A Generalized Method for the Static Analysis of a Monodimensional Prestressed Continuum

Dario Genovese[1]
[1]*Department of Architecture, Constructions and Structures, Università Politecnica delle Marche, Italy*
E-mail: dario79@email.it

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SUMMARY. We present a general method for the static analysis of a monodimensional prestressed continuum under external loads, using the virtual work principle.

In the classic continuum mechanics ([1],[2]) internal virtual work is the product of a stress, which is linear in the displacement, and a virtual strain, which is linear in the virtual displacement. A similar term is added when the continuum is prestressed. This term comes from a second order congruence and it is the product of a constant prestress and a virtual strain which is linear both in the displacement and in the virtual displacement. A typical second order strain is the elongation ϵ caused, in a cable, by transversal dispalcement u ([3]). The relation between stress and strain is $\epsilon = \frac{1}{2}u'^2$.

Boundary forces caused by prestress are defined through integration by parts in the expression of the *principle of virtual work*.

If we consider higher order terms in the expression of $\boldsymbol{\sigma}$, we obtain nonlinear equations between load and displacement. In case of small prestress, some terms in these nonlinear equations can be neglected.

Two examples of this general approach are exposed: the classical compressed Eulero-Bernoulli beam, and a model of cable with transversal, axial and torsional loads and displacements.

1 INTRODUCTION

The static analysis of monodimensional continuums is one of the main branch of mechanics. It provides practical tools for studying the behaviour of tridimensional objects where one dimension is more important than the other two, such as beams or cables.

The approach used in this article is variational. The proposed method is completely generalized, and contemplates these steps:

- Definition of a reference configuration, to which we refer our analysis. This allows us not to confuse the initial configuration with the deformed one, and to adopt only one model of the structure.
- Definition of the kinematic variables displacements and strains, and a second order relation, called congruence, between them. Even if we do not contemplate finite displacements, this second order relation may describe postcritical behaviour of unstable structures.
- Definition of parameters (constitutive law and prestress)
- Application of virtual work principle to obtain addictional relations, such as the equilibrium, between applied forces, displacements, stress and strain.

Even if the structure is stable, non linear analysis can be done to obtain a more precise model. This non linear analysis, under certain conditions, may be simplified by neglecting some terms in the resulting equations.

2 MATHEMATICAL VARIABLES

We define a monodimensional continuum as an interval $[0; L] \subset \mathbb{R}$. A point in this interval is denoted by the variable $s \in [0; L]$.

In this interval we define the following functions:

- The generalized *displacement* vector $\mathbf{d}$
- The generalized applied *force* vector $\mathbf{f}$, of same dimension of $\mathbf{d}$
- The generalized *strain* vector $\boldsymbol{\epsilon}$
- The generalized *stress* vector $\boldsymbol{\sigma}$, of same dimension of $\boldsymbol{\epsilon}$.

We also define the vectors *boundary force* $\boldsymbol{\phi}$ and *boundary displacement* $\boldsymbol{\delta}$, both defined in $\{0; L\}$. Exact definition will be given later.

3 CONGRUENCE

The *congruence* is a relation between displacement and strain. We suppose that the strain is the sum of two terms.

The first term is linear in the displacement $\mathbf{d}$ and its derivates until a finite order, and it's written as

$$\sum_{j,k} \mathscr{C}_{ijk} \; \frac{\partial^{j-1}}{\partial s^{j-1}} \mathbf{d}_k \tag{1}$$

The second term is quadratic:

$$\sum_{j,k,p,q} \frac{1}{2} \mathscr{S}_{ijkpq} \; \left(\frac{\partial^{j-1}}{\partial s^{j-1}} \mathbf{d}_k \right) \left(\frac{\partial^{p-1}}{\partial s^{p-1}} \mathbf{d}_q \right) \tag{2}$$

We can suppose that $\mathscr{S}$ is symmetric in $\{jk\}, \{pq\}$, so that $\mathscr{S}_{ijkpq} = \mathscr{S}_{ipqjk}$. If they are different we can replace them with their arithmetic mean.

Defining the operator

$$\boldsymbol{\partial} = \begin{bmatrix} 1 \\ \partial / \partial s \\ \partial^2 / \partial s^2 \\ \vdots \\ \partial^n / \partial s^n \end{bmatrix} \tag{3}$$

we can write, using Einstein convention,

$$\boldsymbol{\epsilon}_i = \mathscr{C}_{ijk} \; \boldsymbol{\partial}_j \mathbf{d}_k \; + \; \frac{1}{2} \mathscr{S}_{ijkpq} (\boldsymbol{\partial}_j \mathbf{d}_k)(\boldsymbol{\partial}_p \mathbf{d}_q) \tag{4}$$

or, more simply,

$$\boldsymbol{\epsilon} = \mathscr{C}\,\boldsymbol{\partial}\mathbf{d} \;+\; \frac{1}{2}\mathscr{S}\boldsymbol{\partial}\mathbf{d}\,\boldsymbol{\partial}\mathbf{d} \tag{5}$$

Indicating with an underscore a virtual quantity, we write the expression of the virtual strain

$$\underline{\boldsymbol{\epsilon}} = \mathscr{C}\,\boldsymbol{\partial}\underline{\mathbf{d}} \;+\; \frac{1}{2}\mathscr{S}\,\boldsymbol{\partial}\mathbf{d}\,\boldsymbol{\partial}\underline{\mathbf{d}} + \frac{1}{2}\mathscr{S}\,\boldsymbol{\partial}\underline{\mathbf{d}}\,\boldsymbol{\partial}\mathbf{d} = \mathscr{C}\,\boldsymbol{\partial}\underline{\mathbf{d}} \;+\; \mathscr{S}\,\boldsymbol{\partial}\mathbf{d}\,\boldsymbol{\partial}\underline{\mathbf{d}} \tag{6}$$

We have used the assumption that $\mathscr{S}$ is symmetric to obtain the last equation.

Tensors $\mathscr{C}$ and $\mathscr{S}$ can depend on s: for example, if the continuum represents a curved line in space, they may depend on the geometrical curvature and torsion of the line.

4 CONSTITUTIVE LAW

The *constitutive law* is a relation between stress and strain. We suppose this relation linear in $\boldsymbol{\epsilon}$.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\pi} + \mathbf{H}(s)\boldsymbol{\epsilon} \tag{7}$$

In the last equation $\mathbf{H}$ is a square positive symmetric matrix, while $\boldsymbol{\sigma}^{\pi}$ is a state of stress with $\boldsymbol{\epsilon} = 0$. If there is equilibrium with $\mathbf{f} = \mathbf{d} = 0$ and $\boldsymbol{\phi} = 0$, $\boldsymbol{\sigma}^{\pi}$ is called *selfstress*. We always suppose that $\boldsymbol{\sigma}^{\pi}$ is a selfstress.

5 VIRTUAL WORK PRINCIPLE

If $f(s)$ and $g(s)$ are functions $[0; L] \to \mathbb{R}$, we have (omitting the differential $\mathrm{d}s$ for simplicity, and denoting with an apex the operator $\mathrm{d}/\mathrm{d}s$) the integration by parts rule:

$$-\int_0^L f'\,g \;+\; |f\,g|_0^L = \int_0^L f\,g' \tag{8}$$

A generalization of this rule to derivate of n-th order is

$$(-1)^n \int_0^L f^{(n)}\,g \;+\; \left| \sum_{i=0}^{n-1} (-1)^i f^{(i)}\,g^{(n-1-i)} \right|_0^L = \int_0^L f\,g^{(n)} \tag{9}$$

The practical aim of this rule, here, is to 'move' the differential operator from g to f. In doing this we must add boundary terms.

The *virtual work principle* states that a configuration is in equilibrium if and only if the virtual external work is equal to the virtual internal work, for all admissible virtual displacements. This means we have

$$\int_0^L (\mathbf{f}^{\pi} + \mathbf{f}) \cdot \underline{\mathbf{d}} \;+\; |(\boldsymbol{\phi}^{\pi} + \boldsymbol{\phi}) \cdot \underline{\boldsymbol{\delta}}|_0^L = \int_0^L \boldsymbol{\sigma} \cdot \underline{\boldsymbol{\epsilon}} \quad \forall \underline{\mathbf{d}} \tag{10}$$

In the last equation the external and the boundary forces has been expressed as a sum of initial prestress forces (with the apex $^{\pi}$) and other forces. Prestress forces are present also in the reference configuration, when $\mathbf{f} = \mathbf{d} = 0$.

Substituting eqn (5), (6) and (7) in (10) we obtain

$$\int_0^L (\mathbf{f}^{\pi} + \mathbf{f}) \cdot \underline{\mathbf{d}} \;+\; |(\boldsymbol{\phi}^{\pi} + \boldsymbol{\phi}) \cdot \underline{\boldsymbol{\delta}}|_0^L = \int_0^L (\boldsymbol{\sigma}^{\pi} + \mathbf{H}\mathscr{C}\boldsymbol{\partial}\mathbf{d} + \frac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}\mathbf{d}\boldsymbol{\partial}\mathbf{d}) \cdot (\mathscr{C}\boldsymbol{\partial}\underline{\mathbf{d}} + \mathscr{S}\boldsymbol{\partial}\mathbf{d}\boldsymbol{\partial}\underline{\mathbf{d}}) \tag{11}$$

By putting $\mathbf{f} = \mathbf{d} = 0$ and $\boldsymbol{\phi} = 0$:

$$\int_0^L \mathbf{f}^\pi \cdot \underline{\mathbf{d}} \; + \; |\boldsymbol{\phi}^\pi \cdot \underline{\boldsymbol{\delta}}|_0^L = \int_0^L \boldsymbol{\sigma}^\pi \cdot \mathscr{C} \boldsymbol{\partial} \underline{\mathbf{d}} \tag{12}$$

Defining the matrix

$$\begin{aligned} \mathbf{W}[\mathbf{d}]_{jk} = &\quad \boldsymbol{\sigma}_i (\mathscr{C} + \mathscr{S} \boldsymbol{\partial}\mathbf{d})_{ijk} - (\boldsymbol{\sigma}^\pi \mathscr{C})_{jk} \\ = &\quad (\boldsymbol{\sigma}^\pi + \mathbf{H}\mathscr{C}\boldsymbol{\partial}\mathbf{d} + \tfrac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}\mathbf{d}\boldsymbol{\partial}\mathbf{d})_i (\mathscr{C} + \mathscr{S}\boldsymbol{\partial}\mathbf{d})_{ijk} - (\boldsymbol{\sigma}^\pi \mathscr{C})_{jk} \end{aligned} \tag{13}$$

and substituting eqn(12) and (13) in (11) we have:

$$\int_0^L \mathbf{f} \cdot \underline{\mathbf{d}} \; + \; |\boldsymbol{\phi} \cdot \underline{\boldsymbol{\delta}}|_0^L = \int_0^L \mathbf{W} \, \boldsymbol{\partial} \underline{\mathbf{d}} \tag{14}$$

Rule (9) applied to the second member of the last yelds

$$\int_0^L \mathbf{W} \, \boldsymbol{\partial} \underline{\mathbf{d}} = \int_0^L (\boldsymbol{\partial}^* \mathbf{W}) \cdot \underline{\mathbf{d}} \; + \; |(\mathscr{Z} \hat{\boldsymbol{\partial}}^* \mathbf{W}) \cdot (\hat{\boldsymbol{\partial}} \underline{\mathbf{d}})|_0^L \tag{15}$$

where the operator $\hat{\boldsymbol{\partial}}$ is obtained from $\boldsymbol{\partial}$ removing the higher order differential operator. $\boldsymbol{\partial}^*$ and $\hat{\boldsymbol{\partial}}^*$ are obtained respectively from $\boldsymbol{\partial}$ and $\hat{\boldsymbol{\partial}}$ changing the sign of odd order operators:

$$\hat{\boldsymbol{\partial}} = \begin{bmatrix} 1 \\ \frac{\partial}{\partial s} \\ \vdots \\ \frac{\partial^{(n-1)}}{\partial s^{(n-1)}} \end{bmatrix} ; \; \boldsymbol{\partial}^* = \begin{bmatrix} 1 \\ -\frac{\partial}{\partial s} \\ \vdots \\ (-1)^n \frac{\partial^n}{\partial s^n} \end{bmatrix} ; \; \hat{\boldsymbol{\partial}}^* = \begin{bmatrix} 1 \\ -\frac{\partial}{\partial s} \\ \vdots \\ (-1)^{n-1} \frac{\partial^{(n-1)}}{\partial s^{(n-1)}} \end{bmatrix} \tag{16}$$

The tensor $\mathscr{Z} = \mathscr{Z}_{ijk}$ is a third order $n \times (n+1) \times n$ tensor defined as

$$\begin{cases} \mathscr{Z}_{ijk} = 1 \text{ if } \frac{\partial}{\partial s} \hat{\boldsymbol{\partial}}_i \hat{\boldsymbol{\partial}}_k = \boldsymbol{\partial}_j \qquad (\text{i.e.} \quad i + k = j \;) \\ \mathscr{Z}_{ijk} = 0 \text{ elsewhere} \end{cases} \tag{17}$$

Since eqn (14) holds for all $\underline{\mathbf{d}}$, we obtain the *equilibrium*

$$\mathbf{f}_i = \boldsymbol{\partial}_j^* \mathbf{W}_{ji} \tag{18}$$

Boundary terms, as expressed in (15), are a matrix of forces

$$\mathbf{F}_{ij} = \mathscr{Z}_{ihk} \hat{\boldsymbol{\partial}}_k^* \mathbf{W}_{hj} \tag{19}$$

and a matrix of displacements

$$\mathbf{D}_{ij} = \hat{\boldsymbol{\partial}}_i \mathbf{d}_j \tag{20}$$

We can define vectors $\boldsymbol{\phi}$ and $\boldsymbol{\delta}$ by linking columns of $\mathbf{F}$ and $\mathbf{D}$ respectively, to obtain boundary terms as vectors.

Usually $\mathscr{C}$ and $\mathscr{S}$ have only a few non-null terms, so we can neglect boundary forces wich are always equal to 0. We can also multiply these vectors by observing that if the product of the two matrices $\mathbf{Q}_\phi \mathbf{Q}_\delta$ is the identity matrix and if

$$\begin{cases} \tilde{\boldsymbol{\phi}} = \mathbf{Q}_\phi^T \boldsymbol{\phi} \\ \tilde{\boldsymbol{\delta}} = \mathbf{Q}_\delta \boldsymbol{\delta} \end{cases} \tag{21}$$

then $\tilde{\boldsymbol{\phi}} \cdot \underline{\tilde{\boldsymbol{\delta}}} = \boldsymbol{\phi} \cdot \underline{\boldsymbol{\delta}}$.

6 ANALYSIS OF THE CONTINUUM

6.1 *Selfstressability*

Integration by parts of eqn (12) gives the selfstressability condition

$$\mathbf{f}_k^\pi = \boldsymbol{\partial}_j^*(\boldsymbol{\sigma}_i^\pi \mathscr{C}_{ijk}) \tag{22}$$

and the boundary selfstress terms

$$\mathbf{F}_{ij}^\pi = \mathscr{Z}_{ihk} \hat{\boldsymbol{\partial}}_k^*(\boldsymbol{\sigma}_l^\pi \mathscr{C}_{lhj}) \tag{23}$$

6.2 *Mechanisms*

If a displacement $\mathbf{d}^m$ is such that

$$\mathscr{C} \boldsymbol{\partial} \mathbf{d}^m = 0 \tag{24}$$

then $\mathbf{d}^m$ is called *mechanism*. If we have also $\mathscr{S} \, \boldsymbol{\partial} \mathbf{d}^m \, \boldsymbol{\partial} \mathbf{d}^m \neq 0$ then $\mathbf{d}^m$ is an *infinitesimal mechanism*. This definition is analogous to the definition of (infinitesimal) mechanism for a pin-jointed assembly [4].

The set of all mechanisms $\mathbf{d}^m$ is a linear subspace of all displacements $\mathbf{d}$.

6.3 *Linear analysis*

In the expression of $\mathbf{W}$ of eqn (13) we can neglect nonlinear terms in $\boldsymbol{\partial}\mathbf{d}$. From the equilibrium (18) we can write the differential linear equation:

$$\mathbf{f} = \boldsymbol{\partial}^*((\mathbf{H} \mathscr{C} \boldsymbol{\partial} \mathbf{d}) \mathscr{C} + \boldsymbol{\sigma}^\pi \mathscr{S} \boldsymbol{\partial} \mathbf{d}) \tag{25}$$

Expliciting indexes:

$$\mathbf{f}_k = \boldsymbol{\partial}_j^*((\mathbf{H}_{ii'} \mathscr{C}_{i'j'k'} \boldsymbol{\partial}_{j'} \mathbf{d}_{k'}) \mathscr{C}_{ijk} + \boldsymbol{\sigma}_i^\pi \mathscr{S}_{ij'k'jk} \boldsymbol{\partial}_{j'} \mathbf{d}_{k'}) \tag{26}$$

Force $\mathbf{f}$ is the sum of two terms. The first classical term is linear in $\mathbf{H}$, the second is indipendent from $\mathbf{H}$ and linear in the selfstress $\boldsymbol{\sigma}^\pi$. Both terms are linear in $\mathbf{d}$.

If $\mathbf{d} = \mathbf{d}^m$ is an infinitesimal mechanism, it cannot be stiffened by the classical term, but it can be stiffened by selfstress. It is important to notice that, while the classical stiffness is always positive, prestress stiffness may be negative and may cause instability, expecially in the subspace of mechanisms.

Constitutive matrix $\mathbf{H}$ depends on the Young modulus of the material, which is bigger than the yield tension and selfstress. This means that, usually, the selfstress stiffness is smaller than the classic one. This statement, however, depends on the actual configuration of the continuum, and also on L.

6.4 *Nonlinear analysis*

Similar analisys can be done considering all five terms in the expression of $\mathbf{W}$:

$$\begin{array}{rcrc} \mathbf{W} & = & \mathbf{H}\mathscr{C}\boldsymbol{\partial}\mathbf{d}\,\mathscr{C} & \text{(A)} \\ & + & \boldsymbol{\sigma}^{\pi}\ \mathscr{S}\boldsymbol{\partial}\mathbf{d} & \text{(B)} \\ & + & \mathbf{H}\mathscr{C}\boldsymbol{\partial}\mathbf{d}\ \mathscr{S}\boldsymbol{\partial}\mathbf{d} & \text{(C)} \\ & + & \frac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}\mathbf{d}\boldsymbol{\partial}\mathbf{d}\,\mathscr{C} & \text{(D)} \\ & + & \frac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}\mathbf{d}\boldsymbol{\partial}\mathbf{d}\ \mathscr{S}\boldsymbol{\partial}\mathbf{d} & \text{(E)} \end{array} \tag{27}$$

Terms (C-D) are quadratic in the displacement, while term (E) is cubic. Quadratic terms, as we are going to explain, can sometimes be neglected. The basic idea is to do a linear analysis for small non-mechanisms, and nonlinear only for the subspace of mechanisms.

If $|\mathbf{H}^{-1}\boldsymbol{\sigma}^{\pi}| \ll 1$, it will also be, for a general $\mathbf{f} \ll 1$, $\mathbf{d} \simeq \mathbf{d}^m$. We will however suppose $\mathbf{d} \ll 1$.

Constraining the displacement with the condition $\mathbf{d} = \mathbf{d}^m$, we obtain

$$\begin{array}{rcrc} \mathbf{W}\boldsymbol{\partial}\underline{\mathbf{d}}^m & = & \boldsymbol{\sigma}^{\pi}\ \mathscr{S}\boldsymbol{\partial}\mathbf{d}^m\ \boldsymbol{\partial}\underline{\mathbf{d}}^m & \text{(B)} \\ & + & \frac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}\mathbf{d}^m\boldsymbol{\partial}\mathbf{d}^m\ \mathscr{S}\boldsymbol{\partial}\mathbf{d}^m\ \boldsymbol{\partial}\underline{\mathbf{d}}^m & \text{(E)} \end{array} \tag{28}$$

This constrain is used only to calculate the displacement in the space of mechanisms. Also, we suppose that $\mathbf{d}^m$ from eqn(28) is not 0. Integration by parts leads to

$$\begin{cases} \mathbf{f} = \boldsymbol{\partial}^*(\mathbf{W}[\mathbf{d}^m] + \boldsymbol{\sigma}^{\lambda}\mathscr{C}) \\ \mathscr{C}\boldsymbol{\partial}\mathbf{d}^m = 0 \end{cases} \tag{29}$$

The stress $\boldsymbol{\sigma}^{\lambda}$ is a reactive stress, which arises from the constrain $\mathbf{d} = \mathbf{d}^m$. It has the significance of a Lagrange multiplier.

Once calculated $\mathbf{d}^m$ (if calculable) as an approximation of $\mathbf{d}$ in the subspace of mechanisms, we replace eqn(27) with

$$\begin{array}{rcrc} \mathbf{W}[\tilde{\mathbf{d}}] & = & \mathbf{H}\mathscr{C}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m)\,\mathscr{C} & \text{(A)} \\ & + & \boldsymbol{\sigma}^{\pi}\ \mathscr{S}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m) & \text{(B)} \\ & + & \mathbf{H}\mathscr{C}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m)\ \mathscr{S}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m) & \text{(C)} \\ & + & \frac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m)\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m)\,\mathscr{C} & \text{(D)} \\ & + & \frac{1}{2}\mathbf{H}\mathscr{S}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m)\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m)\ \mathscr{S}\boldsymbol{\partial}(\tilde{\mathbf{d}} + \mathbf{d}^m) & \text{(E)} \end{array} \tag{30}$$

We can approximate last equation considering only linear terms in $\tilde{\mathbf{d}}$. Thus we can find $\mathbf{d} \simeq \mathbf{d}^m + \tilde{\mathbf{d}}$ by the equilibrium (18). By doing this we can neglect the part of $\tilde{\mathbf{d}}$ which is a mechanism, since $\mathbf{d}^m \neq 0$ is already a good approximation of $\mathbf{d}$ in the space of mechanisms.

6.5 *Change of variable*

Sometimes is convenient to refer our model to a continuum with unit length. To do so, we have to replace all integrals $\int_0^L$ with integrals $\int_0^1$. The differential $\mathrm{d}s$ is replaced by $L\mathrm{d}x$, where $x = s/L$.

We also have to replace the operator $\boldsymbol{\partial}$ with $\mathbf{L}\boldsymbol{\partial}^x$, where $\mathbf{L}$ is a diagonal matrix with $\mathbf{L}_{ii} = L^{1-i}$, and $\boldsymbol{\partial}^x = [1; \partial/\partial x; \ldots; \partial^n/\partial x^n]^T$. Similar substitutions are made on $\hat{\boldsymbol{\partial}}$, $\boldsymbol{\partial}^*$, and $\hat{\boldsymbol{\partial}}^*$

7 EXAMPLES

For further references on examples (7.1) see [2]. Further references for the behaviour of a suspended cable (7.2) can be found, for example, in [5] and [6]. A precise model of a cable, expecially under torsional sollecitations, is being developed for an accurate determination of the universal gravitational constant, as well as for other precise measurements [7].

7.1 *Bernoulli compressed beam*

In this classic examples we consider a continuum in which there is only one kind of displacement, the transversal displacement u.

$$\mathbf{d} = [u] \tag{31}$$

The dual force of u is the transversal load q.

Strain is a vector of size two. Its components are the elongation $\boldsymbol{\epsilon}_1 = e$ and the classic curvature $\boldsymbol{\epsilon}_2 = \chi$. Dual variables are normal force $N = N^\pi + EAe$ and bending moment $M = EJ\chi$ (we suppose that there is no prebending).

$$\boldsymbol{\epsilon} = \begin{bmatrix} e \\ \chi \end{bmatrix} = \begin{bmatrix} 1/2\ u'^2 \\ u'' \end{bmatrix} \tag{32}$$

The elongation term is a second order strain. It's physical meaning can be understood by observing Fig.1.

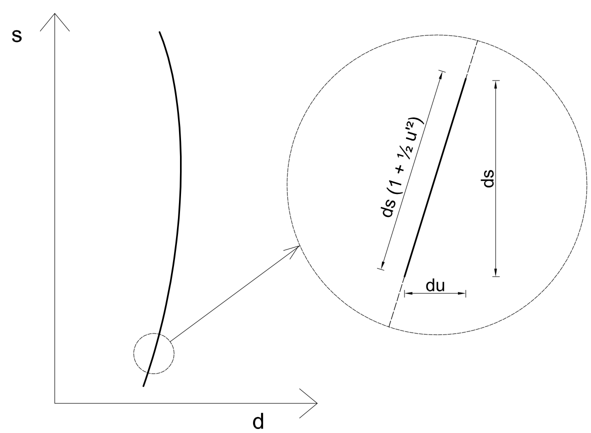

Figure 1: Second order elongation

The elongation of the element is $(\sqrt{\mathrm{d}s^2 + \mathrm{d}u^2} - \mathrm{d}s)/\mathrm{d}s \simeq (1/2)u'^2$. Thus we can set:

$$\begin{aligned} &\mathscr{C}_{231} = 1 \\ &\mathscr{S}_{12121} = 1 \\ &\mathscr{C} = 0\ ;\ \mathscr{S} = 0 \text{ elsewhere} \end{aligned} \tag{33}$$

Size of tensors $\mathscr{C}$ and $\mathscr{S}$ are respectively $2 \times 3 \times 1$ and $2 \times (3 \times 1)^2$.

Using eqn (25) and supposing $\mathbf{H}$ constant, we obtain the linear equation

$$EJu^{IV} - N^\pi u'' = q \tag{34}$$

If prestress is low, the beam has a behaviour similar to the classic beam. However, if, for example, $q = 0$ and $-N^\pi > (\pi/L)^2 EJ$ we have to consider higher order terms of $\mathbf{W}$ to make a nonlinear postbuckling analysis, since the structure is unstable.

Classic boundary force are moment M and shear $-M'$. Prestress causes an addictional boundary force $N^\pi u'$ which does work for u (thus is a shear). Physical interpretation is in Fig.2

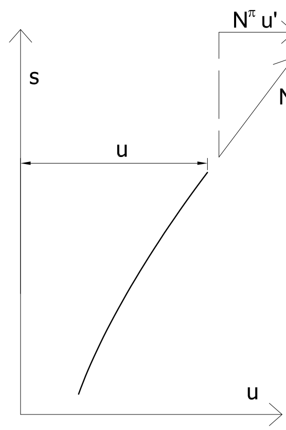

Figure 2: Prestress shear

7.2 *Nonlinear behaviour of a cable*

In the symplified mathematical model of a planar cable presented here there are three displacement: transversal displacement (u), parallel to the axis displacement (v), and rotation (ϕ) of the section of the cable along its axis. Dual forces are respectively q, p, and c.

We suppose the section circular and constant. Strains are the elongation of the cable $e = v' + \frac{1}{2}u'^2 + \frac{1}{2}\frac{J_0}{A}\phi'^2$ (J_0, for a circular section, is the polar and torsional inertia) and the twist $\theta = \phi'$.

The normal force is $N = N^\pi + k(v' + \frac{1}{2}u'^2 + \frac{1}{2}\frac{J_0}{A}\phi'^2)$, where $k = EA$ is the axial stiffness, which we suppose constant. Supposing $T^\pi = 0$, the torsion is expressed by $T = GJ_0\phi' = g\theta$. We set as boundary conditions $u(0) = u(L) = 0$, $v(0) = v(L) = 0$ and $\phi(0) = \phi(L) = 0$. See Fig.3.

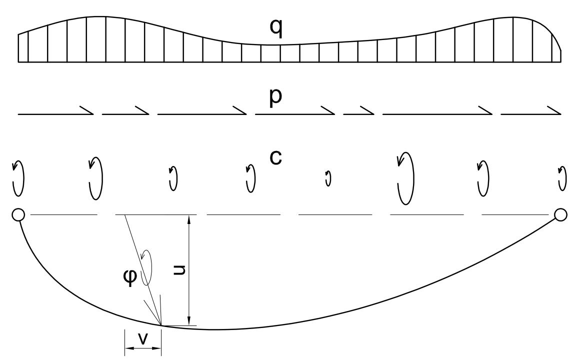

Figure 3: Load and displacement of a cable

The justification of the term $\frac{1}{2}\frac{J_0}{A}\phi'^2$ is in [8]. See also Fig.4.

The elongation of a fiber at distance r from the center is $\frac{1}{2}\phi'^2 r^2$. Integrating in the section of the cable, we obtain the average elongation due to twist as $\frac{1}{2}\frac{J_0}{A}\phi'^2$.

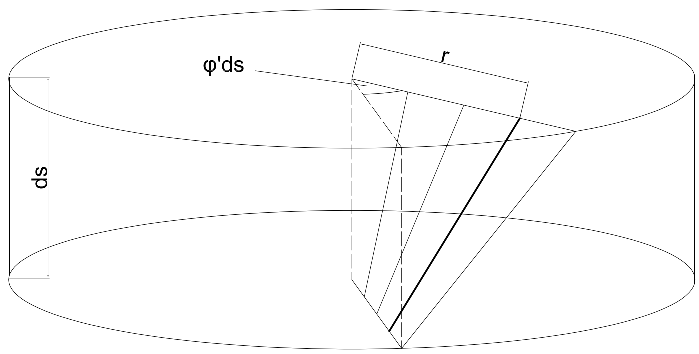

Figure 4: Nonlinear elongation due to torsion

Hence we have:

$$\begin{array}{l} \mathbf{d} = [u; v; \phi]^T \\ \boldsymbol{\epsilon} = [e; \theta]^T \\ \boldsymbol{\sigma} = [N; T]^T \\ \mathbf{f} = [q; p; c]^T; \end{array} \tag{35}$$

Tensors $\mathscr{C}$ and $\mathscr{S}$, of size $2 \times 2 \times 3$ and $2 \times (2 \times 3)^2$, have non-null terms

$$\mathscr{C}_{122} = \mathscr{C}_{223} = \mathscr{S}_{12121} = 1 \quad ; \quad \mathscr{S}_{12323} = J_0/A \tag{36}$$

Mechanisms are all transversal displacements $\mathbf{d}^m = [u; 0; 0]$. Orthogonal space is the set of all displacements $\mathbf{d}^K = [0; v; \phi]$.

Defining $j = J_0/A$, the expression $\mathbf{W}\boldsymbol{\partial}\underline{\mathbf{d}}$ is

$$\begin{array}{rcl} \mathbf{W}\boldsymbol{\partial}\underline{\mathbf{d}} & = & (N^\pi u' + kv'u' + \frac{1}{2}ku'^3 + \frac{1}{2}kj\phi'^2 u')\ \underline{u}' \\ & + & (kv' + \frac{1}{2}ku'^2 + \frac{1}{2}kj\phi'^2)\ \underline{v}' \\ & + & (g\phi' + N^\pi j\phi' + kjv'\phi' + \frac{1}{2}kju'^2\phi' + \frac{1}{2}kj^2\phi'^3)\ \underline{\phi}' \end{array} \tag{37}$$

Integrating by parts the expression of virtual work principle we obtain

$$\left\{ \begin{array}{rl} q = & -(N^\pi u' + kv'u' + \frac{1}{2}ku'^3 + \frac{1}{2}kj\phi'^2 u')' \\ p = & -(kv' + \frac{1}{2}ku'^2 + \frac{1}{2}kj\phi'^2)' \\ c = & -(g\phi' + N^\pi j\phi' + kjv'\phi' + \frac{1}{2}kju'^2\phi' + \frac{1}{2}kj^2\phi'^3)' \end{array} \right. \tag{38}$$

This non linear system is coupled in u,v and ϕ. While calculating u from eqn (38-a), we can neglect terms with v and ϕ. In eqn(38-b,c) we can neglect non linear terms in v and ϕ:

$$\left\{ \begin{array}{rl} q = & -(N^\pi u' + \frac{1}{2}ku'^3)' \\ p = & -(kv' + \frac{1}{2}ku'^2)' \\ c = & -(g\phi' + N^\pi j\phi' + \frac{1}{2}kju'^2\phi')' \end{array} \right. \tag{39}$$

The first equation is non linear in u, while second and third equation, once known u, is linear in v and ϕ.

Notice that prestress N^π, if positive, gives linear stiffness to transversal displacement, and also increases linear torsional stiffness.

8 CONCLUSIONS

A general third order analytical theory for the static and kinematic analysis of a prestressed monodimensional continuum has been proposed. Further work will concern a similar description of its dynamical behaviour.

The layout of the differential equations which describe the continuum are completely determined when tensors $\mathscr{C}$ and $\mathscr{S}$ are defined. Qualitative and quantitative behavior is determined by parameters $\mathbf{H}$ and $\boldsymbol{\sigma}^{\pi}$.

A tensor concerning inertial terms will be defined for dynamical analysis, and a parameter to describe the viscous damping.

The mathematical enviroment proposed has been developed in the optic of an easy numerical implementation.

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