Annular shear driven instability for a compressible elastic tube

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SUMMARY. We study the possibility of axially periodic toroidal twist-like bifurcations for an isotropic compressible elastic tube subject to an annular shear fundamental deformation. We propose a procedure for the determination of the critical load corresponding to such bifurcations.

1 INTRODUCTION

In [1] we investigated the possibility for an isotropic elastic solid to support bifurcating periodic displacements induced by shear stress which are reminiscent of the planar Couette sinusoidal instability pattern observed in the flow of viscous fluids (cf. [2]). Specifically, we studied the planar case of an infinitely long block of generalized Blatz-Ko material confined between, and attached to, parallel plates which are subject to a relative shear displacement. Our bifurcation analysis allowed us to determine a critical value of the applied shearing strain which corresponds to the occurrence of the planar Couette instability.

The analysis developed in [1] represents a precursor to the problem of the annular shear between two concentric cylinders, which is the subject of the present paper. Here, we draw on the classical Taylor-Couette axially periodic cellular instability pattern observed in the laminar shearing flow of a viscous fluid confined between two concentric cylinders, each rotating with different angular velocity (cf. [3]).

To model the analogous for solids of this instability, we consider an isotropic compressible elastic tube whose strain energy is defined by a function of the second and the third principal invariants of the left Cauchy-Green strain tensor. We prescribe a relative annular shear at the inner and outer boundaries of the tube, and assume that the axisymmetric annular shear deformation is the fundamental equilibrium solution. Then, in order to analyze the possibility of bifurcating solutions from a pure annular shear to an axially periodic toroidal twist-like deformation, we restrict the study of the linearized equilibrium equations to the class of axisymmetric incremental periodic displacements defined by three unknown functions of the radial coordinate and with a periodic dependence on the axial coordinate.

The differential problem coming out from this bifurcation analysis is studied by employing some results from the ordinary differential equations theory (cf. [4]). In particular, we obtain a bifurcation condition which allows us to determine the critical value of the loading parameter during an annular shear loading process. Finally, we briefly describe a procedure to perform the bifurcation analysis, and apply this procedure in a numerical example.

Remark: We inform the reader that the present study on the compressible elastic tube is preliminary to case of the incompressible elastic tube, which we mentioned in the AIMETA 2009 extended summary.

2 THE PURE ANNULAR SHEAR EQUILIBRIUM SOLUTION

Let $\mathcal{C}$ denote the natural reference configuration of a homogeneous, isotropic, compressible, hyperelastic tube in a cylindrical coordinate system with orthonormal basis $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$:

$$\mathcal{C} \equiv \{(R, \Theta, Z) \mid R_1 < R < R_2,\ \ 0 \le \Theta < 2\pi,\ \ 0 < Z < H\}; \tag{1}$$

the boundary of $\mathcal{C}$ is divided into two disjoint parts:

$$\begin{aligned} \partial_1\mathcal{C} &\equiv \{(R, \Theta, Z) \in \mathcal{C} \mid R = R_1 \text{ or } R_2\}, \\ \partial_2\mathcal{C} &\equiv \{(R, \Theta, Z) \in \mathcal{C} \mid Z = 0 \text{ or } H\}. \end{aligned} \tag{2}$$

The deformation

$$\mathbf{f}: \mathbf{X} \in \mathcal{C} \mapsto \mathbf{x} = \mathbf{f}(\mathbf{X}) \in \mathbf{f}(\mathcal{C}) \tag{3}$$

is assumed to be a smooth function which satisfies the standard requirement of being a homeomorphism and the orientation-preserving condition $\det \mathbf{F}(\mathbf{X}) > 0$, where $\mathbf{F}(\mathbf{X}) := \nabla \mathbf{f}(\mathbf{X})$ is the deformation gradient. We consider the following class C^2 strain energy function:

$$W(\mathbf{F}) := 2\alpha(II_{\mathbf{B}} + 1)^{1/2} + \psi(III_{\mathbf{B}}), \tag{4}$$

where

$$\mathbf{B} := \mathbf{F}\mathbf{F}^T, \quad II_{\mathbf{B}} := \frac{1}{2}\left((\operatorname{tr}\mathbf{B})^2 - \operatorname{tr}\mathbf{B}^2\right), \quad III_{\mathbf{B}} := \det\mathbf{B}, \tag{5}$$

$\alpha \neq 0$ is a material parameter and ψ is a smooth scalar function. The motivation of the choice (4) for the stored energy function will be justified in the following. For the constitutive class (4), the Piola-Kirchhoff stress tensor takes the form

$$\mathbf{S}(\mathbf{F}) := DW(\mathbf{F}) = 2\alpha(II_{\mathbf{B}} + 1)^{-1/2}(I_{\mathbf{B}}\mathbf{F} - \mathbf{B}\mathbf{F}) + 2\psi'(III_{\mathbf{B}})(\det\mathbf{B})\,\mathbf{F}^{-T}, \tag{6}$$

where $I_{\mathbf{B}} := \operatorname{tr}\mathbf{B}$ and ψ' is the first derivative ψ; the Cauchy stress tensor $\mathbf{T}(\mathbf{F})$ is consequently given by

$$\mathbf{T}(\mathbf{F}) = (\det\mathbf{B})^{-1/2}\,\mathbf{S}\,\mathbf{F}^T = 2\alpha(II_{\mathbf{B}} + 1)^{-1/2}(\det\mathbf{B})^{-1/2}\left(I_{\mathbf{B}}\mathbf{B} - \mathbf{B}^2\right) + 2\psi'(III_{\mathbf{B}})(\det\mathbf{B})^{1/2}\,\mathbf{I}, \tag{7}$$

where $\mathbf{I}$ is the second order identity tensor. Because $\mathcal{C}$ is a natural reference configuration, the condition $\mathbf{T}(\mathbf{I}) = \mathbf{O}$ yields

$$\psi'(1) = -\alpha. \tag{8}$$

Furthermore, in view of (4)-(8), the elasticity tensor at the origin may be written as

$$\mathbb{C}(\mathbf{I}) = 2\alpha(\mathbf{I}\boxtimes\mathbf{I}) + \left[4\psi''(1)-3\alpha\right]\,(\mathbf{I}\otimes\mathbf{I}), \tag{9}$$

where the symbols $\boxtimes$ and $\otimes$ denote, respectively, the tensor products $(\mathbf{A}\boxtimes\mathbf{L})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{L}^T$ and $(\mathbf{A}\otimes\mathbf{L})\mathbf{C} = (\mathbf{C}\cdot\mathbf{L})\mathbf{A}$, defined for each $\mathbf{A}$, $\mathbf{L}$, $\mathbf{C}\in$ Lin. (9) allows to write the classical requirements of positive definiteness and strong ellipticity for $\mathbb{C}(\mathbf{I})$ as follows:

$$\psi''(1) > \frac{7}{12}\alpha, \quad \alpha > 0, \tag{10}$$

$$\psi''(1) > \frac{1}{4}\alpha, \quad \alpha > 0. \tag{11}$$

Notice that the material parameter α corresponds to the shear modulus at the origin.

We assume that the inner cylinder is kept fixed, whereas the outer is subject to a rotating displacement; on the bases of $\mathcal{C}$ only tangential displacements are admitted by applying normal tractions. This leads to the following mixed boundary-value problem:

$$\operatorname{Div}\mathbf{S}(\mathbf{F}) = \mathbf{0} \qquad \text{in } \mathcal{C}, \tag{12}$$

$$\left.\begin{aligned} &(\mathbf{f}(\mathbf{X})-\mathbf{X})\cdot\mathbf{E}_\Theta = \begin{cases} 0 & \text{at } R = R_1 \\ \lambda > 0 & \text{at } R = R_2 \end{cases} \\ &(\mathbf{f}(\mathbf{X})-\mathbf{X})\cdot\mathbf{E}_\mathbf{R} = (\mathbf{f}(\mathbf{X})-\mathbf{X})\cdot\mathbf{E}_\mathbf{Z} = 0 \ \text{ at } R = R_1,\ R_2 \end{aligned}\right. \qquad \text{on } \partial_1\mathcal{C}, \tag{13}$$

$$(\mathbf{f}(\mathbf{X})-\mathbf{X})\cdot\mathbf{E}_Z = 0, \qquad (\mathbf{S}(\mathbf{F})\ \mathbf{E}_Z)\times\mathbf{E}_Z = \mathbf{0} \qquad \text{on } \partial_2\mathcal{C}. \tag{14}$$

A major aim of the present paper is that of exploring the possibility of adjacent bifurcating fields superposed upon a primary equilibrium *pure annular shear* $\mathbf{x} = \tilde{\mathbf{f}}(\mathbf{X})$ deformation, which maps the material point (R, Θ, Z) to

$$r = R, \qquad \theta = \Theta + \omega(R), \qquad z = Z\,, \tag{15}$$

where ω is a C^1 function satisfying

$$\omega(R_1 = r_1) = 0, \qquad \omega(R_2 = r_2) = \lambda. \tag{16}$$

We easily check that the displacement boundary conditions (13) and $(14)_1$ are satisfied. By (15), the deformation gradient $\tilde{\mathbf{F}}$ is found to be

$$\tilde{\mathbf{F}} = \mathbf{e}_r\otimes\mathbf{E}_R + r\,\omega'\,\mathbf{e}_\theta\otimes\mathbf{E}_R + \mathbf{e}_\theta\otimes\mathbf{E}_\Theta + \mathbf{e}_z\otimes\mathbf{E}_Z\,, \tag{17}$$

where ω' is the derivative of ω with respect to r and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is the cylindrical orthonormal basis in the current configuration. In particular, (17) shows that (15) is a isochoric deformation, i.e.,

$$\det \tilde{\mathbf{F}} = \left(\det \tilde{\mathbf{B}}\right)^{1/2} = 1\,; \tag{18}$$

thus, by (6), (17) and (5), the traction boundary condition $(14)_2$ is trivially satisfied too.

It remains to study the equilibrium field equation (12). By following a result obtained in [5], it is possible to show that for the choice (4) of the strain energy function, the pure annular shear (15) is a *universal* deformation. In particular, by (6), (17), (5) and (18), the three ordinary differential equations for ω coming out from (12) reduce to one, which yields

$$\omega(\mathrm{r}) = -\mathrm{ArcCot}\left(\sqrt{4\,\mathrm{C}_1\,\mathrm{r}^4 - 1}\right) + \mathrm{C}_2\,; \tag{19}$$

then, by (16) we get

$$\omega(\mathrm{r}) = -\mathrm{ArcCot}\left(\sqrt{\frac{\mathrm{r}^4\left(\mathrm{r}_1^4 + \mathrm{r}_2^4 - 2\,\mathrm{r}_1^2\,\mathrm{r}_2^2\,\cos\lambda\right)}{\sin^2\lambda\,\mathrm{r}_1^4\,\mathrm{r}_2^4} - 1}\right) + \mathrm{ArcCot}\left(\frac{\mathrm{r}_2^2\,\cos\lambda - \mathrm{r}_1^2}{\sin\lambda\,\mathrm{r}_2^2}\right), \tag{20}$$

where, according to the boundary condition $(16)_2$, the loading parameter $\lambda > 0$ cannot be chosen arbitrarily, but it must be bounded above:

$$\lambda \in \left]0,\ \frac{\pi}{2} - \mathrm{ArcCot}\left(\sqrt{\mathrm{r}_2^4\,\mathrm{r}_1^{-4} - 1}\right)\right]. \tag{21}$$

In the following we assume that $\tilde{\mathbf{f}}$ given by (15), (20) and (21) is the "fundamental" equilibrium field; in the next Section, we analyze the possibility of bifurcations from $\tilde{\mathbf{f}}$ which satisfy the adjacent equilibrium equations based upon (12)-(14).

3 THE INCREMENTAL BOUNDARY-VALUE PROBLEM

We are now interested in additional non-zero solutions of (12)-(14), which bifurcate from $\tilde{\mathbf{f}}$ as the loading parameter λ increases. Such incremental solutions must satisfy the adjacent equilibrium equations and the incremental boundary conditions, here obtained by linearizing (12)-(14) around the fundamental deformation $\mathbf{x} = \tilde{\mathbf{f}}(\mathbf{X})$, which is conveniently assumed as the independent variable:

$$\mathrm{div}\left(\mathbb{A}\left(\tilde{\mathbf{F}}(\mathrm{r},\lambda)\right)[\mathrm{grad}\;\mathbf{u}]\right) = \mathbf{0} \qquad \text{in } \mathcal{C}, \tag{22}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial_1\mathcal{C}, \tag{23}$$

$$\mathbf{u}\cdot\mathbf{e}_Z = 0, \qquad \mathbb{A}\left(\tilde{\mathbf{F}}(\mathrm{r},\lambda)\right)[\mathrm{grad}\;\mathbf{u}]\mathbf{e}_Z \times \mathbf{e}_Z = \mathbf{0} \qquad \text{on } \partial_2\mathcal{C}. \tag{24}$$

In (22)-(24) $\mathbf{u}(\mathbf{x}) : \mathcal{C} \to \mathbb{R}^3$ represents an incremental displacement field, "div" and "grad" are the divergence and gradient operators with respect to **x**, respectively, and $\mathbb{A}$ is the fourth-order instantaneous elasticity tensor (see (18)):

$$\mathbb{A}\left(\tilde{\mathbf{F}}(\mathrm{r},\lambda)\right)[\mathbf{H}] := \mathrm{D}^2\mathrm{W}\left(\tilde{\mathbf{F}}(\mathrm{r},\lambda)\right)\left[\mathbf{H}\,\tilde{\mathbf{F}}(\mathrm{r},\lambda)\right]\tilde{\mathbf{F}}^{\mathrm{T}}(\mathrm{r},\lambda), \qquad \forall\,\mathbf{H}\in \mathrm{Lin}. \tag{25}$$

Notice that (22)-(24) are written in $\mathcal{C}$, $\partial_1\mathcal{C}$ and $\partial_2\mathcal{C}$ because the *pure annular shear* $\tilde{\mathbf{f}}$ maps $\mathcal{C}$, $\partial_1\mathcal{C}$ and $\partial_2\mathcal{C}$ into themselves.

For the constitutive class (4), it follows from (25), (17), (5) and (18) that

$$\begin{aligned}
\mathbb{A}\left(\tilde{\mathbf{F}}(\mathrm{r},\lambda)\right)[\operatorname{grad}\mathbf{u}] = {} & 2\alpha\left(4+(\mathrm{r}\,\omega')^2\right)^{-1/2}\Big[2\left(\tilde{\mathbf{B}}\cdot\operatorname{grad}\mathbf{u}\right)\tilde{\mathbf{B}}+\left(\tilde{\mathbf{B}}\cdot\mathbf{I}\right)\left((\operatorname{grad}\mathbf{u})\,\tilde{\mathbf{B}}\right) \\
& -\tilde{\mathbf{B}}\,(\operatorname{grad}\mathbf{u})^{\mathrm{T}}\,\tilde{\mathbf{B}}-(\operatorname{grad}\mathbf{u})\tilde{\mathbf{B}}^2-\tilde{\mathbf{B}}\,(\operatorname{grad}\mathbf{u})\tilde{\mathbf{B}}\Big] \\
& -2\alpha\left(4+(\mathrm{r}\,\omega')^2\right)^{-3/2}\left[\left(\tilde{\mathbf{B}}\cdot\mathbf{I}\right)\left(\tilde{\mathbf{B}}\cdot\operatorname{grad}\mathbf{u}\right)-\left(\tilde{\mathbf{B}}^2\cdot(\operatorname{grad}\mathbf{u})^{\mathrm{T}}\right)\right]\left[\left(\tilde{\mathbf{B}}\cdot\mathbf{I}\right)\tilde{\mathbf{B}}-\tilde{\mathbf{B}}^2\right] \\
& +2\psi'(1)\left[2\,\left(\mathbf{I}\cdot(\operatorname{grad}\mathbf{u})\right)\mathbf{I}-(\operatorname{grad}\mathbf{u})^{\mathrm{T}}\right]+4\psi''(1)\left[\left(\mathbf{I}\cdot(\operatorname{grad}\mathbf{u})\right)\mathbf{I}\right],
\end{aligned} \tag{26}$$

where

$$\tilde{\mathbf{B}} = \mathbf{e}_{\mathrm{r}}\otimes\mathbf{e}_{\mathrm{r}}+\mathrm{r}\,\omega'\,\left(\mathbf{e}_{\mathrm{r}}\otimes\mathbf{e}_{\theta}+\mathbf{e}_{\theta}\otimes\mathbf{e}_{\mathrm{r}}\right)+\left(1+(\mathrm{r}\,\omega')^2\right)\mathbf{e}_{\theta}\otimes\mathbf{e}_{\theta}+\mathbf{e}_{\mathrm{z}}\otimes\mathbf{e}_{\mathrm{z}}\,. \tag{27}$$

4 TAYLOR-LIKE BIFURCATIONS

Consider now a class of adjacent displacement fields which are reminiscent of certain instability patterns that are observed in the classical Taylor-Couette shear flow of viscous fluids. In particular, we focus our attention on the class $\mathcal{A}$ of *periodic displacements* $\mathbf{u}$ defined by

$$\mathcal{A} := \left\{\begin{array}{l} \mathbf{u}:\mathcal{C}\to\mathbb{R}^3 \;\mid\; \mathrm{u_r}=\mathrm{v}_1(\mathrm{r})\cos\kappa\,\mathrm{z},\;\; \mathrm{u}_\theta=\mathrm{v}_2(\mathrm{r})\cos\kappa\,\mathrm{z},\;\; \mathrm{u_z}=\mathrm{v}_3(\mathrm{r})\sin\kappa\,\mathrm{z}, \\ \kappa=\mathrm{n}\dfrac{\pi}{\mathrm{H}},\;\; \mathrm{n}=0,\,1,\,2,..,\quad \mathbf{u}(\mathrm{r}_1)=\mathbf{u}(\mathrm{r}_2)=\mathbf{0} \end{array}\right\}, \tag{28}$$

where $\mathrm{u_r}$, u_θ and $\mathrm{u_z}$ are the components of $\mathbf{u}$ in the coordinate system (r, θ, z), and v_1, v_2 and v_3 are smooth functions from $[\mathrm{r}_1,\,\mathrm{r}_2]$ to $\mathbb{R}$. Notice that (28) models the occurrence of an axially periodic cellular pattern in the gap between the inner and outer cylinders (n represents the number of possibly forming cells in the axial direction); in particular, inside each of the n possible forming cells, (28) describes a twist-like displacement perpendicular to the $\mathbf{e}_\theta$ - direction of primary annular shear, so that the vector lines of the incremental displacement $\mathbf{u}$ are similar to the streamlines of the twisting Taylor-like effects for fluids. Moreover, the choice (28) allows to reduce the set of partial differential adjacent equilibrium equations to three ordinary differential equations for the functions v_1 (r), v_2 (r) and v_3 (r). In the following, we will drop the dependence of v_1 (r), v_2 (r) and v_3 (r) on r.

The displacement boundary conditions (23) and $(24)_1$ and the traction boundary condition $(24)_2$

(whose checking has been omitted for the sake of brevity) are trivially satisfied by each $\mathbf{u} \in \mathcal{A}$. For what concerns the determination of the left hand side of (22), we now outline the main steps and give only the final result.

By (28), we may decompose grad $\mathbf{u}$ as follows:

$$\operatorname{grad} \mathbf{u} = \cos \kappa z \; \mathbf{L}(r) + \sin \kappa z \; \mathbf{J}(r), \tag{29}$$

where

$$\begin{aligned} &\mathbf{L}(r) = v_1' \mathbf{e}_r \otimes \mathbf{e}_r - v_2 \; r^{-1} \mathbf{e}_r \otimes \mathbf{e}_\theta + v_2' \; \mathbf{e}_\theta \otimes \mathbf{e}_r + v_1 \; r^{-1} \; \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \kappa \; v_3 \; \mathbf{e}_z \otimes \mathbf{e}_z, \\ &\mathbf{J}(r) = -\kappa \; v_1 \; \mathbf{e}_r \otimes \mathbf{e}_z - \kappa \; v_2 \; \mathbf{e}_\theta \otimes \mathbf{e}_z + v_3' \; \mathbf{e}_z \otimes \mathbf{e}_r, \end{aligned} \tag{30}$$

(in (30), and henceforth, a prime denotes the derivative with respect to r; moreover, in order to lighten the notation, we do not explicitly indicate the dependence on the parameter κ introduced in (28)). Thus, we have

$$\begin{aligned} \operatorname{div}\Big(\mathbb{A}\big(\tilde{\mathbf{F}}(r,\lambda)\big)[\operatorname{grad} \mathbf{u}]\Big) \; &= \cos \kappa z \; \Big(\operatorname{div}\Big(\mathbb{A}\big(\tilde{\mathbf{F}}(r,\lambda)\big)\big[\mathbf{L}(r)\big]\Big) + \kappa \; \mathbb{A}\big(\tilde{\mathbf{F}}(r,\lambda)\big)\big[\mathbf{J}(r)\big]\mathbf{e}_z\Big) \\ &+ \; \sin \kappa z \; \Big(\operatorname{div}\Big(\mathbb{A}\big(\tilde{\mathbf{F}}(r,\lambda)\big)\big[\mathbf{J}(r)\big]\Big) - \kappa \; \mathbb{A}\big(\tilde{\mathbf{F}}(r,\lambda)\big)\big[\mathbf{L}(r)\big]\mathbf{e}_z\Big), \end{aligned} \tag{31}$$

so that, after some calculation, equations (22), (23) lead to the following boundary value problem:

$$\begin{cases} \mathbf{P}(r,\lambda)\mathbf{v}'' + \big(\mathbf{P}'(r,\lambda) + \mathbf{R}(r,\lambda) - \mathbf{R}^T(r,\lambda)\big)\mathbf{v}' + \big(\mathbf{R}'(r,\lambda) + \mathbf{Q}(r,\lambda)\big)\mathbf{v} = \mathbf{0}, \quad r_1 < r < r_2 \\ \mathbf{v}(r_1) = \mathbf{v}(r_2) = \mathbf{0}, \end{cases} \tag{32}$$

where

$$\mathbf{v} := (v_1, \; v_2, \; v_3), \tag{33}$$

and $\mathbf{P}(r,\lambda)$, $\mathbf{R}(r,\lambda)$ and $\mathbf{Q}(r,\lambda)$ are matrices determined by the components the fourth-order tensor $\mathbb{A}$ calculated in the coordinate system (r, θ, z):

$$\mathbf{P}(r,\lambda) = \begin{pmatrix} r\,A_{rrrr} & r\,A_{rr\theta r} & 0 \\ r\,A_{\theta rrr} & r\,A_{\theta r\theta r} & 0 \\ 0 & 0 & r\,A_{zrzr} \end{pmatrix}, \tag{34}$$

$$
\mathbf{R}(r,\lambda)=\begin{pmatrix} A_{rr\theta\theta} & -A_{rrr\theta} & \kappa\, r\, A_{rrzz} \\ A_{\theta r\theta\theta} & -A_{\theta rr\theta} & \kappa\, r\, A_{\theta rzz} \\ -\kappa\, r\, A_{zrrz} & -\kappa\, r\, A_{zr\theta z} & 0 \end{pmatrix}, \tag{35}
$$

$$
\mathbf{Q}(r,\lambda)=\begin{pmatrix} -\frac{1}{r}\, A_{\theta\theta\theta\theta}-\kappa^2 r\, A_{rzrz} & \frac{1}{r}\, A_{\theta\theta r\theta}-\kappa^2 r\, A_{rz\theta z} & -\kappa\, A_{\theta\theta zz} \\ \frac{1}{r}\, A_{r\theta\theta\theta}-\kappa^2 r\, A_{\theta zrz} & -\frac{1}{r}\, A_{r\theta r\theta}-\kappa^2 r\, A_{\theta z\theta z} & \kappa\, A_{r\theta zz} \\ -\kappa\, A_{zz\theta\theta} & \kappa\, A_{zzr\theta} & -\kappa^2 r\, A_{zzzz} \end{pmatrix}. \tag{36}
$$

Here, we do not report the explicit expression of the components of $\mathbb{A}$, which may be obtained by (26), (27) and (20). We only observe that $\mathbf{P}(r,\lambda)$ and $\mathbf{Q}(r,\lambda)$ are symmetric, since $\mathbb{A}$ is symmetric; moreover, it is easy to check that $\mathbf{P}(r,\lambda)$ is always invertible.

This property of $\mathbf{P}(r,\lambda)$ plays a crucial role in the analysis of the solutions of the three homogeneous linear second order ordinary differential equation $(32)_1$. Indeed, a common practice to examine the existence of solutions of problems of the form (32) is that of transforming the set of three linear second order ordinary differential equations into a system of six linear first order ordinary differential equations, for which well known existence theorems and procedures for constructing the solutions are available in the literature.

Following this approach, we first rewrite (32) as below:

$$
\begin{cases} \mathbf{v}''=-\mathbf{T}\mathbf{v}-\mathbf{K}\mathbf{v}', \quad r_1<r<r_2 \\ \mathbf{v}(r_1)=\mathbf{v}(r_2)=\mathbf{0}, \end{cases} \tag{37}
$$

where the 3 x 3 matrices

$$
\begin{cases} \mathbf{T}:=\mathbf{P}^{-1}(r,\lambda)\big(\mathbf{R}'(r,\lambda)+\mathbf{Q}(r,\lambda)\big) \\ \mathbf{K}:=\mathbf{P}^{-1}(r,\lambda)\big(\mathbf{P}'(r,\lambda)+\mathbf{R}(r,\lambda)-\mathbf{R}^{T}(r,\lambda)\big) \end{cases} \tag{38}
$$

depend continuously on r, λ and κ.

We consider now λ and κ fixed. By introducing

$$
\mathbf{y}:=\begin{Bmatrix} \mathbf{v} \\ \mathbf{v}' \end{Bmatrix}, \qquad \mathbf{A}(r):=\begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{T}(r) & -\mathbf{K}(r) \end{pmatrix}, \tag{39}
$$

where $\mathbf{O}$ is the second order null tensor, we immediately check that $(37)_1$ is equivalent to the following set of six linear ordinary differential equations:

$$\mathbf{y}' = \mathbf{A}(\mathrm{r})\mathbf{y}, \quad \text{for } \mathrm{r}_1 < \mathrm{r} < \mathrm{r}_2. \tag{40}$$

For what concerns the boundary conditions, we introduce the following 6 x 6 matrices:

$$\mathbf{M} := \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{N} := \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{41}$$

so that the six boundary conditions $(37)_2$ are equivalent to

$$\mathbf{My}(\mathrm{r}_1) - \mathbf{Ny}(\mathrm{r}_2) = \mathbf{0}. \tag{42}$$

Finally, we may rewrite (37) as follows:

$$\begin{cases} \mathbf{y}' = \mathbf{A}(\mathrm{r})\mathbf{y}, \quad \text{for } \mathrm{r}_1 < \mathrm{r} < \mathrm{r}_2 \\ \mathbf{My}(\mathrm{r}_1) - \mathbf{Ny}(\mathrm{r}_2) = \mathbf{0}. \end{cases} \tag{43}$$

Let

$$\mathbf{Y}(\mathrm{r}) = \begin{pmatrix} \mathbf{U}_1(\mathrm{r}) & \mathbf{U}_2(\mathrm{r}) \\ \mathbf{U}_1'(\mathrm{r}) & \mathbf{U}_2'(\mathrm{r}) \end{pmatrix} \tag{44}$$

be the transition matrix for (43), i.e. a fundamental matrix for $(43)_1$ such that

$$\mathbf{y}(\mathrm{r}) = \mathbf{Y}(\mathrm{r})\mathbf{y}(\mathrm{r}_1), \qquad \mathbf{Y}(\mathrm{r}_1) = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}. \tag{45}$$

Recall that a fundamental matrix satisfies

$$\det \mathbf{Y}(\mathrm{r}) \neq 0, \qquad \mathbf{Y}' = \mathbf{A}(\mathrm{r})\mathbf{Y}, \qquad \mathrm{r}_1 < \mathrm{r} < \mathrm{r}_2. \tag{46}$$

Now, by using a result of ODE theory (see [4, §12, Lemma 1.1]), we infer that, in the current case, necessary and sufficient condition for (43) to have nontrivial solutions is that

$$\det\left[\mathbf{MY}(\mathrm{r}_1) - \mathbf{NY}(\mathrm{r}_2)\right] = 0, \tag{47}$$

which, by (41), (44) and (45), is equivalent to

$$\det \mathbf{U}_2(\mathrm{r}_2) = 0. \tag{48}$$

Notice that the determination of the matrix $\mathbf{U}_2(\mathrm{r})$ also allows for explicit representation of nontrivial solutions; indeed, in view of $(45)_1$, $(39)_1$, (44) and $(37)_2$, we obtain

$$\mathbf{v}(\mathrm{r}) = \mathbf{U}_2(\mathrm{r})\,\mathbf{v}'(\mathrm{r}_1). \tag{49}$$

In order to determine $\mathbf{U}_2(\mathrm{r})$, we may solve the Cauchy problem

$$\begin{cases} \mathbf{Y}'(\mathrm{r}) = \mathbf{A}(\mathrm{r})\,\mathbf{Y}(\mathrm{r}), & \text{for } \mathrm{r}_1 < \mathrm{r} < \mathrm{r}_2 \\ \mathbf{Y}(\mathrm{r}_1) = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} & \end{cases}, \tag{50}$$

whose unique solution is

$$\mathbf{Y}(\mathrm{r}) = \mathbf{D}(\mathrm{r})\,\mathbf{Y}(\mathrm{r}_1), \tag{51}$$

where

$$\mathbf{D}(\mathrm{r}) = \begin{pmatrix} \mathbf{D}_1(\mathrm{r}) & \mathbf{D}_2(\mathrm{r}) \\ \mathbf{D}_3(\mathrm{r}) & \mathbf{D}_4(\mathrm{r}) \end{pmatrix} = \exp\left[\int_{\mathrm{r}_1}^{\mathrm{r}} \mathbf{A}(\rho)\ \mathrm{d}\rho\right]. \tag{52}$$

Thus, in view of (44), $(50)_2$ and (51), we finally have

$$\mathbf{U}_2(\mathrm{r}) = \mathbf{D}_2(\mathrm{r}). \tag{53}$$

5 A NUMERICAL EXAMPLE

We now have enough tools for investigating on the possibility of twist-like bifurcating solutions of the form (28), which may occur during an annular shear loading process for an elastic tube belonging to the constitutive class (4).

To this aim, given the geometry of the tube (i.e., the inner an outer radii R_1, R_2 and the height H) and the material parameters α and $\psi''(1)$, we found efficient the numerical procedure described below:

- fix the number n of possibly forming cells in the axial direction, which is equivalent to fixing κ (see (28));
- fix the value of the loading parameter λ;
- compute the matrices (34), (35) and (36);
- determine the 6 x 6 matrix **A**(r) defined by $(39)_2$;
- determine the 6 x 6 matrix **D**(r) defined by (52);
- determine the 3 x 3 matrix $\mathbf{U}_2$(r) given by (53);
- compute the determinant of $\mathbf{U}_2(r_2)$, i.e. the left hand side term of (48).

For fixed n, this procedure should be repeated by increasing at each step the loading parameter λ starting from 0, until the condition (48) is satisfied for the first time: we define critical value λ_{cr} of the shearing strain the first value of λ which satisfies the condition (48). Observe that λ_{cr}, if exists, must be compatible to condition (21).

We now show explicit calculations by assigning the material parameter $\alpha = \psi''(1) = 1$ MPa, and by considering a tube of dimensions $H = R_1 = 1$ m; in particular, we seek bifurcating solutions with n = 2 cells in the axial direction, and varying the ratio R_2/R_1 in the range [1.1, 2.0]. In figure 1 we report the critical shearing strain λ_{cr} vs. the ratio R_2/R_1.

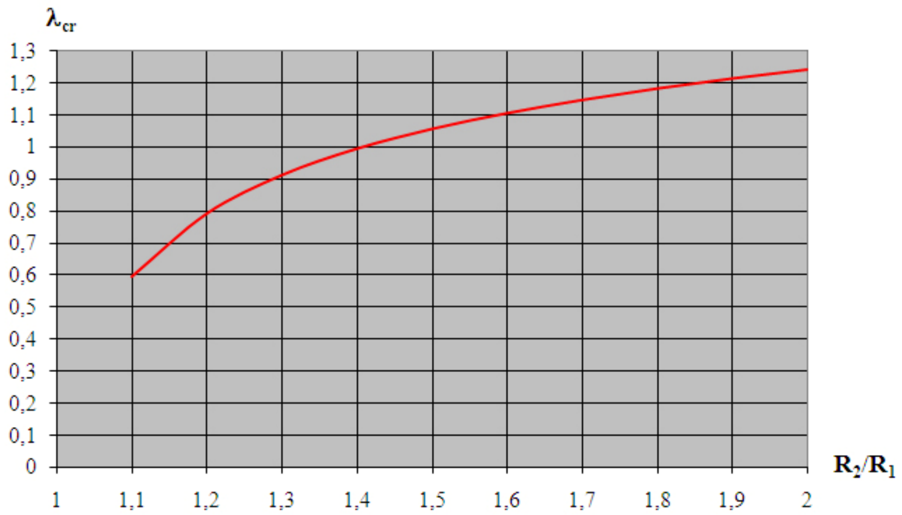

Figure 1: λ_{cr} vs. R_2/R_1, for $\alpha = \psi''(1) = 1$ MPa, $H = R_1 = 1$ m and n = 2.

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