Influence of wrinkling in the structural response of light membranes
Derivation of the incremental equilibrium operator

Massimo Cuomo[1], Leopoldo Greco[1]
[1] *Faculty of Engineering, University of Catania, Italy*
E-mail: mcuomo@dica.unict.it
E-mail: leopoldo.greco@virgilio.it

Keywords: Kirchhoff-Love thin shells, tangent operator, buckling, wrinkling.

SUMMARY. The formulation of the equilibrium equations for Koiter thin shells is revised, both in weak and strong form, in order to rationally obtain a linearized incremental formulation useful for stability analysis both in membrane and bending dominated conditions. The incremental equilibrium operator is obtained considering an arbitrary perturbation process.

1 INTRODUCTION

Wrinkling is a phenomenon that occurs in membrane dominated shells. Therefore it can be predicted performing a non linear geometric analysis or a perturbation analysis. In both cases, it is needed the complete tangent equilibrium operator of the shell, including its bending behavior, that depends strongly on the local curvature.

Linearized stability analysis is often performed employing Von Karman equations. They, however, have been obtained considering for the geometric perturbation an approximated form, that neglects membrane deformations and drilling rotations, [1]. Further expressions of the incremental (tangent) equilibrium equations including drilling rotations have been proposed by Simo [2, 3, 4]. In the literature are also available several kinds of approximations for the behavior of the shell, according to its curvature, sometimes defined shallow or quasi shallow approximations, (see Bazant [5] for a nice discussion). The operator proposed by Timoshenko [6] for linearized stability analysis falls within these categories.

In the present work we derive the equilibrium equations of the shell (under Kirchhoff Love hypotheses) and the complete form of the linearized incremental equilibrium equations without restrictions on the form of the perturbation. While the results for the spatial form of the equilibrium equations are well known, [7, 8], the exact form of the incremental operator was previously proposed by Pietrazkiewicz in 1983 [9], who however, introduced some approximations on the Boundary Conditions. We employ in the derivation a rational procedure that has allowed also to obtain the shallow and quasi-shallow approximations for the same operator. The equation so obtained can be effectively used in incremental non linear analysis of the shell, and in the stability analysis.

2 KINEMATICS

2.1 *Lagrangian configuration of the shell*

Let $\boldsymbol{\vartheta} \in \mathcal{M} \subset \mathbb{R}^2$ be the domain of definition for the reference collocation of the shell and let $\vartheta^3 \in [-h, +h]$ be the quota of the generic layer. The Lagrangian configuration of the shell is defined as

$$\overset{*}{\boldsymbol{P}}(\vartheta^\alpha, \vartheta^3) = \boldsymbol{P}(\vartheta^\alpha) + \vartheta^3 \boldsymbol{N}(\vartheta^\alpha), \quad \alpha = 1, 2. \tag{1}$$

where $\boldsymbol{P}(\vartheta^\alpha) : \mathcal{M} \to \mathcal{S}_o \subset \mathbb{R}^3$ is the position of the reference surface, $\boldsymbol{N}(\vartheta^\alpha) : \mathcal{M} \to \mathcal{R}^3$ is the field of the unit normals to the reference surface.

2.2 *Lagrangian Metrics*

Introducing the base vectors $\boldsymbol{G}_j = \partial_{\theta^j} P^M \mathbf{E}_M$ of the tangent space $T_{\boldsymbol{P}}\mathcal{S}_o$ of the reference surface we will denote the projection of the metric $\boldsymbol{G}$ on the tangent space to the reference surface as $\boldsymbol{G}_{\|} \in L(T_{\boldsymbol{P}}\mathcal{S}_o, T_{\boldsymbol{P}}\mathcal{S}_o)$, with $\boldsymbol{G}_{\|} = G_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta$. The completion of the metric $\boldsymbol{G}_{\|}$ in the space $T_{\boldsymbol{P}}\mathcal{B}_o$ is

$$\boldsymbol{G} = \boldsymbol{G}_{\|} + \boldsymbol{N} \otimes \boldsymbol{N} = G_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta + \boldsymbol{N} \otimes \boldsymbol{N}. \tag{2}$$

The push-forward operator along the normal $\boldsymbol{N}$ is given by the gradient

$$\boldsymbol{Z} = grad(\overset{*}{\boldsymbol{P}}) = \boldsymbol{G}_{\|} + \boldsymbol{N} \otimes \boldsymbol{N} + \vartheta^3 grad_{\|}(\boldsymbol{N}) \otimes \boldsymbol{G}^\alpha \tag{3}$$

Since $\boldsymbol{N} \cdot \boldsymbol{G}_\alpha = 0$ differentiating one has $\partial_\beta(\boldsymbol{N}) \cdot \boldsymbol{G}^\alpha = -\boldsymbol{N} \cdot \partial_{\alpha\beta}\boldsymbol{R}$, that is $grad_{\|}(\boldsymbol{N}) = -\boldsymbol{B}$ the curvature tensor, with $B_{\alpha\beta} = -\boldsymbol{N} \cdot \partial_\beta \boldsymbol{G}_\alpha$. Using (3), the pull-back of the metric of the ϑ^3-lamina on the reference surface is

$$\phi^*_{\boldsymbol{N}}(\overset{*}{\boldsymbol{G}}) = \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Z} = \boldsymbol{G} + \vartheta^3(\boldsymbol{B}^{\mathrm{T}} + \boldsymbol{B}) + (\vartheta^3)^2\boldsymbol{B}^T\boldsymbol{B}. \tag{4}$$

For thin shells the last term can be disregarded.

2.3 *Current configuration of the shell*

The position of the shell at the generic time *t* is represented by

$$\overset{*}{\boldsymbol{p}}(\vartheta^\alpha, \vartheta^3) = \boldsymbol{P}(\vartheta^\alpha) + \boldsymbol{u}(\vartheta^\alpha) + \vartheta^3(\boldsymbol{N}(\vartheta^\alpha) + \boldsymbol{\omega}(\vartheta^\alpha)) \tag{5}$$

where $\boldsymbol{u} : \mathcal{S}_t \to T_{\boldsymbol{P}}\mathcal{B}_o$ is the displacement field of the reference surface that yields its position in the current configuration as $\boldsymbol{p} = \boldsymbol{P} + \boldsymbol{u}$, and $\boldsymbol{\omega}$ is the displacement of the tip of the normal vector $\boldsymbol{N}$, assumed to rotate during deformation. The base vectors on the reference surface transform as

$$\boldsymbol{c}_\alpha = \boldsymbol{G}_\alpha + \boldsymbol{u}_{|\alpha}, \quad \overset{*}{\boldsymbol{c}}_\alpha = \boldsymbol{c}_\alpha + \vartheta^3\boldsymbol{n}_{|\alpha}, \quad \overset{*}{\boldsymbol{n}} = \boldsymbol{n} \tag{6}$$

In the work, since we analyze thin membranes, it is used Kirchhoff-Love hypothesis $\boldsymbol{\omega} = \boldsymbol{n} - \boldsymbol{N}$, where $\boldsymbol{n} = \frac{\boldsymbol{c}_1 \times \boldsymbol{c}_2}{\sqrt{c_{\|}}}$, so that (5) becomes

$$\overset{*}{\boldsymbol{p}}(\vartheta^\alpha, \vartheta^3) = \boldsymbol{P}(\vartheta^\alpha) + \boldsymbol{u}(\vartheta^\alpha) + \vartheta^3\boldsymbol{n}(\vartheta^\alpha). \tag{7}$$

2.4 *Current metric and measure of deformation*

The pull-back of the current metric of the space $T_{\overset{*}{\boldsymbol{p}}}\mathcal{B}_t$ on the space $T_{\boldsymbol{P}}\mathcal{B}_o$ is indicated by $\overset{*}{\boldsymbol{c}}$ and is defined by $\overset{*}{\boldsymbol{c}} := \boldsymbol{F}^T\boldsymbol{F} = \partial_{\boldsymbol{P}}(\overset{*}{\boldsymbol{p}})^T\partial_{\boldsymbol{P}}(\overset{*}{\boldsymbol{p}})$, where $\partial_{\boldsymbol{P}}(\overset{*}{\boldsymbol{p}}) : T_{\boldsymbol{P}}\mathcal{B}_o \to T_{\overset{*}{\boldsymbol{p}}}\mathcal{B}_t$. With reference to fig. 1 it is possible to represent the gradient of deformation as $\boldsymbol{F} := \partial_{\boldsymbol{P}}(\overset{*}{\boldsymbol{p}}) = (\boldsymbol{c}_\alpha + \vartheta^3\boldsymbol{n}_{|\alpha}) \otimes \boldsymbol{G}^\alpha + \boldsymbol{n} \otimes \boldsymbol{N} = \boldsymbol{z}\boldsymbol{F}_o = \overset{*}{\boldsymbol{F}}\boldsymbol{Z}$, where $\boldsymbol{z} = \overset{*}{\boldsymbol{c}}_\alpha \otimes \boldsymbol{c}^\alpha + \boldsymbol{n} \otimes \boldsymbol{n}$ is the push-forward operator along $\boldsymbol{n}$. The pull-back of the metric of the generic lamina in the deformed configuration is then

$$\phi^*(\overset{*}{\boldsymbol{c}}) = [\boldsymbol{c}_\alpha \cdot \boldsymbol{c}_\beta + \vartheta^3(\boldsymbol{n}_{|\alpha} \cdot \boldsymbol{c}_\beta + \boldsymbol{n}_{|\beta} \cdot \boldsymbol{c}_\alpha) + (\vartheta^3)^2\boldsymbol{n}_{|\alpha} \cdot \boldsymbol{n}_{|\beta}]\boldsymbol{G}^\beta \otimes \boldsymbol{G}^\alpha + \boldsymbol{N} \otimes \boldsymbol{N}. \tag{8}$$

In equation (8) the linear term in ϑ^3 is the pull-back of $2\,\mathrm{sym}[\,\boldsymbol{b}\,]$ on $T_{\boldsymbol{P}}\mathcal{B}_o$. Disregarding the quadratic term, the Cauchy-Green deformation tensor for a Kirchhoff-Love shell is

$$2\overset{*}{\mathbf{E}} \approx (\phi^*(\overset{*}{\boldsymbol{c}}_{\|}) - \boldsymbol{G}_{\|}) - \vartheta^3 2\mathrm{sym}((\phi^*(\boldsymbol{b}) - (\boldsymbol{B})). \tag{9}$$

$$
\begin{tikzcd}
T_{\overset{*}{\boldsymbol{P}}}\mathcal{B}_0 \arrow[r, "\overset{*}{\boldsymbol{F}}"] & T_{\overset{*}{\boldsymbol{p}}}\mathcal{B}_t \\
T_{\boldsymbol{P}}\mathcal{B}_0 \arrow[u, "\boldsymbol{Z}"] \arrow[r, "\boldsymbol{F}_o"'] \arrow[ur, "\boldsymbol{F}"] & T_{\boldsymbol{p}}\mathcal{B}_t \arrow[u, "\boldsymbol{z}"']
\end{tikzcd}
$$

Figure 1: Diagram of the tangent spaces and push-forward operators.

2.5 *Up-dated Lagrangian form of the tangent kinematic operators*

The velocity field is $\overset{*}{\boldsymbol{v}} = \boldsymbol{v} + \vartheta^3 \dot{\boldsymbol{n}}$ where $\boldsymbol{v}$ is the velocity of the reference current surface and $\dot{\boldsymbol{n}}$ is the velocity of rotation of the normal vector. The spatial velocity gradient $\overset{*}{\boldsymbol{l}} : T_{\overset{*}{\boldsymbol{p}}}\mathcal{B}_t \to T_{\overset{*}{\boldsymbol{p}}}\mathcal{B}_t$ is

$$
\overset{*}{\boldsymbol{l}} := \partial_{\overset{*}{\boldsymbol{p}}} \overset{*}{\boldsymbol{v}} = \dot{\overset{*}{\boldsymbol{c}}}_\alpha \otimes \overset{*}{\boldsymbol{c}}^\alpha + \dot{\boldsymbol{n}} \otimes \boldsymbol{n} = (\boldsymbol{v} + \vartheta^3 \dot{\boldsymbol{n}})_{|\alpha} \otimes \overset{*}{\boldsymbol{c}}^\alpha + \dot{\boldsymbol{n}} \otimes \boldsymbol{n}. \tag{10}
$$

The velocity of deformation is the symmetric part of the pull-back of $\overset{*}{\boldsymbol{l}}$ on $T_{\boldsymbol{p}}\mathcal{B}_t$ given by $\boldsymbol{l} = \boldsymbol{z}^T \overset{*}{\boldsymbol{l}} \boldsymbol{z}$

$$
\boldsymbol{d} = \mathrm{sym}[(\dot{\overset{*}{\boldsymbol{c}}}_\alpha \cdot \overset{*}{\boldsymbol{c}}_\beta) \boldsymbol{c}^\beta \otimes \boldsymbol{c}^\alpha] \tag{11}
$$

From (11) it is clear that the only non zero terms of the velocity of deformation gradient are in the tangent direction. Linearizing equation (11) in the ϑ^3-direction, for thin shells it is obtained

$$
\{\boldsymbol{d}\}^{lin_{\vartheta^3}} = \boldsymbol{d}_o + \vartheta^3 \dot{\boldsymbol{\chi}}\,. \tag{12}
$$

In (12) have been defined two tangent kinematic operators, the membrane velocity of deformation and the velocity of curvature respectively

$$
\boldsymbol{d}_o := \mathrm{sym}[grad_{\|}(\boldsymbol{v})]^{\|} = \mathrm{sym}[(v_{\beta\|\alpha} - v_3 b_{\beta\alpha}) \boldsymbol{c}^\beta \otimes \boldsymbol{c}^\alpha]. \tag{13}
$$

and

$$
\begin{aligned}
\dot{\boldsymbol{\chi}} &:= -\mathrm{sym}[\boldsymbol{n} \cdot grad_{\|}^2(\boldsymbol{v})] = -\mathrm{sym}[\boldsymbol{n} \cdot (\boldsymbol{v}_{|\beta|\alpha} + \boldsymbol{v}_{|(\boldsymbol{c}_{\beta|\alpha})^{\|}}) \otimes \boldsymbol{c}^\beta \otimes \boldsymbol{c}^\alpha] \\
&= -\mathrm{sym}[((v_3)_{\|\beta\alpha} + v_{\rho\|\beta} b^\rho{}_\alpha + v_{\rho\|\alpha} b^\rho{}_\beta + v_\rho b^\rho{}_{\beta\|\alpha} - v_3 b_{\rho\alpha} b^\rho{}_\beta) \boldsymbol{c}^\beta \otimes \boldsymbol{c}^\alpha].
\end{aligned} \tag{14}
$$

The formulas presented do not carry along any restrictive hypothesis other then those related to the first Kirchhoff-Love approximation. However, simplified formulations have been proposed and are commonly used in the literature, commonly referred to as shallow shell or *quasi*-shallow shell approximations. It is a remarkable result that they can be derived directly from the general theory outlined above, simply modifying the definition of the velocity of curvature with appropriate assumptions, leaving unaltered the deformation tensor and the membrane part of the velocity of deformation.

For the so called theory of *quasi*-shallow shells [5, 6] in the expression (14) it is retained the second geometric term, while the first term containing the second derivative of the velocity is approximated as $\boldsymbol{n} \cdot \boldsymbol{v}_{|\beta|\alpha} \approx (\boldsymbol{n} \cdot \boldsymbol{v}_{|\alpha})_{|\beta}$ (the normal component of the second gradient is substituted by the derivative of the normal component). That is, it is disregarded the term $\mathrm{sym}[(\boldsymbol{n}_{|\alpha} \cdot \boldsymbol{v}_{|\beta})\boldsymbol{c}^{\beta} \otimes \boldsymbol{c}^{\alpha}]$, that represents the influence of the membrane stretching on the velocity of curvature:

$$\begin{aligned}\dot{\boldsymbol{\chi}}_{QSS} &:= \mathrm{sym}[grad_{\|}(\dot{\boldsymbol{n}})]^{\|} = -\mathrm{sym}[(v^{3}{}_{|\beta})_{\|\alpha}\boldsymbol{c}^{\beta} \otimes \boldsymbol{c}^{\alpha}] \\ &= -\mathrm{sym}[((v_3)_{\|\beta\alpha} + v_{\rho\|\alpha}b^{\rho}{}_{\beta} + v_{\rho}b^{\rho}_{\beta\|\alpha})\boldsymbol{c}^{\beta} \otimes \boldsymbol{c}^{\alpha}].\end{aligned} \tag{15}$$

The shallow shell assumption is formally equivalent to approximate also the normal component of the first derivative of the tangent velocity. In addition to the term containing the membrane stretching, then, it is also disregarded the term $(\boldsymbol{n}_{|\beta|\alpha} \cdot \boldsymbol{v})\boldsymbol{c}^{\beta} \otimes \boldsymbol{c}^{\alpha}$. Therefore one has

$$\dot{\boldsymbol{\chi}}_{SS} := -\mathrm{sym}[grad^{2}_{\|}(\boldsymbol{n} \cdot \boldsymbol{v})] = -\mathrm{sym}[(v_3)_{\|\beta\alpha}\boldsymbol{c}^{\beta} \otimes \boldsymbol{c}^{\alpha}]. \tag{16}$$

3 EQUILIBRIUM EQUATIONS

3.1 *Spatial form of the equilibrium equations*

Denoting with $\overset{*}{\boldsymbol{\sigma}}$ the Cauchy stress tensor, the internal virtual work for KL-shells is

$$\mathcal{P}_{int} = \int_{\phi_t(\mathcal{B})} d\mathcal{B}_t(\overset{*}{\boldsymbol{\sigma}} : \overset{*}{\boldsymbol{d}}) = \int_{\phi_t(\mathcal{B})} d\mathcal{B}_t(\boldsymbol{\sigma} : \boldsymbol{d}) \tag{17}$$

where the stress tensor $\boldsymbol{\sigma}$ is the pull-back of the Cauchy stress onto the space $T_{\boldsymbol{p}}\mathcal{B}_t$ defined by $\boldsymbol{\sigma} = \boldsymbol{z}^{-1}\overset{*}{\boldsymbol{\sigma}}\boldsymbol{z}^{-\mathrm{T}}$. For thin shells it is possible to identify $\boldsymbol{\sigma}$ and $\overset{*}{\boldsymbol{\sigma}}$. Splitting the integral and indicating with $\pi(\vartheta^3)$ the jacobian of the transformation along the thickness one has

$$\mathcal{P}_{int} = \int_{\phi_t(\mathcal{B})} d\overset{*}{\mathcal{B}}_t(\overset{*}{\boldsymbol{\sigma}} : \overset{*}{\boldsymbol{d}}) = \int_{\mathcal{S}_t} d\mathcal{S}_t \left(\int_{-h}^{+h} \pi(\vartheta^3)\boldsymbol{\sigma} : \left(\boldsymbol{d}_o + \vartheta^3\dot{\boldsymbol{\chi}}\right) d\vartheta^3 \right) = \int_{\mathcal{S}_t} d\mathcal{S}_t(\boldsymbol{n} : \boldsymbol{d}_o + \boldsymbol{m} : \dot{\boldsymbol{\chi}}). \tag{18}$$

where $\boldsymbol{n} = \int_{-h}^{+h} d\vartheta^3\pi(\vartheta^3)(\boldsymbol{\sigma})$ and $\boldsymbol{m} = \int_{-h}^{+h} d\vartheta^3\pi(\vartheta^3)(\vartheta^3\boldsymbol{\sigma})$, both stress tensors are symmetric.

Let the vector traction on the boundary be $\boldsymbol{\sigma}_{\partial}\tilde{\mathbf{n}} = \sigma^{\alpha\mathrm{n}}_{\partial}\boldsymbol{c}_{\alpha} + \sigma^{3\mathrm{n}}_{\partial}\boldsymbol{n}$. The linearized component of the *Kirchhoff-Love* velocity field along the ϑ^3-direction is

$$\boldsymbol{c}^{\alpha}\{\overset{*}{\boldsymbol{v}} \cdot \overset{*}{\boldsymbol{c}}_{\alpha}\}^{lin_{\vartheta^3}} + v_3\boldsymbol{n} = \boldsymbol{c}^{\alpha}\{(\boldsymbol{v} + \vartheta^3\dot{\boldsymbol{n}})(\boldsymbol{c}_{\alpha} + \underline{\underline{\vartheta^3\boldsymbol{n}_{|\alpha}}})\}^{lin_{\vartheta^3}} + v_3\boldsymbol{n} = \boldsymbol{v} - \vartheta^3(v_{3,\alpha}\boldsymbol{c}^{\alpha} + \underline{2\boldsymbol{v} \cdot \boldsymbol{b}}). \tag{19}$$

Then the external power is

$$\begin{aligned}\mathcal{P}_{ext} &= \int_{\mathcal{S}_t} d\mathcal{S}_t(\boldsymbol{q}^{\|} \cdot \boldsymbol{v}^{\|} + q^3v_3) + \\ &+ \int_{\partial\mathcal{S}_{t,i}} ds^2 \int_{-h}^{+h} d\vartheta^3 \left([(v_{\alpha} - \vartheta^3(v_{3,\alpha} + 2v_{\lambda}b^{\lambda}{}_{\alpha}))\boldsymbol{c}^{\alpha} + v_3\boldsymbol{n}] \cdot [\boldsymbol{\sigma}\tilde{\boldsymbol{n}}] \right) \\ &= \int_{\mathcal{S}_t} d\mathcal{S}_t(\boldsymbol{q}^{\|} \cdot \boldsymbol{v}^{\|} + q^3v_3) + \sum_i (v_3)_iR_i + \\ &+ \int_{\partial\mathcal{S}_{t,i}} ds^2 \left(\boldsymbol{v}^{\|} \cdot [\boldsymbol{f}^{\|}_{\partial} - \underline{2\boldsymbol{b}^{\|}(\boldsymbol{m}_{\partial}\tilde{\boldsymbol{n}})}] + v_3f^3_{\partial} - v_{3,\tilde{n}}m^{\tilde{n}\tilde{n}}_{\partial} + v_3(m^{\tilde{s}\tilde{n}}_{\partial})_{,\tilde{s}} \right) - \left[m^{\tilde{s}\tilde{n}}_{\partial}v_3\right]^{1}_{0}(\mathrm{L}_i).\end{aligned} \tag{20}$$

where with $\boldsymbol{q}^{\,\|}$ and q^3 we intend the external surface forces, with $\boldsymbol{f}^{\|}_{\partial} = (\boldsymbol{n}_{\partial}\tilde{\boldsymbol{n}})^{\|} = \tilde{\boldsymbol{n}} \int_{-h}^{+h} d\vartheta^3[\tilde{\boldsymbol{n}} \cdot \boldsymbol{\sigma}_{\partial} \cdot \tilde{\boldsymbol{n}}] + \tilde{\boldsymbol{s}} \int_{-h}^{+h} d\vartheta^3[\tilde{\boldsymbol{s}} \cdot \boldsymbol{\sigma}_{\partial} \cdot \tilde{\boldsymbol{n}}]$ the membrane boundary traction and with $f^3_{\partial} = \int_{-h}^{+h} d\vartheta^3[\boldsymbol{n} \cdot \boldsymbol{\sigma}_{\partial} \cdot \tilde{\boldsymbol{n}}]$

the boundary shear force; the boundary normal bending moment is $m_\partial^{\tilde{n}\tilde{n}} = \int_{-h}^{+h} d\vartheta^3[\vartheta^3 \tilde{\boldsymbol{n}} \cdot \boldsymbol{\sigma}_\partial \cdot \tilde{\boldsymbol{n}}]$ and the boundary twisting moment is $m_\partial^{\tilde{s}\tilde{n}} = \int_{-h}^{+h} d\vartheta^3[\vartheta^3 \tilde{\boldsymbol{s}} \cdot \boldsymbol{\sigma}_\partial \cdot \tilde{\boldsymbol{n}}]$; $R_i = (\boldsymbol{R}_i \cdot \boldsymbol{n})$ are the normal component of the point-force applied to vertex of the non-regular boundary.

Equating internal and external power the strong equilibrium equations in $\mathcal{S}_t$ are derived

$$\begin{aligned} -div_{\|}(\boldsymbol{n} - \underline{\boldsymbol{bm}}) + \underline{\underline{\boldsymbol{b}\, div_{\|}(\boldsymbol{m})}} &= \boldsymbol{q}^{\|} \\ -div_{\|}(div_{\|}(\boldsymbol{m})) - (\boldsymbol{n} - \underline{\boldsymbol{bm}}) : \boldsymbol{b} &= q^3 \end{aligned} \tag{21}$$

where the operator $div_{\|}(...) : L(T_{\boldsymbol{p}}\mathcal{S}_t, T_{\boldsymbol{p}}\mathcal{S}_t) \to T_{\boldsymbol{p}}\mathcal{S}_t$ is the restriction of the divergence operator to the current reference surface $\mathcal{S}_t$, $\boldsymbol{b}$ is the curvature tensor of this surface, $\tilde{\boldsymbol{n}} := \partial_{s^1}\boldsymbol{p} \in T_{\boldsymbol{p}}\mathcal{S}_t$ and $\tilde{\boldsymbol{s}} := \partial_{s^2}\boldsymbol{p} \in T_{\boldsymbol{p}}\mathcal{S}_t$ are, respectively, the boundary's normal and tangent vectors of the line-boundary $\partial\mathcal{S}_t$, i.e. $\boldsymbol{n} = \tilde{\boldsymbol{n}} \times \tilde{\boldsymbol{s}}$, and i the number of vertices p_i, of edges the L_i.

The BC's on the edges $\partial\mathcal{S}_{t,i}$ and vertices p_i are

$$\begin{aligned} (\boldsymbol{n} - \underline{2\boldsymbol{bm}})\tilde{\boldsymbol{n}} &= \boldsymbol{f}_\partial^{\|} - \underline{2\boldsymbol{b}^{\|}(\boldsymbol{m}_\partial\tilde{\boldsymbol{n}})} \quad or \quad \boldsymbol{v}^{\|} = \boldsymbol{0} \\ -m^{\tilde{n}\tilde{n}} &= -m_\partial^{\tilde{n}\tilde{n}} \quad or \quad \partial_{\tilde{n}}(v_3) = 0 \\ \partial_{\tilde{s}}(m^{\tilde{s}\tilde{n}}) + div_{\|}(\boldsymbol{m}) \cdot \tilde{\boldsymbol{n}} &= f_\partial^3 + \partial_{\tilde{s}}(m_\partial^{\tilde{s}\tilde{n}}) \quad or \quad v_3 = 0 \\ -[(m^{\tilde{s}\tilde{n}})_{p_i}^{L_{i-1}} - (m^{\tilde{s}\tilde{n}})_{p_i}^{L_i}] &= -[(m_\partial^{\tilde{s}\tilde{n}})_{p_i}^{L_{i-1}} - (m_\partial^{\tilde{s}\tilde{n}})_{p_i}^{L_i}] + R_i \quad or \quad (v_3)_i = 0. \end{aligned} \tag{22}$$

In the case of QSS must be neglected the double underlined term, while in the case of SS all underlined terms must be neglected.

4 INCREMENTAL FORMULATION OF THE EQUILIBRIUM EQUATIONS

In this section it is derived the incremental form of the equilibrium equations, that includes the Foppl-Von Kármán formulation for thin shells. Starting from a deformed configuration of the shell whose geometry and differential structure is known, let's consider an increment of the displacement field $\Delta\boldsymbol{u} = \boldsymbol{u}_t - \bar{\boldsymbol{u}}$. No restriction is introduced on the form of $\Delta\boldsymbol{u}$. An updated Lagrangian approach is used, that is the static and kinematic quantities will be projected onto the known configuration by a pull-back operation.

4.1 *Time rate of incremental deformation in up-dated formulation*

We start defining the tangent incremental kinematic operators. As can be observed from formulas (13) and (14) the membrane velocity gradient is a linear function of the velocity field only, while the rate of curvature tensor depends non linearly from the displacement field through the curvature tensor.

The incremental time rate membrane deformation tensor is given by a sum of two addend, the first linear in the velocity field ($\boldsymbol{v} = \dot{\boldsymbol{u}}$), the second linear in the velocity and incremental displacement fields

$$\{\boldsymbol{d}\}_{t+\Delta t}^t = \boldsymbol{d}(\boldsymbol{v}) + \boldsymbol{\Delta d}(\boldsymbol{u}, \boldsymbol{v}) \tag{23}$$

where

$$\boldsymbol{d}(\boldsymbol{v}) = \mathrm{sym}[(\boldsymbol{v}_{|\alpha} \cdot \bar{\boldsymbol{c}}_\beta)\bar{\boldsymbol{c}}^\beta \otimes \bar{\boldsymbol{c}}^\alpha] \tag{24}$$

and

$$\boldsymbol{\Delta d}(\boldsymbol{u},\boldsymbol{v}) = \mathrm{sym}[(\boldsymbol{v}_{|\alpha}\cdot\boldsymbol{u}_{|\beta})\bar{\boldsymbol{c}}^{\beta}\otimes\bar{\boldsymbol{c}}^{\alpha}]. \tag{25}$$

By definition of curvature tensor we have

$$\dot{\boldsymbol{\chi}} = \mathrm{sym}[(\dot{\boldsymbol{n}}_{|\alpha}\cdot\boldsymbol{c}_{\beta} + \boldsymbol{n}_{|\alpha}\cdot\boldsymbol{v}_{|\beta})\boldsymbol{c}_{\beta}\otimes\boldsymbol{c}_{\alpha}] \tag{26}$$

where $\boldsymbol{n}(\boldsymbol{u})$ is a non-rational function of $\boldsymbol{u}$, while $\boldsymbol{c}_{\beta}(\boldsymbol{u})$ and $\boldsymbol{v}_{|\beta}(\boldsymbol{u})$ are linear functions of $\boldsymbol{u}$. In order to linearize the expression (26) with respect to the displacement field we first give the exact $\boldsymbol{u}$-dependency of $\dot{\boldsymbol{n}}_{|\alpha}(\boldsymbol{u})$ and $\boldsymbol{n}_{|\alpha}(\boldsymbol{u})$.

From the definition of $\boldsymbol{n}$ we have the relation

$$\boldsymbol{n}\sqrt{c_{\|}} = \boldsymbol{c}_1\times\boldsymbol{c}_2 \tag{27}$$

deriving (27) respect to ϑ^{α} yields

$$\boldsymbol{n}_{|\alpha}\sqrt{c_{\|}} + \boldsymbol{n}\left(\sqrt{c_{\|}}\right)_{|\alpha} = (\boldsymbol{c}_1\times\boldsymbol{c}_2)_{|\alpha} \tag{28}$$

the product of the first and second members of equation (27) by $\boldsymbol{n}$, enforcing KL hypothesis gives

$$\left(\sqrt{c_{\|}}\right)_{|\alpha} = (\boldsymbol{c}_1\times\boldsymbol{c}_2)_{|\alpha}\cdot\boldsymbol{n}. \tag{29}$$

Analogously deriving equation (27) with respect to time we obtain the time rate

$$\dot{\overline{\sqrt{c_{\|}}}} = \dot{\overline{\boldsymbol{c}_1\times\boldsymbol{c}_2}}\cdot\boldsymbol{n}. \tag{30}$$

The incremental form of the kinematic operators is obtained linearizing the relevant expressions in a neighborhood of the configuration at *t*-instant for a vanishing increment of the displacement.

Linearizing $\boldsymbol{n}(\boldsymbol{u})$ in a neighborhood of the configuration at *t*-instant, defined by $\bar{\boldsymbol{u}}$, gives $\{\boldsymbol{n}(\bar{\boldsymbol{u}})\}^{t}_{t+\Delta t} = \boldsymbol{n}(\bar{\boldsymbol{u}}) + \partial_{\boldsymbol{u}}(\boldsymbol{n}(\boldsymbol{u}))\big|_{\bar{\boldsymbol{u}}}\cdot\boldsymbol{u}$, that in the up-dated representation can be shown to reduce to:

$$\{\boldsymbol{n}(\bar{\boldsymbol{u}})\}^{t}_{t+\Delta t} = \bar{\boldsymbol{n}} - {u^3}_{|\rho}\bar{\boldsymbol{c}}^{\rho} \tag{31}$$

where ${u^3}_{|\rho} = \bar{\boldsymbol{n}}\cdot\boldsymbol{u}_{|\rho}$. The definition of the time rate of the normal $\dot{\boldsymbol{n}}(\boldsymbol{u})$, using equation (30), is

$$\begin{aligned}\dot{\boldsymbol{n}}(\boldsymbol{u}) &= \frac{\dot{\overline{\boldsymbol{c}_1\times\boldsymbol{c}_2}}}{\sqrt{c_{\|}}} - \frac{\dot{\overline{\sqrt{c_{\|}}}}}{\sqrt{c_{\|}}}\boldsymbol{n}(\boldsymbol{u})\\ &= \frac{\dot{\overline{\boldsymbol{c}_1\times\boldsymbol{c}_2}}}{\sqrt{c_{\|}}} - \frac{(\dot{\overline{\boldsymbol{c}_1\times\boldsymbol{c}_2}})\cdot\boldsymbol{n}(\boldsymbol{u})}{\sqrt{c_{\|}}}\boldsymbol{n}(\boldsymbol{u})\end{aligned} \tag{32}$$

Linearizing the expression (32) in a neighborhood of the configuration at the *t*-instant we obtain $\{\dot{\boldsymbol{n}}(\bar{\boldsymbol{u}})\}^{t}_{t+\Delta t} = \dot{\boldsymbol{n}}(\bar{\boldsymbol{u}}) + \partial_{\boldsymbol{u}}(\dot{\boldsymbol{n}}(\boldsymbol{u}))\big|_{\bar{\boldsymbol{u}}}\cdot\boldsymbol{u}$ whose expression in the up-dated representation is

$$\{\dot{\boldsymbol{n}}(\bar{\boldsymbol{u}})\}^{t}_{t+\Delta t} = (-{v^3}_{|\rho} + {u^3}_{|\mu}{v^{\mu}}_{|\rho} + {v^3}_{|\mu}{u^{\mu}}_{|\rho})\bar{\boldsymbol{c}}^{\rho} - {v^3}_{|\lambda}\bar{c}^{\lambda\mu}{u^3}_{|\mu}\bar{\boldsymbol{n}} \tag{33}$$

From the definition of $\boldsymbol{n}(\boldsymbol{u})$ and equation (29) we have the expression of the variation along the ϑ^{α}-direction of the normal vector

$$\begin{aligned}\boldsymbol{n}_{|\alpha}(\boldsymbol{u}) &= \frac{(\boldsymbol{c}_1\times\boldsymbol{c}_2)_{|\alpha}}{\sqrt{c_{\|}}} - \frac{\left(\sqrt{c_{\|}}\right)_{|\alpha}}{\sqrt{c_{\|}}}\boldsymbol{n}(\boldsymbol{u})\\ &= \frac{(\boldsymbol{c}_1\times\boldsymbol{c}_2)_{|\alpha}}{\sqrt{c_{\|}}} - \frac{(\boldsymbol{c}_1\times\boldsymbol{c}_2)_{|\alpha}\cdot(\boldsymbol{c}_1\times\boldsymbol{c}_2)}{c_{\|}\sqrt{c_{\|}}}(\boldsymbol{c}_1\times\boldsymbol{c}_2).\end{aligned} \tag{34}$$

Linearizing the expression (34) in a neighborhood of the configuration at the t-instant we obtain the up-dated expression for $\{\boldsymbol{n}_{|\alpha}(\bar{\boldsymbol{u}})\}^t_{t+\Delta t} = \boldsymbol{n}_{|\alpha}(\bar{\boldsymbol{u}}) + \partial_{\boldsymbol{u}}(\boldsymbol{n}_{|\alpha}(\boldsymbol{u}))\big|_{\bar{\boldsymbol{u}}} \cdot \boldsymbol{u}$, that can be evaluated as

$$\{\boldsymbol{n}_{|\alpha}(\bar{\boldsymbol{u}})\}^t_{t+\Delta t} = (-\bar{b}_{\rho\alpha} + \bar{b}_{\lambda\alpha} u^{\lambda}{}_{|\rho} - u^3{}_{|\rho\alpha})\bar{\boldsymbol{c}}^{\rho} - \bar{b}^{\lambda}{}_{\alpha} u^3{}_{|\lambda} \bar{\boldsymbol{n}} \tag{35}$$

where $\bar{\boldsymbol{c}}^{\rho}$ and $\bar{\boldsymbol{n}}$ are the tangent vector at t-instant, $\bar{\boldsymbol{b}}$ is the curvature tensor of surface at the t-instant, $u^{\lambda}{}_{|\rho} = \bar{\boldsymbol{c}}^{\lambda} \cdot \boldsymbol{u}_{|\rho}$ where $\boldsymbol{u}_{|\rho}$ is the variation of $\boldsymbol{u}$ along $\bar{\boldsymbol{c}}_{\rho}$, and $u^3{}_{|\rho\alpha} = \bar{\boldsymbol{n}} \cdot \boldsymbol{u}_{|\rho\alpha}$ where $\boldsymbol{u}_{|\rho\alpha}$ is the second surface covariant derivative of $\boldsymbol{u}$.

Differentiating the definition (34) of $\boldsymbol{n}_{|\alpha}(\boldsymbol{u})$ we obtain, using equation (30) and the KL hypothesis, the definition of the time rate of the spacial rate of $\boldsymbol{n}$ indicated by $\dot{\boldsymbol{n}}_{|\alpha}(\boldsymbol{u})$

$$\begin{aligned}
\dot{\boldsymbol{n}}_{|\alpha}(\boldsymbol{u}) = {} & \frac{\dot{\overline{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)_{|\alpha}}}}{\sqrt{c_{\|}}} - \frac{\dot{\overline{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)}} \cdot (\boldsymbol{c}_1 \times \boldsymbol{c}_2)}{c_{\|}\sqrt{c_{\|}}} - \frac{\dot{\overline{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)_{|\alpha}}} \cdot (\boldsymbol{c}_1 \times \boldsymbol{c}_2)}{c_{\|}\sqrt{c_{\|}}}(\boldsymbol{c}_1 \times \boldsymbol{c}_2) + \\
& - \frac{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)_{|\alpha} \cdot \dot{\overline{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)}}}{c_{\|}\sqrt{c_{\|}}}(\boldsymbol{c}_1 \times \boldsymbol{c}_2) - \frac{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)_{|\alpha} \cdot (\boldsymbol{c}_1 \times \boldsymbol{c}_2)}{c_{\|}\sqrt{c_{\|}}}\dot{\overline{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)}} + \\
& + \frac{3[\dot{\overline{(\boldsymbol{c}_1 \times \boldsymbol{c}_2)}} \cdot (\boldsymbol{c}_1 \times \boldsymbol{c}_2)][(\boldsymbol{c}_1 \times \boldsymbol{c}_2)_{|\alpha} \cdot (\boldsymbol{c}_1 \times \boldsymbol{c}_2)]}{c_{\|}^2\sqrt{c_{\|}}}(\boldsymbol{c}_1 \times \boldsymbol{c}_2).
\end{aligned} \tag{36}$$

Linearizing the expression (36) in a neighborhood of the configuration at the t-instant we obtain the expression for $\{\dot{\boldsymbol{n}}_{|\alpha}(\boldsymbol{u})\}^t_{t+\Delta t} = \dot{\boldsymbol{n}}_{|\alpha}(\bar{\boldsymbol{u}}) + \partial_{\boldsymbol{u}}(\dot{\boldsymbol{n}}_{|\alpha}(\boldsymbol{u}))\big|_{\bar{\boldsymbol{u}}} \cdot \boldsymbol{u}$, that in up-dated representation is

$$\begin{aligned}
\{\dot{\boldsymbol{n}}_{|\alpha}(\bar{\boldsymbol{u}})\}^t_{t+\Delta t} = {} & [-v^3{}_{|\rho\alpha} + u^3{}_{|\lambda\alpha} v^{\lambda}{}_{|\rho} + u^3{}_{|\lambda} v^{\lambda}{}_{|\rho\alpha} + u^{\lambda}{}_{|\rho\alpha} v^3{}_{|\lambda} + u^{\lambda}{}_{|\rho} v^3{}_{|\lambda\alpha} + \\
& + \bar{b}_{\lambda\alpha} v^{\lambda}{}_{|\rho} - \bar{b}_{\mu\alpha} u^{\mu}{}_{|\lambda} v^{\lambda}{}_{|\rho} - \bar{b}_{\mu\alpha} u^{\lambda}{}_{|\rho} v^{\mu}{}_{|\lambda} \\
& + \bar{b}_{\rho\alpha} u^3{}_{|\mu} \bar{c}^{\mu\lambda} v^3{}_{|\lambda} + \bar{b}^{\lambda}{}_{\alpha} u^3{}_{|\rho} v^3{}_{|\lambda} + \bar{b}^{\lambda}{}_{\alpha} u^3{}_{|\lambda} v^3{}_{|\rho}]\, \bar{\boldsymbol{c}}^{\rho} + \\
& + [-\bar{b}_{\lambda\alpha} \bar{c}^{\lambda\mu} v^3{}_{|\mu} - \bar{c}^{\lambda\mu} u^3{}_{|\mu\alpha} v^3{}_{|\lambda} - \bar{c}^{\lambda\mu} u^3{}_{|\mu} v^3{}_{|\lambda\alpha} + \\
& + \bar{b}_{\lambda\alpha}(u^3{}_{|\mu} \bar{c}^{\mu\rho} v^{\lambda}{}_{|\rho} + u^{\lambda}{}_{|\mu} \bar{c}^{\mu\rho} v^3{}_{|\rho}) + \bar{b}_{\rho\alpha} \bar{c}^{\rho\lambda} u^3{}_{|\mu} v^{\mu}{}_{|\lambda} + \bar{b}_{\rho\alpha} \bar{c}^{\rho\lambda} u^{\mu}{}_{|\lambda} v^3{}_{|\mu} + \\
& + \bar{b}_{\rho\alpha} \bar{c}^{\rho\lambda}(v^1{}_{|1} + v^2{}_{|2}) u^3_{|\lambda} + \bar{b}_{\rho\alpha} \bar{c}^{\rho\lambda}(u^1{}_{|1} + u^2{}_{|2}) v^3_{|\lambda}]\, \bar{\boldsymbol{n}}.
\end{aligned} \tag{37}$$

From equation (26), (35) and (37) the up-dated incremental time rate curvature tensor is given by a sum of two addends, the first linear in the velocity field, the second linear in the velocity and in the incremental displacement fields

$$\{\dot{\boldsymbol{\chi}}\}^t_{t+\Delta t} = \dot{\boldsymbol{\chi}}(\boldsymbol{v}) + \boldsymbol{\Delta}\dot{\boldsymbol{\chi}}(\boldsymbol{u}, \boldsymbol{v}) \tag{38}$$

where

$$\dot{\boldsymbol{\chi}}(\boldsymbol{v}) = \mathrm{sym}[-\bar{\boldsymbol{n}} \cdot \boldsymbol{v}_{|\alpha\beta} \otimes \bar{\boldsymbol{c}}^{\alpha} \otimes \bar{\boldsymbol{c}}^{\beta}] \tag{39}$$

and

$$\begin{aligned}
\boldsymbol{\Delta}\dot{\boldsymbol{\chi}}(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{sym}[& (\bar{\boldsymbol{n}} \cdot \boldsymbol{u}_{|\eta}) \bar{c}^{\eta\rho} (\bar{\boldsymbol{n}} \cdot \boldsymbol{v}_{|\rho}) \bar{b}_{\alpha\beta} \bar{\boldsymbol{c}}^{\alpha} \otimes \bar{\boldsymbol{c}}^{\beta} + \\
& + (\bar{\boldsymbol{n}} \cdot \boldsymbol{v}_{|\rho})(\bar{\boldsymbol{c}}^{\rho} \cdot \boldsymbol{u}_{|\alpha\beta}) \bar{\boldsymbol{c}}^{\alpha} \otimes \bar{\boldsymbol{c}}^{\beta} + \\
& + (\bar{\boldsymbol{n}} \cdot \boldsymbol{u}_{|\rho})(\bar{\boldsymbol{c}}^{\rho} \cdot \boldsymbol{v}_{|\alpha\beta}) \bar{\boldsymbol{c}}^{\alpha} \otimes \bar{\boldsymbol{c}}^{\beta}].
\end{aligned} \tag{40}$$

In the case of QSS the up-dated representation of this tensor reduces to

$$\dot{\boldsymbol{\chi}}_{QSS}(\boldsymbol{v}) = \dot{\boldsymbol{\chi}}(\boldsymbol{v}) + \mathrm{sym}[\,\bar{b}_{\mu\beta} v^{\mu}{}_{|\alpha} \bar{\boldsymbol{c}}^{\alpha} \otimes \bar{\boldsymbol{c}}^{\beta}] = -\mathrm{sym}[(\bar{\boldsymbol{n}} \cdot \boldsymbol{v}_{|\alpha} \otimes \bar{\boldsymbol{c}}^{\alpha})_{\|\beta} \otimes \bar{\boldsymbol{c}}^{\beta}] \tag{41}$$

and

$$\boldsymbol{\Delta}\dot{\boldsymbol{\chi}}_{QSS}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{\Delta}\dot{\boldsymbol{\chi}}(\boldsymbol{u},\boldsymbol{v}) + \mathrm{sym}\{[\overbrace{(u^{3}{}_{|\mu\beta} - \bar{b}_{\rho\beta} u^{\rho}{}_{|\mu})}^{(u_{3|\mu})_{\|\beta}} v^{\mu}{}_{|\alpha} + \bar{b}^{\rho}{}_{\beta} u^{3}{}_{|\rho} v^{3}{}_{|\alpha}] \bar{\boldsymbol{c}}^{\alpha} \otimes \bar{\boldsymbol{c}}^{\beta}\} \tag{42}$$

4.2 *Up-dated Lagrangian incremental equilibrium equations*

The incremental equilibrium equations are obtained via principle of virtual displacements. Let $\mathcal{S}_{t+\Delta t}$ be the middle surface of the shell in an equilibrium state in a neighborhood of the configuration mapped by the surface $\mathcal{S}_t$, subjected to the conservative surface force $\boldsymbol{q}_{t+\Delta t}$ per unit area of surface $\mathcal{S}_t$, and to the conservative boundary force $\boldsymbol{f}_{\partial}^{t+\Delta t}$ and the conservative boundary moment $\boldsymbol{m}_{\partial}^{t+\Delta t}$, both per unit of length of $\partial\mathcal{S}_t$. The principle of virtual displacements is

$$\begin{aligned}\int_{\mathcal{S}_{t+\Delta t}} (\boldsymbol{N} : \boldsymbol{d}_o + \boldsymbol{M} : \dot{\boldsymbol{\chi}}) d\mathcal{S}_{t+\Delta t} = \int_{\mathcal{S}_{t+\Delta t}} (\boldsymbol{q}_{t+\Delta t} \cdot \boldsymbol{v}) d\mathcal{S}_{t+\Delta t} + \\ + \int_{\partial\mathcal{S}_{t+\Delta t}} (\boldsymbol{f}_{\partial}^{t+\Delta t} \cdot \boldsymbol{v}_{\partial} + \boldsymbol{m}_{\partial}^{t+\Delta t} \cdot \dot{\boldsymbol{n}}_{\partial}) d(\partial\mathcal{S}_{t+\Delta t}).\end{aligned} \tag{43}$$

We represent this equation on the configuration at the *t*-instant via a pull-back operation and consider an infinitesimal incremental change of configuration from $\mathcal{S}_t$ to $\mathcal{S}_{t+\Delta t}$. The internal forces become then $\boldsymbol{N} = \boldsymbol{N}_0 + \Delta\boldsymbol{N}(\boldsymbol{u})$, $\boldsymbol{M} = \boldsymbol{M}_0 + \Delta\boldsymbol{M}(\boldsymbol{u})$, where the increments contain only the linear terms in the incremental displacement. The velocities of deformation are calculated using (23) and (38). Retaining only the terms linear in $\boldsymbol{u}$ equation (43) becomes

$$\begin{aligned}\int_{\mathcal{S}_t} (\boldsymbol{\Delta N}(\boldsymbol{u}) : \boldsymbol{d}_o(\boldsymbol{v}) + \boldsymbol{\Delta} M(\boldsymbol{u}) : \dot{\boldsymbol{\chi}}(\boldsymbol{v}) + \boldsymbol{N}_0 : \boldsymbol{\Delta d}_o(\boldsymbol{u},\boldsymbol{v}) + \boldsymbol{M}_0 : \boldsymbol{\Delta}\dot{\boldsymbol{\chi}}(\boldsymbol{u},\boldsymbol{v})) d\mathcal{S}_t - \int_{\mathcal{S}_t} (\Delta\boldsymbol{q} \cdot \boldsymbol{v}) d\mathcal{S}_t + \\ - \int_{\partial\mathcal{S}_t} (\Delta\boldsymbol{f}_{\partial} \cdot \boldsymbol{v}_{\partial} + \Delta\boldsymbol{m}_{\partial} \cdot \{\dot{\boldsymbol{n}}_{\partial}\}^{lin}) d(\partial\mathcal{S}_t) = 0\end{aligned} \tag{44}$$

Equation (44) splits the internal increment of virtual work in two parts, the incremental constitutive part and the incremental geometric part. Applying Green's formula to the incremental geometrical work, from equation (25) the incremental geometric internal membrane work is

$$\begin{aligned}\int_{\mathcal{S}_t} \boldsymbol{N}_0 : \boldsymbol{\Delta d}_o(\boldsymbol{u},\boldsymbol{v}) d\mathcal{S}_t = \int_{\partial\mathcal{S}_t} u^{\rho}{}_{|\alpha} N_0^{\alpha\beta} \tilde{n}_{\beta} v_{\rho} ds^2 + \int_{\mathcal{S}_t} [\bar{b}^{\rho}{}_{\beta}(u^{3}{}_{|\alpha} N_0^{\alpha\beta}) - (u^{\rho}{}_{|\alpha} N_0^{\alpha\beta})_{\|\beta}] v_{\rho} d\mathcal{S}_t + \\ + \int_{\partial\mathcal{S}_t} u^{3}{}_{|\alpha} N_0^{\alpha\beta} \tilde{n}_{\beta} v_3 ds^2 - \int_{\mathcal{S}_t} [(u^{\rho}{}_{|\alpha} N_0^{\alpha\beta}) \bar{b}_{\rho\beta} + (u^{3}{}_{|\alpha} N_0^{\alpha\beta})_{\|\beta}] v_3 d\mathcal{S}_t.\end{aligned} \tag{45}$$

From equation (40) the incremental geometric internal bending work splits in a sum of three addends

$$\int_{\mathcal{S}_t} (\boldsymbol{M}_0 : \boldsymbol{\Delta}\dot{\boldsymbol{\chi}}(\boldsymbol{u},\boldsymbol{v})) d\mathcal{S}_t = \int_{\mathcal{S}_t} M_0^{\alpha\beta} (\bar{b}_{\alpha\beta} u^{3}{}_{|\eta} \bar{c}^{\eta\rho} v^{3}{}_{|\rho} + v^{3}{}_{|\rho} u^{\rho}{}_{|\alpha\beta} + u^{3}{}_{|\rho} v^{\rho}{}_{|\alpha\beta}) d\mathcal{S}_t. \tag{46}$$

Applying Green's formula to the equation (46) the first addend reduces to

$$\int_{\mathcal{S}_t} M_0^{\alpha\beta}\bar{b}_{\alpha\beta}u^3{}_{|\eta}\bar{c}^{\eta\rho}v^3{}_{|\rho}d\mathcal{S}_t = \int_{\mathcal{S}_t}\bar{b}^\rho{}_\lambda\bar{b}_{\alpha\beta}M_0^{\alpha\beta}u^3{}_{|\eta}\bar{c}^{\eta\lambda}v_\rho d\mathcal{S}_t - \int_{\mathcal{S}_t}(\bar{b}_{\alpha\beta}M_0^{\alpha\beta}u^3{}_{|\eta}\bar{c}^{\eta\rho})_{\|\rho}v_3 d\mathcal{S}_t + \int_{\partial\mathcal{S}_t}\bar{b}_{\alpha\beta}M_0^{\alpha\beta}u^3{}_{|\eta}\bar{c}^{\eta\lambda}\tilde{n}_\lambda v_3 ds^2. \tag{47}$$

The second integral of equation (46) reduces to

$$\int_{\mathcal{S}_t} M_0^{\alpha\beta}u^\rho{}_{|\alpha\beta}v^3{}_{|\rho}d\mathcal{S}_t = \int_{\mathcal{S}_t}\bar{b}^\rho{}_\mu u^\mu{}_{|\alpha\beta}M_0^{\alpha\beta}v_\rho d\mathcal{S}_t - \int_{\mathcal{S}_t}(u^\mu{}_{|\alpha\beta}M_0^{\alpha\beta})_{\|\mu}v_3 d\mathcal{S}_t + \int_{\partial\mathcal{S}_t}u^\mu{}_{|\alpha\beta}M_0^{\alpha\beta}\tilde{n}_\mu v_3 ds^2. \tag{48}$$

The third integral of equation (46) reduces to

$$\begin{aligned}\int_{\mathcal{S}_t} M_0^{\alpha\beta}u_{3|\rho}v^\rho{}_{|\alpha\beta}d\mathcal{S}_t = &\int_{\mathcal{S}_t}(M_0^{\alpha\beta}u_{3|\eta})_{\|\alpha\beta}\bar{c}^{\eta\rho}v_\rho d\mathcal{S}_t - \int_{\mathcal{S}_t}M_0^{\alpha\beta}u_{3|\mu}\bar{b}^\mu{}_\alpha\bar{b}^\rho{}_\beta v_\rho d\mathcal{S}_t\\ &+ \underline{\int_{\partial\mathcal{S}_t}M_0^{\alpha\beta}u_{3|\rho}v^\rho{}_{|\beta}\tilde{n}_\alpha ds^2} - \int_{\partial\mathcal{S}_t}(M_0^{\alpha\beta}u_{3|\eta})_{\|\alpha}\tilde{n}_\beta\bar{c}^{\eta\rho}v_\rho ds^2\\ &+ \int_{\mathcal{S}_t}[(M_0^{\alpha\beta}u_{3|\rho}\bar{b}^\rho{}_\beta)_{\|\alpha} + (M_0^{\alpha\beta}u_{3|\rho}\bar{b}^\rho{}_\alpha)_{\|\beta} - M_0^{\alpha\beta}u_{3|\rho}\bar{b}^\rho{}_{\alpha\|\beta}]v_3 d\mathcal{S}_t\\ &+ \int_{\partial\mathcal{S}_t}M_0^{\alpha\beta}u_{3|\rho}\bar{b}^\rho{}_\alpha\tilde{n}_\beta v_3 ds^2\end{aligned} \tag{49}$$

where $(M_0^{\alpha\beta}u_{3|\eta})_{\|\alpha\beta} = M_0^{\alpha\beta}{}_{\|\alpha\beta}u_{3|\eta} + M_0^{\alpha\beta}{}_{\|\alpha}(u_{3|\eta})_{\|\beta} + M_0^{\alpha\beta}{}_{\|\beta}(u_{3|\eta})_{\|\alpha} + M_0^{\alpha\beta}(u_{3|\eta})_{\|\alpha\beta}$, with $(u_{3|\eta})_{\|\beta} = (u_{3|\eta})_{,\beta} - u_{3|\lambda}\bar{\Gamma}^\lambda_{\eta\beta}$, and $(u_{3|\rho})_{\|\alpha\beta} = (u_{3|\rho})_{,\alpha\beta} - (u_{3|\eta}\bar{\Gamma}^\eta_{\rho\alpha})_{,\beta} - (u_{3|\xi})_{,\alpha}\bar{\Gamma}^\xi_{\rho\beta} + u_{3|\eta}\bar{\Gamma}^\eta_{\xi\alpha}\bar{\Gamma}^\xi_{\rho\beta} - ((u_{3|\eta})_{,\rho} - u_{3|\lambda}\bar{\Gamma}^\lambda_{\eta\rho})\bar{\Gamma}^\eta_{\alpha\beta}$.

In the case of QSS the internal geometric bending work contains some additional terms:

$$\int_{\mathcal{S}_t} M_0^{\alpha\beta}(u^3{}_{|\mu})_{\|\beta}v^\mu{}_{|\alpha}d\mathcal{S}_t = \int_{\partial\mathcal{S}_t}M_0^{\alpha\beta}(u^3{}_{|\mu})_{\|\beta}c^{\mu\rho}\tilde{n}_\alpha v_\rho ds^2 - \int_{\mathcal{S}_t}[M_0^{\alpha\beta}(u^3{}_{|\mu})_{\|\beta}]_{\|\alpha}\bar{c}^{\mu\rho}v_\rho d\mathcal{S}_t - \int_{\mathcal{S}_t}M_0^{\alpha\beta}(u^3{}_{|\mu})_{\|\beta}\bar{b}^\mu{}_\alpha v_3 d\mathcal{S}_t \tag{50}$$

and

$$\int_{\mathcal{S}_t} M_0^{\alpha\beta}\bar{b}^\rho{}_\beta u^3{}_{|\rho}v^3{}_{|\alpha}d\mathcal{S}_t = \int_{\partial\mathcal{S}_t}M_0^{\alpha\beta}\bar{b}^\rho{}_\beta u^3{}_{|\rho}\tilde{n}_\alpha v_3 ds^2 - \int_{\mathcal{S}_t}(M_0^{\alpha\beta}\bar{b}^\rho{}_\beta u^3{}_{|\rho})_{\|\alpha}v_3 d\mathcal{S}_t + \int_{\mathcal{S}_t}M_0^{\alpha\beta}\bar{b}^\mu{}_\beta u^3{}_{|\mu}\bar{b}^\rho{}_\alpha v_\rho d\mathcal{S}_t. \tag{51}$$

Remembering that $\Delta\boldsymbol{n} = -\boldsymbol{n}\cdot grad_{\|}(\boldsymbol{u})$, the local incremental equilibrium equation on $\mathcal{S}_t$ are then

$$
\begin{aligned}
&- div_{\|}[\Delta \boldsymbol{N} - \underline{\bar{\boldsymbol{b}}\Delta\boldsymbol{M}} + (grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{N}_0] + \\
&\quad + \bar{\boldsymbol{b}}\left(div_{\|}(\Delta\boldsymbol{M}) - \Delta\boldsymbol{n}\boldsymbol{N}_0 - \Delta\boldsymbol{n}(\bar{\boldsymbol{b}}:\boldsymbol{M}_0) + grad_{\|}^2(\boldsymbol{u}):\boldsymbol{M}_0 + \underline{\Delta\boldsymbol{n}\bar{\boldsymbol{b}}\boldsymbol{M}_0}\right) + \\
&\qquad - div_{\|}\left(div_{\|}(\boldsymbol{M}_0)\otimes\Delta\boldsymbol{n}\right) - \underline{(grad_{\|}(\Delta\boldsymbol{n}))^{\|}div_{\|}(\boldsymbol{M}_0)} - \underline{grad_{\|}^2(\Delta\boldsymbol{n}):\boldsymbol{M}} = \Delta\boldsymbol{q}^{\|},
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&- div_{\|}[div_{\|}(\Delta\boldsymbol{M}) + \Delta\boldsymbol{n}\,\boldsymbol{N}_0 + \Delta\boldsymbol{n}\,\bar{\boldsymbol{b}}:\boldsymbol{M}_0] - div_{\|}[grad_{\|}^2(\boldsymbol{u}):\boldsymbol{M}_0] + \\
&\quad - \underline{div_{\|}(\Delta\boldsymbol{n}\,\bar{\boldsymbol{b}}\boldsymbol{M}_0)} - \Delta\boldsymbol{n}\,div_{\|}(\bar{\boldsymbol{b}}\boldsymbol{M}_0) - \underline{\bar{\boldsymbol{b}}\boldsymbol{M}_0 : (grad_{\|}(\Delta\boldsymbol{n}))^{\|}} - \Delta\boldsymbol{n}(grad_{\|}(\bar{\boldsymbol{b}}))^{\|}:\boldsymbol{M}_0 + \\
&\qquad - \bar{\boldsymbol{b}}[\Delta\boldsymbol{N} - \underline{\bar{\boldsymbol{b}}\Delta\boldsymbol{M}} + (grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{N}_0] = \Delta q^3.
\end{aligned} \tag{53}
$$

The boundary conditions on $\partial\mathcal{S}_t$are

$$
\begin{aligned}
&(\Delta\boldsymbol{N} + (grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{N}_o - \underline{2}\bar{\boldsymbol{b}}\Delta\boldsymbol{M})\tilde{\boldsymbol{n}} + \Delta\boldsymbol{n}\,div_{\|}(\boldsymbol{M}_0\tilde{\boldsymbol{n}}) + \underline{(grad_{\|}(\Delta\boldsymbol{n}))^{\|}\boldsymbol{M}_0\tilde{\boldsymbol{n}}} + \\
&\quad + \bar{\boldsymbol{b}}(grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{M}_0\tilde{\boldsymbol{n}} = \Delta\boldsymbol{f}_{\partial}^{\|} - \underline{2}\bar{\boldsymbol{b}}\Delta\boldsymbol{m}_{\partial}\tilde{\boldsymbol{n}} \quad or \quad \boldsymbol{v}_{\partial}^{\|} = \boldsymbol{0}
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
&\partial_{\tilde{s}}\{\tilde{\boldsymbol{s}}\cdot[\Delta\boldsymbol{M} + (grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{M}_0]\cdot\tilde{\boldsymbol{n}}\} + div_{\|}(\Delta\boldsymbol{M})\cdot\tilde{\boldsymbol{n}} - \Delta\boldsymbol{n}\cdot(\boldsymbol{N}_0 - \bar{\boldsymbol{b}}\boldsymbol{M}_0)\cdot\tilde{\boldsymbol{n}} + \\
&\quad - (\bar{\boldsymbol{b}}:\boldsymbol{M}_0)\Delta\boldsymbol{n}\cdot\tilde{\boldsymbol{n}} + \tilde{\boldsymbol{n}}\cdot grad_{\|}^2(\boldsymbol{u}):\boldsymbol{M}_0 = \Delta f_{\partial}^3 + \partial_{\tilde{s}}(\Delta m_{\partial})^{\tilde{s}\tilde{n}}, \quad or \quad v_{\partial}^3 = 0
\end{aligned} \tag{55}
$$

$$
-\tilde{\boldsymbol{n}}\cdot[\Delta\boldsymbol{M} + (grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{M}_0]\cdot\tilde{\boldsymbol{n}} = -(\Delta m_{\partial})^{\tilde{n}\tilde{n}}, \quad or \quad \partial_{\tilde{n}}(v^3) = 0 \tag{56}
$$

and on vertices $\boldsymbol{p}_i$

$$
\begin{aligned}
&- \tilde{\boldsymbol{s}}\cdot[(\boldsymbol{M}_0)_{p_i}^{L_{i-1}} - (\boldsymbol{M}_0)_{p_i}^{L_i}]\cdot\tilde{\boldsymbol{n}} \\
&- \tilde{\boldsymbol{s}}\cdot\{[(grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{M}_0]_{p_i}^{L_{i-1}} - [(grad_{\|}(\boldsymbol{u}))^{\|}\boldsymbol{M}_0]_{p_i}^{L_i}\}\cdot\tilde{\boldsymbol{n}} = \Delta R^i, \quad or \quad v_3(p_i) = 0.
\end{aligned} \tag{57}
$$

The underlined terms must be disregarded in the case of the QSS.

5 CONCLUSIONS

The exact non linear kinematic of a thin shell has been developed and has been employed for deriving the weak and strong forms of the spatial equilibrium equations and of the incremental equilibrium operator. It has been shown that the approximated forms of the equilibrium equations commonly denoted as shallow and quasi-shallow approximations are obtained disregarding some terms in the rate of curvature tensor.

The incremental equilibrium equations have been derived considering an arbitrary form of the increment of displacement, contrary to what is usually done in the literature, where a restricted perturbation is considered, neglecting the part responsible for membrane stretching. The boundary conditions have also been rationally derived. Also the quasi shallow approximations has been discussed.

The latter equations have been used by the authors for closed form stability analysis of thin shells with membrane dominated behavior, undergoing wrinkling, and the influence of the curvature, as well as the limit of application of different types of approximations have been analyzed. These results are reported in separated papers.

References

[1] Mason, J. *Variational, Incremental and Energy Methods in Solids Mechanics and Shell Theory*. Elsevier, New York, (1980).

[2] Fox, D. D. and Simo, J. C. "A drill rotation formulation for geometrically exact shells". *Comp. Meth. APPL. Mech. Engng.* **98**, 329–343 (1992).

[3] Simo, J. C. and Fox, D. "On a stress resultant geometrically exact shell model. Part I: Formulation and optimal parametrization". *Computer Methods in Applied Mechanics and Engineering* **72**, 267–304 (1989).

[4] J. C. Simo, D. F. and Rifai, M. S. "On a stress resultant geometrically exact shell model. Part II: The linear theory". *Computer Methods in Applied Mechanics and Engineering* **73**, 53–62 (1989).

[5] Bazant, Z. P. and Cedolin, L. *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*. Dover Publications, New York, (1991).

[6] Timoshenko, S. P. and Gere, J. M. *Theory of Elastic Stability*. McGraw Hill, New York, (1988).

[7] Ciarlet, P. *Mathematical Elasticity*, volume III: Theory of Shells. North-Holland, Amsterdam, (2000).

[8] Wempner, G. *Mechanics of Solids with Applications to Thin Bodies*. SiJthoff and Noordhoff, Alphen aan den Rijn, (1981).

[9] Pietraszkiewicz, W. "Lagrangian description and incremental formulation in the non-linear theory of thin shells". *Int. J. Non-Linear Mechanics* **19**(2), 115–140 (1983).