the events, each related to a design point, generally allows to determine good approximations of the failure probability also in the critical range of very small probabilities.

When MCS is applied, one exploits the approximate performance function to get experiments with extremely low computational cost. The accuracy of the results depends on how well the approximate response surface fits the exact implicit one on the neighborhood of the design points. Since the analytical structure of the limit state surface of the RPS is close to the target one, if used with the MCS it gives quite accurate results also if the principal point of the RPS does not coincide with the global design point. Consequently, it does not require the coordinate transformation toward the standard normal space and experiments can be conducted in the original space. So operating, correlation distortions related to probabilistic transformations are not introduced.

The paper is organized as follows. At first, the PBA is presented in the framework of structural reliability analysis, then the RPS together with the proposed experimental plan is described. Finally, some simple numerical examples show the accuracy and the effectiveness of the proposed procedure.

2 PROBABILISTIC BUCKLING ANALYSIS

The problem of stability of a discrete or discretized elastic structural system is of concern. The potential energy is defined as [17] $\Pi = U - W$, where U is the elastic strain energy and W is the work of loads. The loading may in general be considered to change as a function of a control parameter λ . Let $\delta q_1, \delta q_2, \ldots, \delta q_n$ be small variations of the n generalized displacements from the equilibrium state assumed to occur at constant λ . Assume that Π is a smooth function, developing a Taylor series expansion around the equilibrium state $\Delta\Pi = \delta\Pi + \delta^2\Pi + \ldots$, where $\delta\Pi$, $\delta^2\Pi$ are the first and the second variation of the potential energy, that is $\delta\Pi = \sum_{i=1}^{n} \left(\partial\Pi/\partial q_i\right)\delta q_i$, $\delta^2\Pi = \frac{1}{2}\sum_{i=1}^{n}\left(\partial^2\Pi/\partial q_i \partial q_j\right)\delta q_i \delta q_j$.

The conditions of equilibrium are $\delta\Pi = 0$ for any δq_i , or $\partial\Pi/\partial q_i = 0$ for each *i*. According to the Lagrange-Dirichel theorem, the equilibrium state is stable for those values of λ for which $\delta^2\Pi > 0$, for any $\delta q_i , \delta q_j$. In compact matrix form it can be written as $\frac{1}{2}\boldsymbol{\delta q}^T \boldsymbol{K} \boldsymbol{\delta q} > 0$ for any $\boldsymbol{\delta q} \neq \boldsymbol{0}$, where $\boldsymbol{K}$ is the tangent stiffness matrix, defined as $K_{ij} = \partial^2\Pi/\partial q_i\, \partial q_j$,and that must be positive definite. At the limit of stability, the second variation ceases to be positive definite, which implies $Det\left(\boldsymbol{K}\right) = 0$.

Each element K_{ij} of the stiffness matrix, without loss of generality, can be defined as a sum of individual stiffnesses k_i, $i = 1, 2, \ldots, N$, that is $K_{ij} = \sum \left(-1\right)^{r_q} k_q$, where r_q is an exponent related to the stiffness k_q , giving a positive or negative contribute to the stiffness K_{ij} . Let us keep the fundamental hypotheses of the classical buckling analysis, but assume that the N stiffnesses k_i, are uncertain. Let us consider now the N fluctuations $\alpha_1, \alpha_2, \ldots, \alpha_N$ (collected in an $N-$vector $\boldsymbol{\alpha}$) of the uncertain stiffnesses with respect to their mean value $\bar{k}_i$, that is $k_i = \bar{k}_i \left(1 + \alpha_i\right)$; α_i are random variables with zero mean, while their standard deviation give the coefficient of variation of the random parameters k_i . Let us assume that M random imperfection parameters $\zeta_1, \zeta_2, \ldots, \zeta_M$ are present and collected in an $M-$vector $\boldsymbol{\zeta}$. The set of the $N + M$ basic variables is then given as $\left\{\boldsymbol{\alpha}, \boldsymbol{\zeta}\right\}$. The critical load $\Lambda = \Lambda\left(\boldsymbol{\alpha}, \boldsymbol{\zeta}\right)$ is the derived random variable, λ is the load parameter. The aim of the PBA is the evaluation of the Conditional Probability of Buckling (CPB), $P_f\left(\lambda\right)$, that is the probability that buckling occurs for assigned value λ of the acting loads

$$P_f\left(\lambda\right) = Prob\left[\Lambda\left(\boldsymbol{\alpha}, \boldsymbol{\zeta}\right) \le \lambda\right] \tag{1}$$

In the structural reliability theory, the probability of failure with respect to a given limit state $G\left(\boldsymbol{\alpha}, \boldsymbol{\zeta}; \lambda\right) = 0$ is given as

$$P_f(\lambda) = Prob\left[G(\boldsymbol{\alpha},\boldsymbol{\zeta};\lambda) \le 0\right] = \int_{G\le 0} f_{\alpha\zeta}(\boldsymbol{\alpha},\boldsymbol{\zeta})\, d\boldsymbol{\alpha} d\boldsymbol{\zeta} = \int Prob\left[G(\boldsymbol{\alpha},\boldsymbol{\zeta};\lambda) \le 0 \middle| \boldsymbol{\zeta}\right] f_{\zeta}(\boldsymbol{\zeta}) d\boldsymbol{\zeta} \tag{2}$$

where $f_{\alpha\zeta}(\boldsymbol{\alpha},\boldsymbol{\zeta})$ is the joint pdf of the basic variables $\boldsymbol{\alpha},\boldsymbol{\zeta}$ and the last equality has been obtained using the definition of conditional probability. If we define the random quantity $q{=}Prob\left[G(\boldsymbol{\alpha},\boldsymbol{\zeta};\lambda) \le 0 \middle| \boldsymbol{\zeta}\right]$ then the probability of failure can be estimated as the sample mean of q

$$\hat{P}_f(\lambda) = \frac{1}{N_s}\sum_{k=1}^{N_s} q_k = \frac{1}{N_s}\sum_{k=1}^{N_s} Prob\left[G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta}_k) \le 0\right] \tag{3}$$

where N_s is the number of samples of the imperfection parameters. In the last equality of eq.(3) it is underlined that the $k-$th sample q_k is the failure probability of the structural system when the basic variables are only the random structural parameters, while the imperfections are fixed. For this reason, in the following, it will be analyzed in detail the evaluation of the conditional probability for assigned values of λ and $\boldsymbol{\zeta}$, $P_f(\lambda,\boldsymbol{\zeta}) = Prob\left[\Lambda(\boldsymbol{\alpha};\boldsymbol{\zeta}) \le \lambda\right]$, given as

$$P_f(\lambda,\boldsymbol{\zeta}) = Prob\left[G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta}) \le 0\right] = \int_{G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta})\le 0} f_{\alpha}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \tag{4}$$

where $f_\alpha(\boldsymbol{\alpha})$ denotes the joint probability density function of $\boldsymbol{\alpha}$, and $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta}) = \Lambda(\boldsymbol{\alpha};\boldsymbol{\zeta}) - \lambda$. The evaluation of the multidimensional integral appearing in Eq.(4) over the failure set is a cumbersome task, for this reason several approximation methods have been developed. As described in the introduction, a particular type of Response Surface Method (RSM) has been here adopted, and it will be presented in detail in section 3.

3 RESPONSE SURFACE METHOD BASED ON RATIO OF POLYNOMIALS

The basic idea of the RSM is to replace the original implicit limit state function $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta})$ by a simple and approximate explicit function $\hat{G}(\boldsymbol{\alpha})$, named Response Surface (RS), whose values can be easily computed [18, 19]. Once that the surrogate model is built, it is no longer necessary to run demanding usually non-linear analyses. The failure probability in Eq.(4) is strictly related to the limit state equation $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta}) = 0$ which defines the bounds of the integration. Therefore it is necessary that the RS $\hat{G}(\boldsymbol{\alpha})$ approximates this boundary quite well, in fact a small deviation of $\hat{G}(\boldsymbol{\alpha})$ from the target limit state function $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta})$ in this region can lead to significant estimation errors. To obtain a well approximated response surface it is most important to obtain sampling points very close to or exactly on the limit state surface $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta}) = 0$. Therefore, a really effective response surface should fit as better as possible the limit state surface $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta}) = 0$, and have the same sign of the limit state function $G(\boldsymbol{\alpha};\lambda,\boldsymbol{\zeta})$.

3.1 *The Response Surface based on Ratio of Polynomials*

The proposed Response Surface is based on Ratio of Polynomials (RPS) and it is defined as

$$\hat{G}(\boldsymbol{\alpha}) = a_0 + \sum_{i=1}^{N} \frac{\alpha_i}{a_i + b_i \alpha_i} \tag{5}$$

where $a_0 = \hat{G}(\boldsymbol{0})$, a_i and b_i, $i = 1, 2, \ldots, N$ are the $2N+1$ coefficients of the RPS to be determined. The RPS has many desirable properties, in fact several numerical applications have shown that it is not very sensitive to the choice of the central point, and it guarantees good accuracy and stability. It is worth noting that the coefficients to be determined are functions of the parameters λ and $\boldsymbol{\zeta}$, that is $a_0 = a_0(\lambda, \boldsymbol{\zeta})$, $a_i = a_i(\lambda, \boldsymbol{\zeta})$, $b_i = b_i(\lambda, \boldsymbol{\zeta})$; moreover, geometrical consideration imply $a_i > 0$, $b_i > 0$, $a_i > b_i$, for further details see [15]. Using Eq.(5), it is seen that the RPS gives an N-order polynomial approximation of the limit state equation, that is $\hat{G}(\boldsymbol{\alpha}) = 0$ can be expressed as

$$\hat{G}(\boldsymbol{\alpha}) = \hat{c}_0 + \sum_{s=1}^{N} \sum_{k_1+\ldots+k_N=s} \hat{c}_{k_1 \ldots k_N} \alpha_1^{k_1} \ldots \alpha_N^{k_N} = 0, \; k_i = \{0,1\} \tag{6}$$

The second summation in the above equation is extended to all the combinations such that $k_1 + \ldots + k_N = s$, with s ranging from zero to the degree N of the polynomial and $k_1, k_2, \ldots, k_N$ which can assume only the values "0" and "1", while the coefficients $\hat{c}_0(\lambda, \boldsymbol{\zeta})$ and $\hat{c}_{k_1 \ldots k_N}(\lambda, \boldsymbol{\zeta})$ depend upon the specific buckling problem; the number n_{tot} of the coefficients increases with the number of uncertain parameters N. For clarity's sake, let us consider the case $N = 3$, Eq.(6) gives

$$\hat{G}(\alpha_1, \alpha_2, \alpha_3) = \hat{c}_0 + \hat{c}_{100}\alpha_1 + \hat{c}_{010}\alpha_2 + \hat{c}_{001}\alpha_3 + \hat{c}_{110}\alpha_1\alpha_2 + \hat{c}_{101}\alpha_1\alpha_3 + \hat{c}_{011}\alpha_2\alpha_3 + \hat{c}_{111}\alpha_1\alpha_2\alpha_3 = 0 \tag{7}$$

where the $\hat{n}_{tot}$ approximated coefficients depend upon the $2N+1$ parameters of the RPS, that is $\hat{c}_{k_1 k_2 \ldots k_N} = \hat{c}_{k_1 k_2 \ldots k_N}(a_0, a_i, b_i)$.

3.2 *Choice and determination of the support points of the RPS*

It has been previously underlined that in order the RPS to give an accurate solution of the PBA the following issues should be of concern: *i*) a good fit of the exact limit state function must be assured, as a consequence the $2N+1$ support points of the RPS required for a saturated design have to be located around the limit state; *ii*) the exact and approximate performance functions must have the same sign.

The tasks are accomplished detecting for each uncertain parameter α_i two values $\alpha_i^{(+)}$ and $\alpha_i^{(-)}$ such that the actual performance function $G(\boldsymbol{\alpha}; \lambda, \boldsymbol{\zeta})$ at the points $\boldsymbol{\alpha}_i^{(+)} = \{0 \; \ldots \; \alpha_i^{(+)} \; \ldots \; 0\}^T$, $\boldsymbol{\alpha}_i^{(-)} = \{0 \; \ldots \; \alpha_i^{(-)} \; \ldots \; 0\}^T$ will have the signs $G(\boldsymbol{\alpha}_i^{(+)}; \lambda, \boldsymbol{\zeta}) \geq 0$, $G(\boldsymbol{\alpha}_i^{(-)}; \lambda, \boldsymbol{\zeta}) \leq 0$ and the distance $\Delta\alpha_i = \alpha_i^{(+)} - \alpha_i^{(-)}$ should be as small as possible. The points $\boldsymbol{\alpha}_i^{(+)}$ and $\boldsymbol{\alpha}_i^{(-)}$ play the role of the first $2N$ support points and can be located by numerical techniques devoted to find the zero of a function of a single variable $g(\alpha_i) = G(0, 0, \ldots, \alpha_i, \ldots, 0; \lambda, \boldsymbol{\zeta})$. Let say that the function has to be evaluated at N_i points in order to detect the zero and locate points $\boldsymbol{\alpha}_i^{(+)}$ and $\boldsymbol{\alpha}_i^{(-)}$. A crucial point for obtaining a good approximation of the CPB is given from the choice of the remaining support point, here called "principal point" of the experimental plan. The best choice is obviously represented by the design point.

The RPS allows an efficient and quick evaluation of the design point. In fact, as a first step a tentative principal point $\boldsymbol{\alpha}^{(0)}$ is chosen, and a first RPS is built. The last support point $\boldsymbol{\alpha}^{(0)}$ has to be at the limit state so that $G\left(\boldsymbol{\alpha}^{(0)}; \lambda, \boldsymbol{\zeta}\right) = 0$, and for example it can been chosen equidistant from all the axes, namely $\boldsymbol{\alpha}^{(0)} = \alpha^{(0)}\boldsymbol{\chi}^{(0)}$, where $\boldsymbol{\chi}^{(0)} = \left(1/\sqrt{N}\right)\{1 \quad 1 \quad \ldots \quad 1\}^T$. The last support point is obtained as the zero of $G\left(\alpha^{(0)}\boldsymbol{\chi}^{(0)}\right)$, function of the variable $\alpha^{(0)}$.

Once that the $2N+1$ support points $\boldsymbol{\alpha}^{(0)}$, $\boldsymbol{\alpha}_i^{(+)}$ and $\boldsymbol{\alpha}_i^{(-)}$ have been determined the $2N+1$ coefficients a_0, a_i, b_i of the RPS at the first iteration are obtained as the only solution of the following non-linear system

$$\hat{G}\left(\boldsymbol{\alpha}^{(0)}\right) = 0, \quad \hat{G}\left(\boldsymbol{\alpha}_i^{(+)}\right) = G\left(\boldsymbol{\alpha}_i^{(+)}; \lambda, \boldsymbol{\zeta}\right), \quad \hat{G}\left(\boldsymbol{\alpha}_i^{(-)}\right) = G\left(\boldsymbol{\alpha}_i^{(-)}; \lambda, \boldsymbol{\zeta}\right) \tag{8}$$

which satisfies the constraint $a_i > 0, \quad b_i > 0$. The RPS at the first iteration gives already a good approximation of the actual performance function so that first iteration design point $\boldsymbol{\alpha}^{*(0)}$ determined on the RPS will be very close to the target one $\boldsymbol{\alpha}^*$ and very close to the exact limit state. The first iteration design $\boldsymbol{\alpha}^{*(0)}$ is obtained by a classical FORM method applied to the RPS, which does not require the numerical gradient evaluation, because the first derivatives of the RPS are known in closed form. The RPS may be further improved by a second iteration where the principal point is set along the radial direction pointing to $\boldsymbol{\alpha}^{*(0)}$, obtaining so $\boldsymbol{\alpha}^{(1)}$. The new coefficients a_0, a_i, b_i are given by the feasible solution of the nonlinear system in Eq.(8) with the first equation substituted by $\hat{G}\left(\boldsymbol{\alpha}^{(1)}\right) = 0$. Numerical investigations have shown that the second iteration design $\boldsymbol{\alpha}^{*(1)}$ converges almost exactly toward the exact MPP $\boldsymbol{\alpha}^*$ in the space of the basic variables, guaranteeing in this way a very good approximation of the target CPB. The accuracy of the RPS is not very sensitive to the choice of the principal point (as long as it stays on the limit state). This is why good results are also obtained at the first iteration.

3.3 *Buckling Reliability approaches applied to the RPS*

In this section the classical structural reliability analysis techniques are exploited in conjunction to the proposed RPS. At first the support points $\boldsymbol{\alpha}_i^{(+)}$ and $\boldsymbol{\alpha}_i^{(-)}$ are determined by sampling the true performance function $G(\boldsymbol{\alpha}; \lambda, \boldsymbol{\zeta})$.

Then, a tentative principal point $\boldsymbol{\alpha}^{(0)}$ is assumed and the first iteration RPS $\hat{G}^{\boldsymbol{\alpha}^{(0)}}(\boldsymbol{\alpha})$ is obtained along with its design point $\boldsymbol{\alpha}^{*(0)}$. Eventually a second iteration is performed which furnishes a more accurate RPS $\hat{G}^{\boldsymbol{\alpha}^{(1)}}(\boldsymbol{\alpha})$, around the limit state region, and the design point $\boldsymbol{\alpha}^{*(1)}$.

The FORM approximation for the failure probability is given by $P_{f,FORM} = \Phi\left(-\hat{\beta}\right)$, where $\hat{\beta} = \|\hat{\boldsymbol{y}}\|$ is the reliability index. For improving FORM solution, it is possible to use SORM. Note that also the second-order numerical derivatives are not required, because they are known in closed form. Anyway, the RPS shows its effectiveness especially when applied together with MCS, due to low computational effort related to samples evaluation by its explicit expression.

4 NUMERICAL APPLICATIONS

In this section two numerical applications are presented in order to show the effectiveness of the proposed procedure. In the first one a perfect beam with two random springs is considered, and the buckling bifurcation is analyzed. The second application refers to a discrete Roorda's frame, where asymmetric bifurcation is exhibited.

4.1 *Perfect beam with two random springs*

Let us consider an articulated beam with two random springs subjected to a compressive load λP (Figure 1a). The two springs are modeled by the random variables $k_1 = \bar{k}_1\left(1+\alpha_1\right)$ and $k_2 = \bar{k}_2\left(1+\alpha_2\right)$, where $\bar{k}_1$ and $\bar{k}_2$ represent their mean values, while α_1 and α_2 are the fluctuations, assumed to have a normal distribution with mean value $\mu_{\alpha_1} = \mu_{\alpha_2} = 0$ and standard deviation $\sigma_{\alpha_1} = \sigma_{\alpha_2} = 0.20$. For this kind of problem, the second-order theory can be used giving rise to $\Pi_2 = \frac{1}{2}k_1\vartheta_1^2 + \frac{1}{2}k_2(\vartheta_2 - \vartheta_1)^2 - \lambda\frac{PL}{2}(\vartheta_1^2 + \vartheta_2^2)$. Taking account of $K_{ij} = \partial^2\Pi/\partial q_i\,\partial q_j$, the elements of the tangent stiffness matrix are $K_{11} = k_1 + k_2 - \lambda$, $K_{12} = -k_2$, $K_{22} = k_2 - \lambda$, where it has assumed $P=1$, $L=1$. It is seen that the elements of $\boldsymbol{K}$ linearly depend upon k_1, k_2, λ and then also upon $\alpha_1, \alpha_2, \lambda$. The limit state equation is then $\lambda^2 + \alpha_1\alpha_2 - \alpha_1\lambda - 2\alpha_2\lambda + \alpha_1 + \alpha_2 - 3\lambda + 1 = 0$. In the following analyses the CPB $P_f\left(\lambda\right)$ is evaluated assuming $\bar{k}_1 = \bar{k}_2 = 1$, i.e. with reference to a nominal structure ($\alpha_1 = \alpha_2 = 0$) having a buckling load $\lambda = \lambda_E = (3-\sqrt{5})/2$. In a first reliability analysis it has been chosen the load factor $\lambda = \lambda_1 = 0.5\lambda_E$ and the basic variables are assumed uncorrelated. The "exact" failure probability $P_f = 2.39\times10^{-3}$ is obtained by a crude MCS with 1'000'000 samples. The target reliability index $\beta_1 = 2.8599$, evaluated by the HL-RF algorithm is related to the design point $\boldsymbol{\alpha}_0^* = \{-0.571 \quad -0.034\}^T$. FORM gives $P_f = 2.118\times10^{-3}$ with a relative error of 11.45%. The first iteration RPS is represented in Fig.1b. Note that, although the exact and approximate limit state functions $G\left(\boldsymbol{\alpha};\lambda\right)$ and $\hat{G}\left(\boldsymbol{\alpha}\right) = \hat{G}^{\alpha^{(0)}}\left(\boldsymbol{\alpha}\right)$, respectively, are quite different, the exact and approximate limit state, $G\left(\boldsymbol{\alpha};\lambda\right) = 0$ and $\hat{G}\left(\boldsymbol{\alpha}\right) = 0$, practically coincide, thick lines of Fig.1b. In this simple case with $N=2$, the limit state surface is accurately reproduced by a first iteration RPS so that a second iteration is not needed.

This issue is confirmed by the MCS applied to the first iteration RPS, adopting the same sampling points utilized to evaluate the target failure probability, which gives $P_f = 2.386\times10^{-3}$ with a relative error of 0.29%. In Fig.1b the exact and approximate limit state are plotted in the original space of the basic variables. In the figure the design point and the principal point are evidenced. It is worth underlining that the RPS describes an explicit relationship between the performance function and the basic variables, which does not depend upon degree of correlation of the uncertain parameters, their coefficient of variation, or the underlying probability distribution.

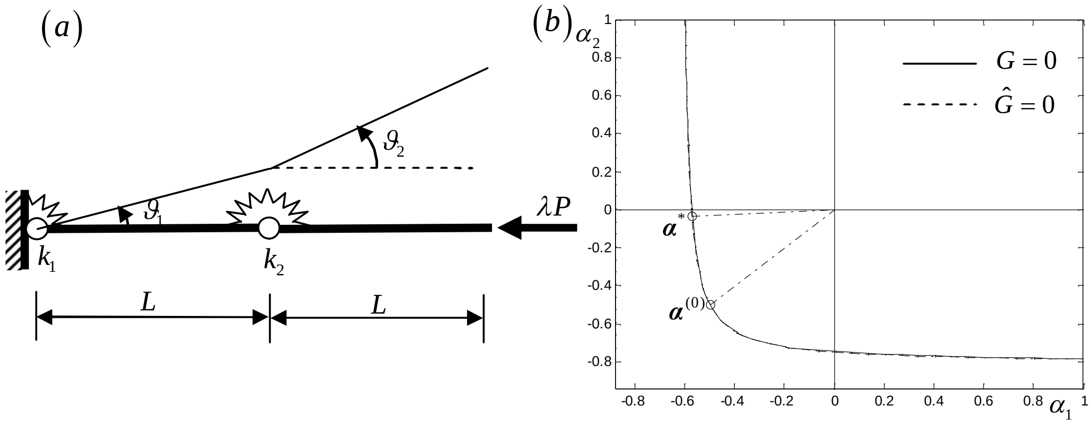

Figure 1: (a) Perfect beam with two random springs, (b) : Exact and approximate limit state function for $\lambda = 0.5\lambda_E$

Let us now assume that α_1 and α_2 are correlated, with correlation coefficient ρ . In the original space, the limit state functions for the two cases (uncorrelated and correlated) will be the same and only the design point will change. On the other hand, in the standard normal space both the limit state and the design point depend upon the correlation coefficient. As already stated, in the range of the very small probabilities ($10^{-4} \div 10^{-5}$) multiple design points may be present which contribute to the overall failure probability. The effectiveness of the RPS is not affected by the value of the failure probability neither by the presence of multiple design points. In fact for, $\lambda = \lambda_2 = 0.30\lambda_E$ and $\rho = 0$, the "exact" failure probability obtained by a crude MCS with 10^7 samples is $P_f = 8.851 \times 10^{-5}$. The target reliability index (with reference to the global design point) is $\beta_1 = 3.7794$, and the design point has coordinates $\boldsymbol{\alpha}^* = \{-0.756 \quad -0.0130\}^T$. The FORM gives $P_f = 7.860 \times 10^{-5}$ with a relative error of 11.20% . The error increases with respect to the case $\lambda = \lambda_1 = 0.5\lambda_E$ because the contribution of the local design point on the failure probability, which is missed by FORM, is not negligible. The MCS applied on the first iteration RPS gives a failure probability of $P_f = 8.94 \times 10^{-5}$ with a relative error of 0.99% .

4.2 *L-shaped Rigid-bar frame (Roorda's frame)*

Let us consider an L-shaped frame modeled with two random springs joining two rigid segments, such that the column is subjected to a compressive load λP (Figure 2a). The two springs are modeled by the random variables $k_1 = \bar{k}_1(1+\alpha_1)$ and $k_2 = \bar{k}_2(1+\alpha_2)$, where $\bar{k}_1$ and $\bar{k}_2$ represent their mean values, while α_1 and α_2 are the fluctuations, assumed to have a normal distribution with mean value $\mu_{\alpha_1} = \mu_{\alpha_2} = 0$ and standard deviation $\sigma_{\alpha_1} = \sigma_{\alpha_2} = 0.20$. It is known that this kind of frame exhibits asymmetric bifurcation, which implies that it may be imperfection-sensitive, and then it is necessary to consider the fourth-order approximation of the potential energy

$\Pi_4 = \frac{1}{2}k_1\vartheta^2 + \frac{1}{2}k_2(2\vartheta + \frac{H}{L}\vartheta^2)^2 - \lambda P(\frac{H}{2}\vartheta^2 + \zeta\vartheta)$. The equilibrium condition reads as $\partial\Pi/\partial q_i = 0$, which implies $\vartheta = \vartheta(k_1, k_2; \lambda, \zeta)$, ϑ being a nonlinear function of k_1, k_2, λ, ζ. The stiffness matrix becomes $K = k_1 + k_2 - \lambda + \left[6k_2\left(2\vartheta + \vartheta^2\right)\right]$ where it has been assumed $P = 1$, $H = 1$, $L = 1$.

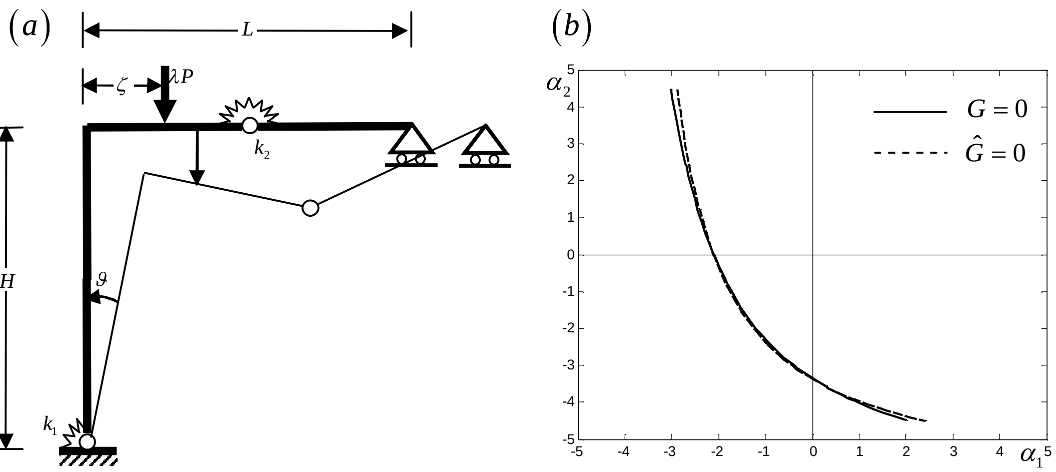

Figure 2: (a) L-shaped frame with two random springs, (b) Exact and approximate limit state for $\lambda = 0.50\lambda_E, \zeta = -0.05$,

In the following analyses the CPB $P_f(\lambda)$ is evaluated assuming $\bar{k}_1 = \bar{k}_2 = 1$, i.e. with reference to a nominal structure ($\alpha_1 = \alpha_2 = 0$) having a buckling load $\lambda = \lambda_E = 5$. In a sample reliability analysis it has been chosen the load factor $\lambda = \lambda_1 = 0.5\lambda_E$, $\zeta = -0.05$ and the basic variables are assumed uncorrelated. The "exact" failure probability $P_f = 2.78\times10^{-2}$ is obtained by a crude MCS with 100'000 samples. The target reliability index $\beta_1 = 1.9877$, is related to the design point $\boldsymbol{\alpha}_0^* = \{-0.372 \quad -0.139\}^T$. FORM gives $P_f = 2.34\times10^{-2}$ with a relative error of 15.91%. In Fig. 2b is shown that the approximated limit state surface fits quite well the target limit state surface. The accuracy of the approximation is also validated by the MCS applied to the RPS, which gives $P_f = 2.72\times10^{-2}$ with a relative error of 2.23%. For design purposes, the proposed method should be applied for different values of the load parameter λ and the imperfection parameter ζ. It is noted that the computational effort is not excessive, since that only a reduced number of full nonlinear analyses are required.

5 CONCLUSIONS

A novel Response Surface method suitable for the evaluation of the Probabilistic Buckling Analysis has been presented. It is used as a reproduction of the target limit state function, indeed the MCS applied to the RS gives very good approximation of the exact failure probability. Further numerical experimentation has to be developed, in order to assess the effectiveness and robustness of the proposed procedure as applied to more complicated structures.

References

[1] Ditlevsen O., Madsen H.O., *Structural Reliability Methods*, Wiley (1999).

[2] Elishakoff I., "Buckling of a stochastically imperfect finite column on a nonlinear elastic foundation", *Journal of Applied Mechanics*, **46**, 411–416 (1979).

[3] Elishakoff, I, Arbocz, J., "Reliability of axially compressed cylindrical shells with random axisymmetric imperfections", *International Journal of Solids and Structures*, **18**, 563–585 (1982).

[4] Schenk, C.A., Schuëller, G.I., "Buckling analysis of cylindrical shells with random geometric imperfections", *International Journal of Non-Linear Mechanics*, **38,** 1119–1132 (2003).

[5] Schafer, B.W., Graham-Brady, L., "Stochastic post-buckling of frames using Koiter's method", *International Journal of Structural Stability and Dynamics*, **6**, 1-26 (2006).

[6] Papadopoulos V., Papadrakakis M., "The effect of material and thickness variability on the buckling load of shells with random initial imperfections", *Computer Methods in Applied Mechanics and Engineering*, **194**, 1405-1426 (2005).

[7] Zhang, J., Ellingwood, B., "Effects of uncertain material properties on structural stability", *Journal of structural engineering*, **121**, 705-716 (1995).

[8] Alibrandi, U., *Computational Methods in Stochastic Mechanics and Structural Reliability*, Ph.D Dissertation, University of Messina –Italy (2006).

[9] Falsone, G., Impollonia, N., "A New Approach for the Stochastic Analysis of Finite Element Modeled Structures with Uncertain Parameters", *Computer Methods in Applied Mechanics and Engineering*, **191**, 5067-5085 (2002). Erratum, **192**, 2187-2188 (2003).

[10] Impollonia, N., Sofi, A. "A Response Surface Approach for the Static Analysis of Stochastic Structures with Geometrical Nonlinearities", *Computer Methods in Applied Mechanics and Engineering*, **192,** 4109-4129 (2003).

[11] Impollonia, N., Ricciardi, G.. "Explicit solutions in the stochastic dynamics of structural systems", *Probabilistic Engineering Mechanics*, **21**, 171-181 (2006)

[12] Impollonia, N., "A method to derive approximate explicit solutions for structural mechanics problems", *International Journal of Solids and Structures*, **43**, 7082-7098 (2006)

[13] Alibrandi, U., Impollonia, N., Ricciardi, G., "A non linear performance function for probabilistic buckling analysis", *Proc. ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering* (2007)

[14] Alibrandi, U, Impollonia, N., Ricciardi, G., "Probabilistic buckling analysis through a novel non-linear response surface", *Proc. 18th AIMETA Congress of Theoretical and Applied Mechanics* (2007).

[15] Alibrandi, U., Impollonia, N., Ricciardi, G., "Probabilistic Eigenvalue Buckling Analysis solved through the Ratio of Polynomials Response Surface", submitted to Computer Methods in Applied Mechanics and Engineering

[16] Der Kiureghian, A., Dakessian, T., "Multiple design points in first and second-order reliability", *Structural Safety*, **20**, 37-49 (1998)

[17] Bazant, Z.P., Cedolin, L., *Stability of structures, Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press (1991)

[18] Faravelli, L., "Response-surface approach for reliability analysis", *Journal of Engineering Mechanics*, **115**, 2763-2781 (1989).

[19] Bucher, C., Burgound, U., "A fast and efficient response surface approach for structural reliability problems", *Structural Safety*, **7**, 57-66 (1990).