Bounds for plastic multipliers in combined loading: some applications to solids and structures

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SUMMARY. In the framework of classical Plasticity, even when limit multipliers and collapse mechanisms associated to different loads independently acting on a solid or structure are known, no much can be inferred on the limit multiplier of the combined loading. With the aim of some new theorems developed to this purpose, a few examples are presented and discussed.

1 INTRODUCTION

An important aspect of structural analysis, especially for ultimate safety assessment or design, consists in evaluating the maximum load that a structure can sustain. This is at the heart of many structural codes and the earthquake which has recently shaken the central regions of Italy, wrecking homes and causing a considerable death toll, is just the most recent example of the utmost importance of a reliable evaluation of the collapse load in many structural engineering problems.

An attempt to find a sort of superposition principle has been recently done by Puzrin and Randholph [1] with reference to particular yield criteria and in the framework of upper bound limit analysis. They investigated the implications of combining two arbitrary kinematically admissible velocity fields and showed that, under certain conditions, a sort of superposition can be applied and produces the upper bound limiting value for the surface traction of the combined mechanism. Nonetheless the applicability of their results is limited by the strict assumptions at the base of the treatment.

Frame structures under the action of dead loads and seismic forces, soil-foundation interaction problems, tunnels under a variety of loads, deepwater pipelines subject to bending and pressure, constitute a few selected examples for which some sort of superposition rule would result extremely useful. In all these cases bounding techniques can be extremely helpful. From a purely engineering standpoint, lower bounds to the limit carrying capacity generally result more relevant than upper bounds, since in many practical applications safety factors are needed. However, upper bounds may be employed to estimate the inaccuracy with respect to the actual limit multiplier.

This said, on the sole basis of the tools of the classical theory of limit analysis and making resort to a set of inequalities, the present paper exploits a new set of theorems for bounding limit multipliers for combined loads [2] to present and discuss a few examples showing their practical value.

2 LIMIT MULTIPLIER BOUNDS FOR COMBINED LOADS

As stated before, in the present work some new theorems which can yield relevant bounds on the overall limit multiplier in case of combined loading (Figure 1) will be employed.

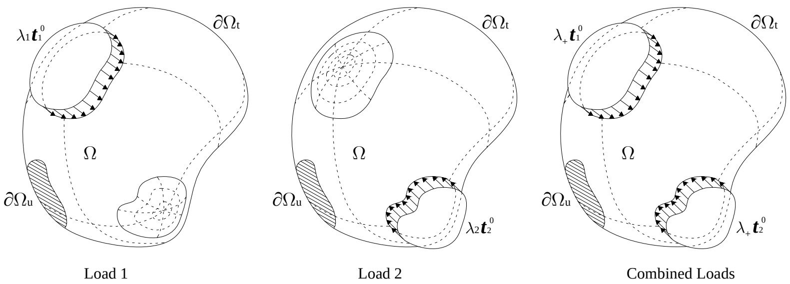

Figure 1: An elastic-plastic body subject to simple and combined loading.

On the basis of the findings in [2], it is possible to summarise all the procedures to bound the limit multiplier in case of combined loads, showing their practical value. The presentation is restricted, for the sake of clarity, to the case of two different loadings but it can be directly extended to the case of n loadings. ranging from a minimum knowledge about the critical state of the particular loading to a reasonable guess of the kinematics of the problem under combined loads, more and more refined bounds for the overall limit multiplier can be derived, as summarised in Table 1.

$(\boldsymbol{t}_2^0, \dot{\boldsymbol{u}}_1) \geq$	$(\boldsymbol{t}_1^0, \dot{\boldsymbol{u}}_2) \geq$	$(\boldsymbol{t}_2^0, \dot{\boldsymbol{u}}_+) \leq$	$(\boldsymbol{t}_1^0, \dot{\boldsymbol{u}}_+) \leq$	λ_1	λ_2	**Safety factor**	**Overload factor**
×	×	×	×	•	×	$\lambda_1/2$	×
×	×	×	×	•	•	$(\lambda_1^{-1}+\lambda_2^{-1})^{-1}$	×
•	×	×	×	•	•	$(\lambda_1^{-1}+\lambda_2^{-1})^{-1}$	λ_1
×	•	×	×	•	•	$(\lambda_1^{-1}+\lambda_2^{-1})^{-1}$	λ_2
×	•	•	×	•	•	λ_1	λ_2
×	×	•	×	•	•	λ_1	×
×	×	×	•	•	•	λ_2	×

Table 1: Safety and overload multipliers for the combined loading,. Without loss of generality, it is assumed: $\lambda_1 \leq \lambda_2$.

3 EXAMPLES

Two examples showing the actual value of the new theorems in engineering problems are presented and discussed in this section. The examples are chosen in order to present quantitative estimates of the actual carrying capacity of common structural problems.

3.1 *Frame structure under the action of combined horizontal and vertical point loads.*

With reference to the Figure 2, the frame structure can be subject to two different loading conditions, that is a horizontal force , a vertical force and a combined loading given by both the previous forces exerted on the structure at the same time.

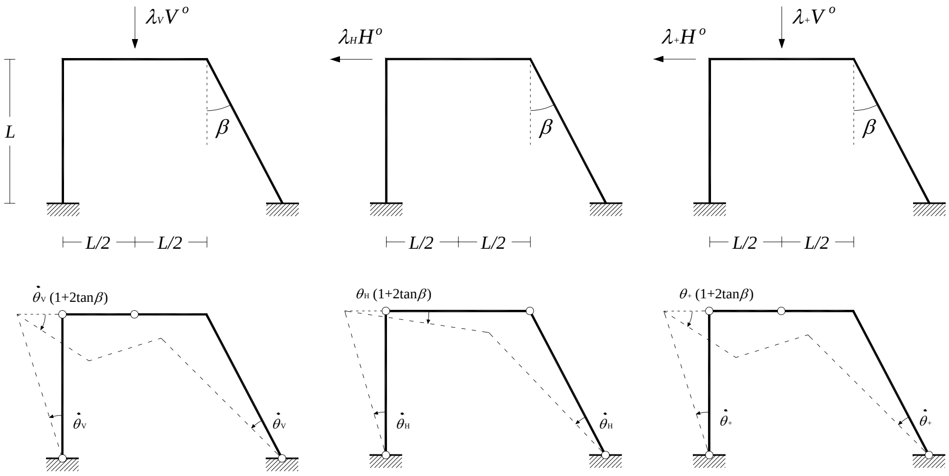

Figure 2: Frame structure under the action of vertical, horizontal and combined point loads conditions (top) and relative critical mechanisms (bottom).

The positions of possible plastic hinges can be immediately located and, therefore, all the collapse mechanisms are known (Massonet and Save, 1980). Thus, the following limit multipliers are obtained

$$\lambda_V = \frac{4\bar{M}(3+2\tan\beta)}{V^0 L(1+2\tan\beta)}, \qquad \lambda_H = \frac{2\bar{M}(2+\tan\beta)}{H^0 L}, \qquad \lambda_+ = \frac{4\bar{M}(3+2\tan\beta)}{[H^0+V^0(1+2\tan\beta)]L} \tag{1}$$

where $\bar{M} = \sigma_Y W^{pl}$ is the plastic moment, σ_Y is the yield stress and W^{pl} is the plastic section modulus of the cross section, which is assumed the same for all the elements. Theorem 2 gives the following lower bound, λ_L, for the combined loading limit multiplier, λ_+,

$$\lambda_+ \geq \lambda_L = (\lambda_V^{-1} + \lambda_H^{-1})^{-1} \quad \Rightarrow \quad \frac{H^0+V^0(1+2\tan\beta)}{(3+2\tan\beta)} \leq \frac{2H^0}{(2+\tan\beta)} + \frac{V^0(1+2\tan\beta)}{(3+2\tan\beta)} \tag{2}$$

The inequality (2) can be trivially verified since $4+3\tan\beta>0,\ \forall\beta\in[0,\pi/2[$.

Making reference to the collapse mechanisms illustrated in Figure 2, it is also immediate to recognise that the mutual dissipation is positive,

$$H^0 \times \dot{u}_V = H^0 \times \dot{\vartheta}_V L > 0, \quad V^0 \times \dot{u}_H = V^0 \times \dot{\vartheta}_H \frac{L}{2} \tan\beta > 0 \tag{3}$$

where $\dot{u}_V$ and $\dot{u}_H$ denote the velocity fields at the critical states associated with the vertical and horizontal point loads, respectively, and $\dot{\vartheta}$ is the corresponding angular velocity. As a consequence, the upper bound , λ_U , for the combined limit multiplier, λ_+ , is

$$\lambda_+ \leq \lambda_U \equiv \min\{\lambda_V, \lambda_H\} = \begin{cases} \lambda_V, & \beta \big| \{\tan\beta < A - B, \tan\beta > A + B\} \\ \lambda_H, & \beta \big| \{A - B < \tan\beta < A + B\} \end{cases} \tag{4}$$

where

$$A = \frac{H^0}{V^0} - \frac{5}{4}, \quad B = \sqrt{\left(\frac{H^0}{V^0}\right)^2 + \frac{H^0}{2V^0} + \frac{9}{16}} \tag{5}$$

3.2 *Limit carrying capacity of an elliptical cross section subject to combined bending.*

Figure 3 shows the elliptical cross section of a cylinder subject to combined bending $\boldsymbol{M} = [M_1^0, M_2^0]^T$. a and $b \geq a$ are the main semi-diameters along the principal axes $\{x_1, x_2\}$.

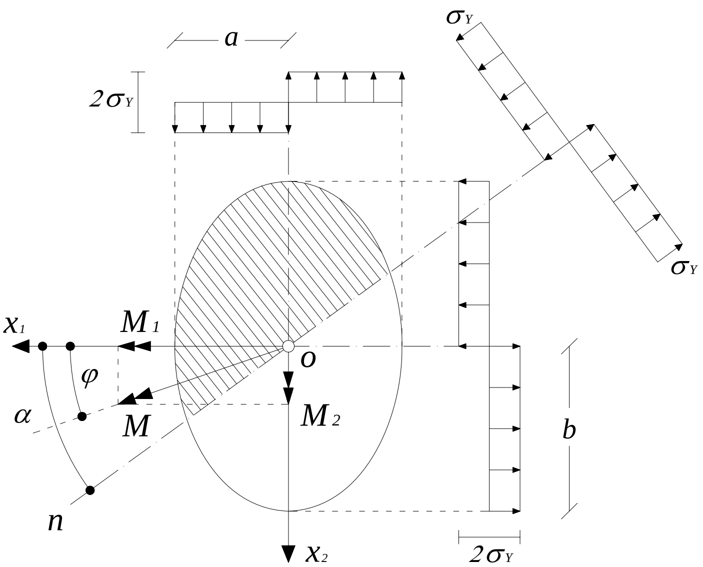

Figure 3: Elliptical cross section of a beam subject to biaxial bending.

By assuming that, within the cross section, the evolution of the normal stress from the purely elastic to the fully plastic state yields a bi-rectangular distribution of the yield stress σ_Y , with a discontinuity around the neutral axis

$$x_2 = \tan\alpha\ x_1, \quad \tan\alpha = \frac{b^2}{a^2}\tan\varphi, \quad \tan\varphi = \frac{M_2^0}{M_1^0} \tag{6}$$

the plastic bending can be easily computed by taking into account the static moments of the regions delimited by the neutral axis [3]. Indeed, after some algebraic calculations, it is easy to obtain

$$M_1 = \frac{4}{3}ab^2\sigma_Y, \quad M_2 = \frac{4}{3}a^2b\sigma_Y, \quad M_+ = \frac{4}{3}ab\sqrt{\frac{b^4 + a^4\tan^2\alpha}{b^2 + a^2\tan^2\alpha}}\sigma_Y \tag{7}$$

Thus, the corresponding plastic multipliers are defined as follows

$$\lambda_1 \equiv \frac{M_1}{M_1^0} = \frac{4ab^2\sigma_Y}{3M_1^0}, \quad \lambda_2 \equiv \frac{M_2}{M_2^0} = \frac{4a^2b\sigma_Y}{3\tan\varphi M_1^0}, \quad \lambda_+ \equiv \frac{M_+}{M_+^0} = \frac{4a^2b^2\sigma_Y}{3M_1^0\sqrt{a^2 + b^2\tan^2\varphi}} \tag{8}$$

A lower bound, λ_L , for the limit multiplier of the combined loading, λ_+, can be obtained by writing the inequality

$$\lambda_+ \geq \lambda_L \equiv (\lambda_1^{-1} + \lambda_2^{-1})^{-1} \tag{9}$$

Then, substitution of equation (8) into equation (9) gives

$$\lambda_+ \geq \lambda_L \quad \Rightarrow \quad \frac{c}{\sqrt{a^2 + b^2\tan^2\varphi}} \geq \frac{c}{a + b\tan\varphi}, \qquad c = \frac{4a^2b^2\sigma_Y}{3M_1^0} \tag{10}$$

which is trivially satisfied if the physically manifest hypotheses $M_1^0 > 0$ and $\tan\varphi > 0$ are assumed to hold true.

The mutual dissipation written in terms of generalized stresses (bending moments M_i^0) and generalised plastic strains (rate of plastic curvatures $\dot{\chi}_j$) , gives

$$\{(M_1^0, \dot{\chi}_2) = 0, \quad (M_2^0, \dot{\chi}_1) = 0\} \quad \Rightarrow \quad \lambda_+ \leq \min\{\lambda_1, \lambda_2\} \tag{11}$$

Thus, the most accurate upper bound λ_U is given by

$$\lambda_+ \leq \lambda_U = \min\{\lambda_1, \lambda_2\} = \begin{cases} \lambda_1, & 0 < \tan\varphi < \dfrac{a}{b} \\ \lambda_2, & \tan\varphi > \dfrac{a}{b} \end{cases} \tag{12}$$

4 CONCLUSIONS

The work has presented examples of application of some new theorems which yield relevant bounds on the overall limit multiplier in case of combined loads. The results can be regarded as a sort of rule of superposition of the load multipliers in classical limit analysis. The findings have been shown to be useful in cases of combined loading.

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