Methods of stress analysis of laminated glass

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SUMMARY. With the development of nanoscale semiconductor devices (nanostructures), quantum transport has become a major tool of solid state physics and a very active area of research among physicists and mathematicians.

1 INTRODUCTION

The paper focuses on stratified glass plates, subjected to in plane and out of plane loads. The design of glass panels is largely driven by the requirement of keeping as small as possible the relative displacements in order to reduce the strain energy and to avoid the propagation of fractures from existing defects. The analysis of the stress and deformation fields is therefore preliminary to the safety analysis of the structure.

Stress analysis in laminated glass requires the consideration of discontinuous displacement fields, as well as of the non linear and viscous behaviour of the polymeric interstial material [1]. It is well known that the distribution of the stresses through the thickness of the laminate strongly depends on the degree of coupling between the glass layers induced by the interlayer [2,3]. Disregarding local effects due to the external connections, it is possible to derive the equilibrium equations of the element using some simplified but realistic assumption on the deformation field [2]. The equations can be solved in closed form in the case of a one-dimensional response. In this way it is possible to perform a non-dimensional analysis, enlightening the main aspects of the phenomenon. The quality of the response changes according to the value of a factor that will be shown to depends on the ratio of the rigidity of the glass and of the interlayer and also on geometrical factors. Estimates about the insurgence of fracture phenomena and whether delamination can anticipate glass fracture can thus be obtained. In order to evaluate this possibility, the model is enriched with an additional degree of freedom, that allows different curvatures in the glass layers. In this sense the model presented extends those already proposed in [2,4].

2 THE MODEL

The model described in the paper is limited only to the case of one-dimensional behaviour. In this way it is possible to obtain close-form solutions that allow to perform comparisons and parametric analyses. However the model can be easily extended to bidimensional structures, following the methodology introduced by Asik [5].

Let's consider then a beam composed by *n* layers of glass, bonded by (*n-1*) polymeric interlayers. Each layer has its own thickness, while the interlayer has thickness t_i. The depth of the beam is considered constant. The model applies to the membrane and bending response of the glass sheet.

2.1 *Material models*

The glass is considered linear elastic, characterized by Young's modulus E. The non linear behaviour of the polymeric material is disregarded in the model, so it is considered linear elastic, characterized by different shear modulus G and Young modulus E_P for short or long duration loads. A linear viscoelastic model has been considered in a forthcoming paper.

2.2 *Kinematics*

The following hypotheses are made:

1. shear deformation of glass layers is neglected;
2. perfect bonding is assumed between glass and polymer;
3. shear deformation is considered in the interlayer;
4. the transverse section of each glass layer remains straight.

The adhesion between glass and polymer (PVB, for instance), is a chemical bond, that modifies the surface of both components. While this effect can be disregarded in glass, it is not so in the polymer, given its small thickness and its high deformability. Therefore, shear sliding produces an *S*-shaped type of deformation, so that the strain and stress fields are not constant within the interlayer. It is possible to account for this effect, typical of rubber in elastomeric bearings, in a rather simplified way [6], however in this paper it is considered a simpler kinematic model (level 0), assuming that shear deformation is constant within the interlayer. In order to account for the effects of adhesion, however, axial strains are added.

Further hypotheses made on the deformation field are that the glass layers are in a plane stress state, while the interlayer is in a plane strain state, with zero strain in the direction of the depth of the beam.

The model so described is kinematically compatible, but, because of the assumption that the strain be constant across the layers, is not locally equilibrated. In the interlayer vertical stresses develop, that are neglected in the glass layers. This seems a reasonable approximation, since normal stresses in the transverse direction would be small and rapidly vanishing away from the glued boundary.

With reference to fig.1 the degrees of freedom that describe the model are:

- u_i , i=1,…n, axial displacement of the centre line of the i-th layer;
- v_i , i=1,…n, transverse displacement of the centre line of the i-th layer.

According to the hypothesis of negligible shear deformation in the glass layer, on has:

$$\left.\begin{aligned} \varphi_i &= \frac{\partial v_i}{\partial x} \\ u_{xi}(x,y) &= u_i - \varphi_i\, y \end{aligned}\right\} \quad i = 1,\ldots,n \tag{1}$$

The standard Euler approximation of the Lagrange deformation tensor is used ($E \cong 1/2(H + H^T) + 1/2\,\Omega^T\Omega$), so that the strain fields within the glass layers are given by:

$$\varepsilon_{xi} = \frac{\partial u_i}{\partial x} - \frac{\partial^2 v_i}{\partial x^2} y + \underline{\frac{1}{2}\left(\frac{\partial v_i}{\partial x}\right)^2}, \quad i = 1, \ldots, n$$
$$\varepsilon_{yi} \cong 0 \tag{2}$$
$$\gamma_j = \frac{1}{t_j}\left[u_{j+1} - u_j + \left(\varphi_{j+1}\frac{h_{j+1}}{2} + \varphi_j \frac{h_j}{2}\right)\right]$$

In the interlayer the deformations are obtained considering the following displacement field

$$u_{Pxj} = \frac{\bar{u}_{j+1} - \bar{u}_j}{t} y + \frac{\bar{u}_{j+1} + \bar{u}_j}{2} \qquad \bar{u}_j = u_j - \varphi_j \frac{h_j}{2} \;, \quad \bar{u}_{j+1} = u_{j+1} + \varphi_{j+1}\frac{h_{j+1}}{2}$$
$$v_{Pj} = \frac{v_{j+1} - v_j}{t} y + \frac{v_{j+1} + v_j}{2} \tag{3}$$

It is found

$$\varepsilon_{Pxj} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x} + \frac{\partial u_{j+1}}{\partial x} - \frac{\partial^2 v_j}{\partial x^2}\frac{h_j}{2} + \frac{\partial^2 v_{j+1}}{\partial x^2}\frac{h_{j+1}}{2}\right] + \frac{1}{t}\left[\frac{\partial u_{j+1}}{\partial x} - \frac{\partial u_j}{\partial x} + \frac{\partial^2 v_j}{\partial x^2}\frac{h_j}{2} + \frac{\partial^2 v_{j+1}}{\partial x^2}\frac{h_{j+1}}{2}\right] y$$
$$\varepsilon_{Pyj} = \frac{v_{j+1} - v_j}{t} \tag{4}$$
$$\gamma = \frac{1}{t}\left[u_{j+1} - u_j + \frac{\partial v_j}{\partial x}\frac{h_j}{2} + \frac{\partial v_{j+1}}{\partial x}\frac{h_{j+1}}{2}\right] + \frac{1}{2}\left[\frac{\partial v_j}{\partial x} + \frac{\partial v_{j+1}}{\partial x}\right]$$

Only the linear terms have been considered. The shear deformation has been taken constant over the interlayer, and equal to its mean value.

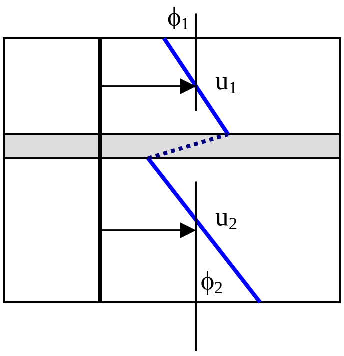


Figure 1: Kinematics of the model

Linear elastic constitutive equations are considered, assuming plane stress state. The interlayer, instead, is considered in a state of plane strain, due to the confining effect of the bonding surface,

$$
\sigma_{Pxj} = E^{*}\left[(1-\nu)\varepsilon_{Pxj} + \nu\varepsilon_{Pyj}\right] \qquad E^{*} = \frac{E_P}{(1+\nu)(1-2\nu)} = \frac{2G^2}{3G - E_P}
$$
$$
\sigma_{Pyj} = E^{*}\left[(1-\nu)\varepsilon_{Pyj} + \nu\varepsilon_{Pxj}\right] \tag{5}
$$

The model presented differs from those of [2,4,5] in the fact that here it is not assumed a common rotation for all the glass layers. Furthermore, deformation other than shear is assumed in the interlayer.

3 FORMULATION

The equations of the model are obtained from the Total Potential Energy of the system, that is written for simplicity for the case of a 2-layers system. The generalization to more layers is obvious:

$$
\begin{aligned}
\Pi = {} & \frac{1}{2}\int_0^l \int_{A_1} E\left[\left(u'_1 + \frac{1}{2}v'^2_1\right) - v''_1\, y\right]^2 dx + \frac{1}{2}\int_0^l \int_{A_2} E\left[\left(u'_2 + \frac{1}{2}v'^2_2\right) - v''_2\, y\right]^2 dx + \frac{Gbt}{2}\int_0^l \gamma^2\, dx + \\
& + \frac{b}{2}\int_0^l \int_{-t/2}^{t/2} E^{*}\left[\varepsilon_{Px}^2 + \varepsilon_{Py}^2 - \nu\left(\varepsilon_{Px} - \varepsilon_{Py}\right)^2\right]
\end{aligned} \tag{6}
$$

After some lengthy calculation the set of non-linear equations accounting for the complete deformation of the interlayer is obtained:

$$
\begin{aligned}
& EI_1 \frac{\partial^4 v_1}{\partial x^4} - \frac{\partial}{\partial x}\left(N_1 \frac{\partial v_1}{\partial x}\right) - Gb\frac{h_1^{*}}{2}\frac{\partial \gamma}{\partial x} - \underline{\frac{E^{*}bt}{3}(1-\nu)\frac{h_1}{2}\frac{\partial^3 \kappa_1}{\partial x^3}} - \frac{E^{*}b}{t}(1-\nu)(v_2 - v_1) - \underline{\frac{E^{*}b}{2}\nu\frac{\partial \psi_1}{\partial x}} = q_{1y} \\
& EI_2 \frac{\partial^4 v_2}{\partial x^4} - \frac{\partial}{\partial x}\left(N_2 \frac{\partial v_2}{\partial x}\right) - Gb\frac{h_2^{*}}{2}\frac{\partial \gamma}{\partial x} + \underline{\frac{E^{*}bt}{3}(1-\nu)\frac{h_2}{2}\frac{\partial^3 \kappa_2}{\partial x^3}} + \frac{E^{*}b}{t}(1-\nu)(v_2 - v_1) + \underline{\frac{E^{*}b}{2}\nu\frac{\partial \psi_2}{\partial x}} = q_{2y} \\
& -\frac{\partial N_1}{\partial x} - Gb\gamma - \underline{\frac{E^{*}bt}{3}(1-\nu)\frac{\partial^2 \kappa_1}{\partial x^2}} - \underline{\frac{E^{*}b}{2}\nu\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial x}\right)} = q_{1x} \\
& -\frac{\partial N_2}{\partial x} + Gb\gamma - \underline{\frac{E^{*}bt}{3}(1-\nu)\frac{\partial^2 \kappa_2}{\partial x^2}} - \underline{\frac{E^{*}b}{2}\nu\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial x}\right)} = q_{2x} \\
& \kappa_1 = u_1 + \frac{1}{2}u_2 - \frac{\partial v_1}{\partial x}\frac{h_1}{2} + \frac{1}{2}\frac{\partial v_2}{\partial x}\frac{h_2}{2} \quad ; \quad \kappa_2 = \frac{1}{2}u_1 + u_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\frac{h_1}{2} + \frac{\partial v_2}{\partial x}\frac{h_2}{2} \\
& \psi_1 = u_1 + u_2 - 3\frac{\partial v_1}{\partial x}\frac{h_1}{2} + \frac{\partial v_2}{\partial x}\left(h_1 + \frac{h_2}{2}\right) \quad ; \quad \psi_1 = u_1 + u_2 - \frac{\partial v_1}{\partial x}\left(h_1 + \frac{h_2}{2}\right) + 3\frac{\partial v_2}{\partial x}\frac{h_2}{2} \\
& \gamma = \frac{1}{t}\left[u_2 - u_1 + \frac{\partial v_1}{\partial x}\frac{h_1^{*}}{2} + \frac{\partial v_2}{\partial x}\frac{h_2^{*}}{2}\right] \qquad h_1^{*} = \frac{h_1 + t}{2} \quad h_2^{*} = \frac{h_2 + t}{2}
\end{aligned} \tag{7}
$$

Equations (7) are difficult to treat analytically, so some simpler models are examined.

3.1 *Simplified formulation*

The first approximation concerns in neglecting the longitudinal stress in the interlayer, but not the transverse stress, that thus turns out to be equal to $\sigma_{Py} = E^{*}(1-\nu)\varepsilon_{Py}$. The relevant equations are obtained setting to zero the underlined terms in (7) (so that $\kappa_1 = \kappa_2 = \psi_1 = \psi_2 = 0$), and adding the BC's:

$$
\begin{aligned}
&N_k = 0 && or \quad u_k = 0 \\
&EI_k \frac{\partial^2 v_k}{\partial x^2} = 0 && or \quad \frac{\partial v_k}{\partial x} = 0 \quad k = 1,2 \\
&-EI_k \frac{\partial^3 v_k}{\partial x^3} + N_k \frac{\partial v_k}{\partial x} + Gb \frac{h_k^*}{2} \frac{1}{t}\left[u_2 - u_1 + \frac{\partial v_1}{\partial x}\frac{h_1^*}{2} + \frac{\partial v_2}{\partial x}\frac{h_2^*}{2} \right] = 0 && or \quad v_k = 0
\end{aligned}
\tag{8}
$$

The solution is sought in a dimensionless form. The following groups are introduced

$$
\begin{aligned}
&\xi = \frac{x}{l} \qquad \bar{u}_1 = \frac{u_1}{l} \qquad \bar{u}_2 = \frac{u_2}{l} \qquad \bar{\varpi}_1 = \frac{h_1^*}{2l}\frac{v_1}{l} \qquad \bar{\varpi}_2 = \frac{h_2^*}{2l}\frac{v_2}{l} \\
&\bar{k} = \frac{G}{E}\frac{l^2}{h_1^2}\frac{h_1}{t} \qquad \lambda = \frac{l}{h_1} \qquad \zeta = \frac{h_1}{h_2} \qquad \varepsilon_1 = \frac{h_1^*}{h_1} \qquad \varepsilon_2 = \frac{h_2^*}{h_2} = 1 + (\varepsilon_1 - 1)\zeta \\
&\bar{q}_{1x} = \frac{q_{1x}\lambda}{Eb} \qquad \bar{q}_{2x} = \frac{q_{1x}\lambda\zeta}{Eb} \qquad \bar{q}_{1y} = 6\frac{q_{1y}\lambda^2\varepsilon_1}{Eb} \qquad \bar{q}_{2y} = 6\frac{q_{2y}\zeta^2\lambda^2\varepsilon_2}{Eb}
\end{aligned}
\tag{9}
$$

With this position the equilibrium equations take the form, indicating with a prime the derivation w.r.t. ξ,

$$
\begin{aligned}
&-\underbrace{\left(\bar{u}_1' + \underline{\frac{2\lambda^2}{\varepsilon_1^2}\bar{\varpi}_1'^2} \right)}_{\bar{N}_1}{}' - \bar{k}\left(\bar{u}_2 - \bar{u}_1 + \bar{\varpi}_1' + \bar{\varpi}_2'\right) = \bar{q}_{1x} \\
&-\underbrace{\left(\bar{u}_2' + \underline{\frac{2\lambda^2\zeta^2}{\varepsilon_2^2}\bar{\varpi}_2'^2} \right)}_{\bar{N}_2}{}' + \bar{k}\zeta\left(\bar{u}_2 - \bar{u}_1 + \bar{\varpi}_1' + \bar{\varpi}_2'\right) = \bar{q}_{2x} \\
&\bar{\varpi}_1'''' \underline{-12\lambda^2\left(\bar{N}_1\bar{\varpi}_1'\right)'} - 3\bar{k}\,\varepsilon_1^2\left(\bar{u}_2 - \bar{u}_1 + \bar{\varpi}_1' + \bar{\varpi}_2'\right)' - 24\frac{1-\nu}{1-2\nu}\bar{k}\,\lambda^2\left(\bar{\varpi}_2\frac{\varepsilon_1}{\varepsilon_2}\zeta - \bar{\varpi}_1\right) = \bar{q}_{1y} \\
&\bar{\varpi}_2'''' \underline{-12\lambda^2\zeta^2\left(\bar{N}_2\bar{\varpi}_2'\right)'} - 3\bar{k}\zeta\varepsilon_2^2\left(\bar{u}_2 - \bar{u}_1 + \bar{\varpi}_1' + \bar{\varpi}_2'\right)' + 24\frac{1-\nu}{1-2\nu}\bar{k}\,\lambda^2\zeta^2\frac{\varepsilon_2}{\varepsilon_1}\left(\bar{\varpi}_2\frac{\varepsilon_1}{\varepsilon_2}\zeta - \bar{\varpi}_1\right) = \bar{q}_{2y}
\end{aligned}
\tag{10}
$$

With dimensionless BC's (for the second layer there are slight straightforward modifications)

$$\begin{aligned}
&\overline{N}_1 = 0 && or \quad \overline{u}_1 = 0 \\
&\frac{\partial^2 \overline{\omega}_1}{\partial \xi^2} = \frac{M_1}{Ebh_1^2}\frac{\varepsilon_1}{2} && or \quad \frac{\partial \overline{\omega}_1}{\partial \xi} = 0 \\
&-\frac{\partial^3 \overline{\omega}_1}{\partial \xi^3} + 12\overline{N}_k \lambda^2 \frac{\partial \overline{\omega}_1}{\partial \xi} + 3\overline{k}\varepsilon_1^2 \left(\overline{u}_2 - \overline{u}_1 + \frac{\partial \overline{\omega}_1}{\partial \xi} + \frac{\partial \overline{\omega}_2}{\partial \xi} \right) = 6 \frac{Q_1}{Ebh_1} \varepsilon_1 \lambda_1 && or \quad \overline{\omega}_1 = 0
\end{aligned} \tag{11}$$

Equations (10) are non linear, the non linear terms having been indicated by an underline. They allow stability analysis for different kinds of BC's. It has to be observed that, even in the case of natural BC's for the axial forces (i.e., for simply supported beams), the non linear terms do not disappear, as it happens in the case the curvature of the two layers is taken constant ([4]). Indeed, form the first two of (10)one gets $\overline{N}_1' + \overline{N}_2' = 0$, but the variation of the individual axial forces remains unknown. Therefore the equations obtained allow a richer stability analysis, that includes the possibility of delamination, that can occur at high temperatures. On the other side, close form solution of (10) is quite impervious, so some further approximation is considered.

3.2 *Linear formulation*

An insight in the performance of the laminated structure can be obtained from the linear equations, obtained from (10) disregarding the underlined terms. They are amenable to a closed form solution, that can be obtained summing the 3rd and 4th equations and subtracting the 3rd from the 4th, after multiplying the 4th by $\zeta\varepsilon_1 / \varepsilon_2$. Then the sum $(\overline{\omega}_1 + \overline{\omega}_2)'$ is obtained from the 1st of (10), and from the second it is obtained the condition

$$\overline{k}\,\gamma = -\frac{\partial^2 \overline{u}_1}{\partial \xi^2} - \overline{q}_{1x} = \frac{1}{\zeta}\left(\frac{\partial^2 \overline{u}_2}{\partial \xi^2} + \overline{q}_{2x} \right) \tag{12}$$

The set of equations obtained is

$$\begin{aligned}
&-\frac{\overline{u}_1'''''}{\overline{k}} + \left(1 + \zeta + 3(\varepsilon_1^2 + \zeta\varepsilon_2^2)\right)\overline{u}_1''' + \alpha(\mu - 1)\Delta = \overline{q}_{1y} + \overline{q}_{2y} - \left(\zeta + 3(\varepsilon_1^2 + \zeta\varepsilon_2^2)\right)\overline{q}_{1x}' - \overline{q}_{2x}' + \frac{\overline{q}_{1x}'''}{\overline{k}} \\
&\Delta'''' + 3\left(\beta\,\zeta\varepsilon_2^2 - \varepsilon_1^2\right)\left(\overline{u}_1''' - \overline{q}_{1x}\right) + \alpha\left[1 + \mu\,\beta\right]\Delta = \beta\,\overline{q}_{2y} - \overline{q}_{1y} \\
&\alpha = 24\frac{1-\nu}{1-2\nu}\overline{k}\lambda^2 \qquad \beta = \frac{\varepsilon_1}{\varepsilon_2}\zeta \qquad \mu = \frac{\varepsilon_2}{\varepsilon_1}\zeta^2
\end{aligned} \tag{13}$$

The solution is sought obtaining $\overline{u}_1'''$ form the 2nd of (13) and substituting into the 1st, so to get a 6th order equation in Δ, that can be solved analytically. However, the coefficients of the solution are quite involved, and of difficult interpretation. Therefore it is examined a simpler case, that is the two layers are of equal thickness. In this case one has $\varepsilon_1 = \varepsilon_2$, $\zeta = \beta = \mu = 1$ and equations (13) get uncoupled, and reduce to

$$\begin{aligned}
&-\frac{\overline{u}_1'''''}{\overline{k}} + \left(2 + 6\varepsilon^2\right)\overline{u}_1''' = \overline{q}_{1y} + \overline{q}_{2y} - \left(1 + 6\varepsilon^2\right)\overline{q}_{1x}' - \overline{q}_{2x}' + \frac{\overline{q}_{1x}'''}{\overline{k}} \\
&\Delta'''' + \alpha\,2\Delta = \overline{q}_{2y} - \overline{q}_{1y}
\end{aligned} \tag{14}$$

(however, they remain coupled in the BC's (11)). The solution is then given by, in the case only transverse load is present (horizontal plate, or dominant external pressure),

$$\begin{aligned}
&\bar{u}_1 = \{c_1, c_2, c_3, c_4, c_5\} \cdot \left\{1, \xi, \xi^2, e^{\sqrt{\eta}\xi}, e^{-\sqrt{\eta}\xi}\right\} + \bar{u}_{1P} \qquad \eta = \bar{k}\left(2 + 6\varepsilon^2\right) \\
&\bar{u}_2 = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\} \cdot \left\{-1, -\xi, -\xi^2, -e^{\sqrt{\eta}\xi}, -e^{-\sqrt{\eta}\xi}, 1, \xi\right\} - \bar{u}_{1P} \\
&\Delta = \{c_1, c_2, c_3, c_4\} \cdot \left\{e^{\bar{\alpha}\xi}\mathrm{Cos}\bar{\alpha}\xi . e^{-\bar{\alpha}\xi}\mathrm{Cos}\bar{\alpha}\xi, e^{\bar{\alpha}\xi}\mathrm{Sin}\bar{\alpha}\xi, e^{-\bar{\alpha}\xi}\mathrm{Sin}\bar{\alpha}\xi\right\} + \Delta_P \qquad \bar{\alpha} = (2\alpha)^{-1/4}
\end{aligned} \tag{15}$$

From The 1^{st} of (12) the sum of the transverse displacements is obtained in order to complete the solution. The bending moments and the other relevant stresses are obtained from (4). It is underlined that the resultant bending moment is given by the sum of the bending moments in the two layers plus the contribution of the normal forces.

It is observed that while the longitudinal displacements depend on the parameter $\bar{k}$, the differential transverse displacement is of oscillatory nature and depends on the parameter α, which accounts for the ratio of the Young modulus of the interlayer and the Young modulus of glass. If this ratio gets very small (order of 10^{-5} or less), the term becomes relevant.

The main part of the displacement depends, on the contrary, on the parameter $\bar{k}$, defined in (9), that accounts for the ratio *G*/*E*, but also on some geometrical factors, primarily the slenderness of the plate. It is expected, then, that the stress distribution in the beam does not depend only on the stiffness of the interlayer.

4 RESULTS

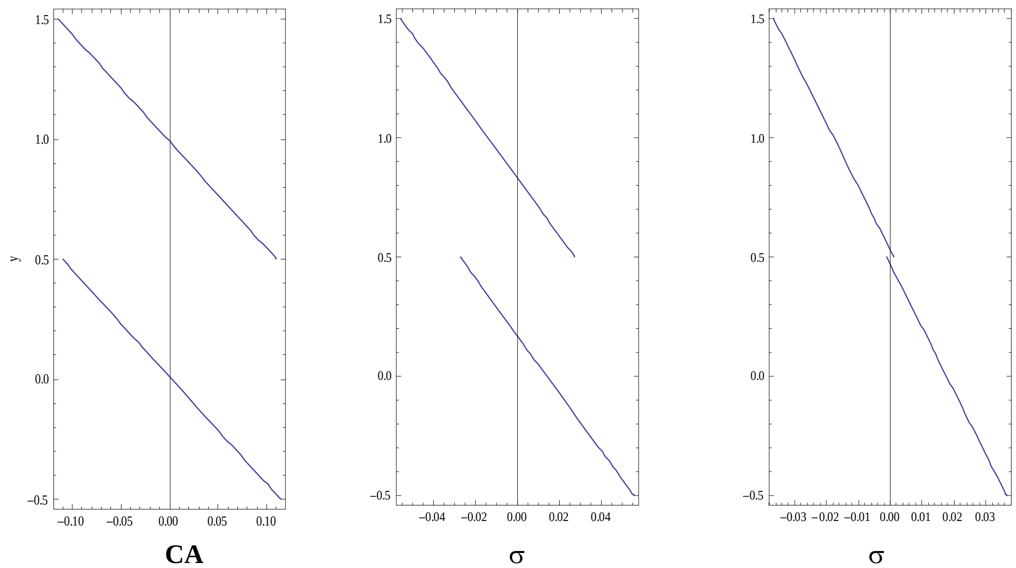
In order to investigate the influence of the parameters defined in section 3 on the response of a 2-layers laminated glass, it is first evaluated the case that the parameter a be negligible, so that the two layers have essentially the same transverse displacement. The response is then influenced by the parameter $\bar{k}$. In table 1 are summarized the geometrical and material parameters used for 3 cases examined; it can be seen that the parameter ranges in 5 orders of magnitude, and is extremely influenced by the slenderness of the plate. Figure 2 shows that the parameter $\bar{k}$ rules the degree of coupling between the two layers.

Table 1. Geometric and materials parameters used in the calculations

	G (MPa)	E (MPa)	L (mm)	h_1 (mm)	t (mm)	k
CASE 1	0.53 (T=30°C long duration load)	73000	500	10	1.52	0.12
CASE 2	7.6 (T=10°C long duration load)	70000	500	6	0.76	5.95
CASE 3	212.8 (T=10°C short duration load)	62000	1000	2.5	0.37	3710.55

As k increases, the beam tends to behave as a monolithic one, and this can happen either for stiff interlayer, or for a very slender plate. Figure 3 presents the shear occurring between the glass layers. For monolithic beam, they are concentrated at the ends of the beam. The end conditions are quite important, then, on the response of the beam.

Let's consider a very slender ($\bar{k} \approx 10^2$) simply supported beam pinned at the ends, in such a way that longitudinal displacements are not allowed. The solution is presented in fig.4, where the stress σ_x at the centre section and in a section close to the support is compared. The beam behaves mostly like a monolithic one, but at the beam ends the stress changes sign, as would happen in a built in section. This effect is due to the fact that the longitudinal displacement is constrained at both ends, while the rotation is not. Note that in order to have this effect it is sufficient that only the lower layer be constrained against longitudinal displacements. A further effect is that the degree of coupling between the layers decreases near the supports.



CASE 1	**CASE 2**	**CASE 3**
k = 0.119412	**k= 5.95238**	**k=3710.55**

Figure 2: Simply supported beam with distributed load: stresses at centre section.

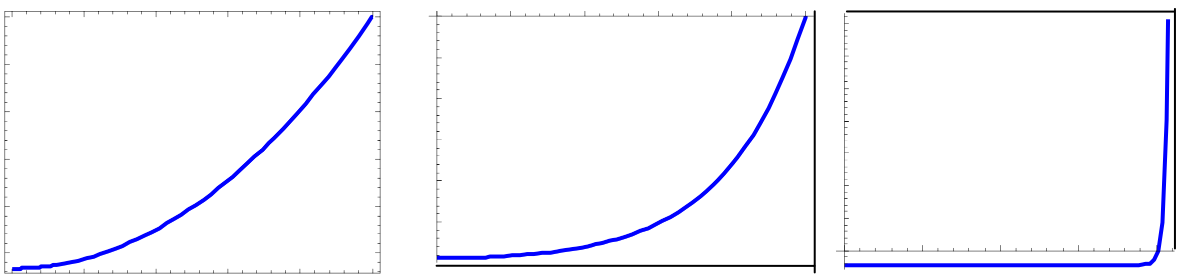


Figure 3: Simply supported beam with distributed load: shear deformation on half beam.

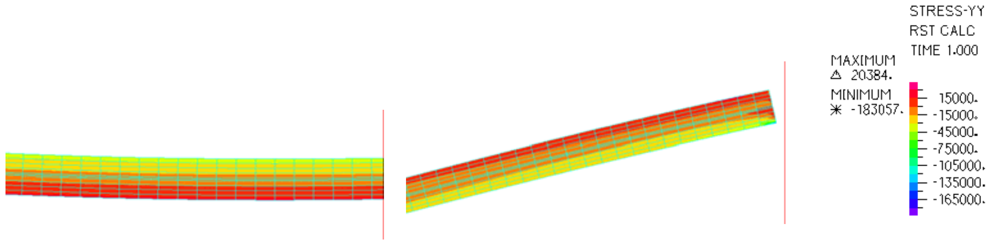


Figure 4: Simply supported beam with distributed load pinned ends: longitudinal stress in the centre section and in a section close to the support

The effect of the partial debonding between the layers can be seen in figure 5, that presents the stresses in a simply supported beam relative to case 1. The maximum stress is reached in the upper layer, contrary to what happened in the previous case. This is due to the fact that the two layers have different curvature, as can be observed from the deformed shape of the plate. Low degree of coupling occurs in this case. The observation can give some explanation of why in same cases fracture occurs in the upper layer rather than in the lower one.

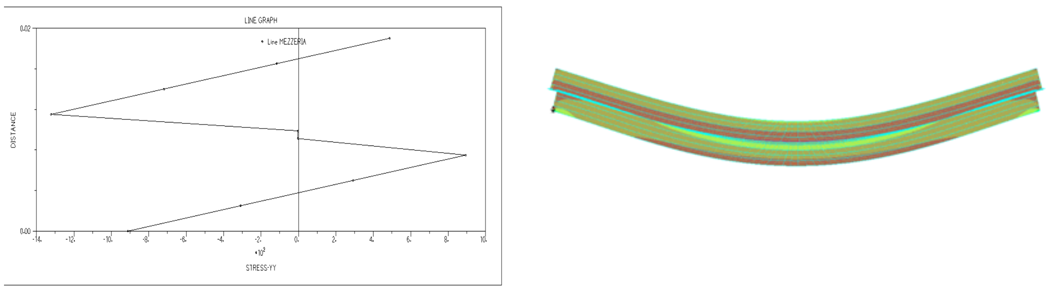


Figure 5: Simply supported beam with distributed load: uncoupled case.

5 CONCLUSIONS

It has been presented a simplified model for the bending behaviour of laminated glass. Only the case of cylindrical bending has been considered, although the model can be extended to biaxial bending. It has been considered the possibility of different curvatures in the two layers, as has been shown that occurs in short beams. The non linear equations of the beam have been presented, in a form useful for stability analysis. It has been presented a complete closed form solution for the linear case. The response of the structure is dominated by a dimensionless parameter, $\bar{k}$, that depends on the ratio between the shear modulus of the polymer and the Young modulus of the glass, and on the square of the slenderness ratio of the beam. It rules the degree of coupling between the two layers. The debonding is ruled by another dimensionless parameter, that depends on the ratio of the confined Young modulus of the polymer with respect to the Ypoung modulus of the glass and also on the square of the slenderness ratio. Finally it has been stressed the strong effects of the BC's on the type of stress distribution. The BC's are coupled between the transverse and the longitudinal displacements, and different results are obtained whether the actual constraints is applied to the lower or the upper plate or to both.

Only the linear elastic case has been considered, since main purpose of the work was to analyze the limiting conditions. However viscoelastic constitutive equations for the polymer could be considered without modifying the model, for transient analysis.

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