A finite element cable for the analysis of cable nets

Massimo Cuomo[1], Leopoldo Greco[1]
[1] *Faculty of Engineering, University of Catania, Italy*
E-mail: mcuomo@dica.unict.it
E-mail: leopoldo.greco@virgilio.it

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SUMMARY. The paper addresses the exact formulation of a deformable catenary element permitting the numerical simulation of cable net systems. The formulation proposed, stemming as a modification of the conventional equations for inextensible cables, ensures exact equilibrium after deformation of cable and includes the case of follower loads. The use of analytical expression for both the geometrical description of the element and the corresponding linearized problem provides high numerical stability in the calculation procedure. The accuracy and efficiency of this approach are assessed by comparison with simple results obtained in literature using alternative analytical and numeric approaches.

1 INTRODUCTION

Object of the present study are structures made by cables. Many engineering structures make use of this typology, among them, suspension and cable stayed bridges, tenso-structures, light roof systems [1]. The latter are often built with double-curvature membranes, which however, are in some case modelled as cable nets.

Cables are intrinsically non linear elements, with constitutive non linearity due to the no-compression behavior, and geometrical non linearities. The first property has long been recognized and has sometimes been used also for modelling masonry structures such as arches and columns. Recently Andreu [2] et alt. have proposed a numerical model for cables in order to analyze the equilibrium configurations of historical buildings.

The classical solution of the catenary for a single cable due to J. Bernoulli has been extended to account for several other effects. Irvine [3] used the equation of deformable catenary for studying plane cable nets. He obtaines an expression for the tangent stiffness matrix of an elastic cable, that, however, was not symmetric. Following his work, a number of authors have developed numerical procedures for the analysis of complex type of cable-structures. Ahmadi-Kashani and Bell [4] with the aim of studying cable trusses, extended Irvine's model to account for follower loads, like these due to snow and wind, following the original work of Peyrot [5], who introduced the so called associated catenary element. They also presented a numerical strategy of solution with the aim of obtaining an exact expression for the equivalent nodal forces. Further contributes toward a general finite element cable are due to Tibert [6]. Lacarbonara and Pacitti [7] who extended the formulation to the case of cables with non negligible bending stiffness.

Most of the available solutions, however, show numerical instability and poor efficiency, or are available only for particular structural typologies.

Aim of the authors is to develop an exact formulation of a cable element, that includes also the case of follower loads, like those due to fluid interactions (the later are relevant in sub-marine cables). Particular attention will be devoted to the efficiency of the numerical procedure, in order to develop a non linear finite element cable that can be used also in conjunction with much stiffer standard elements, as occurs in many structures.

2 EQUILIBRIUM EQUATIONS

2.1 *Variational principle of a cable-element*

Let $\boldsymbol{p} = \boldsymbol{p}(s)$ be the parametric configuration of the cable at a generic instant, with s the arc-length. The tangent space $T_{\boldsymbol{p}}\mathcal{B}_t$ at point $\boldsymbol{p}$ is generated by the unitary triad constituted by the tangent vector $\hat{\boldsymbol{t}} := \partial_s \boldsymbol{p}$, the unit normal $\hat{\boldsymbol{n}} := \frac{\partial_s \boldsymbol{p}}{\|\partial_s \boldsymbol{p}\|}$ and the unit bi-nomal vector $\hat{\boldsymbol{b}} := \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$. We denote with $\boldsymbol{\tau}$ the normal resultant stress vector defined by $\boldsymbol{\tau} := \tau\hat{\mathbf{t}}$, where $\tau = \|\boldsymbol{\tau}\|$. For any virtual displacement $\boldsymbol{v}$ the principle of virtual work is given by

$$\int_0^L \boldsymbol{\tau} \cdot \partial_s(\boldsymbol{v})\ ds = \int_0^L \boldsymbol{q} \cdot \boldsymbol{v}\ ds + \mathbf{F}_0 \cdot \boldsymbol{v}_0 + \mathbf{F}_L \cdot \boldsymbol{v}_L \tag{1}$$

integrating the first term of (1) we have

$$[\boldsymbol{\tau} \cdot \boldsymbol{v}]_0^L - \int_0^L \partial_s(\boldsymbol{\tau}) \cdot \boldsymbol{v}\ ds = \int_0^L \boldsymbol{q} \cdot \boldsymbol{v}\ ds + \mathbf{F}_0 \cdot \boldsymbol{v}_0 + \mathbf{F}_L \cdot \boldsymbol{v}_L. \tag{2}$$

The field equations in $[0, L]$, are

$$-\partial_s(\boldsymbol{\tau}) = \boldsymbol{q} \tag{3}$$

and the boundary conditions are

$$\begin{aligned} \boldsymbol{\tau}(0) = -\mathbf{F}_0 \quad & or \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 \\ \boldsymbol{\tau}(L) = \mathbf{F}_L \quad & or \quad \boldsymbol{v}(L) = \boldsymbol{v}_L. \end{aligned} \tag{4}$$

From the last condition the boundary forces must be tangent to the the boundary configuration of the cable.

2.2 *Intrinsic representation of plane equilibrium equations*

Let $\boldsymbol{v} \in T_{\boldsymbol{p}}\mathcal{B}_t$ be a generic tangent vector, $\boldsymbol{v} = (\boldsymbol{v} \cdot \hat{\boldsymbol{t}})\hat{\boldsymbol{t}} + (\boldsymbol{v} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + (\boldsymbol{v} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$. The intrinsic line gradient of $\boldsymbol{v}$ is indicated by $grad_{\|}(\boldsymbol{v})$ and is defined by

$$\begin{aligned} grad_{\|}(\boldsymbol{v}) &= \partial_{\boldsymbol{s}}(\boldsymbol{v}) \otimes \hat{\boldsymbol{t}} \\ &= (\partial_s \boldsymbol{v} \cdot \hat{\boldsymbol{t}})\hat{\boldsymbol{t}} \otimes \hat{\boldsymbol{t}} + (\partial_s \boldsymbol{v} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{t}} + (\partial_s \boldsymbol{v} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}} \otimes \hat{\boldsymbol{t}}. \end{aligned} \tag{5}$$

Projecting the equilibrium equation (3) in the intrinsic tangent space we have

$$-\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{t}} = \boldsymbol{q} \cdot \hat{\boldsymbol{t}} = q_{\hat{t}}, \quad -\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} = \boldsymbol{q} \cdot \hat{\boldsymbol{n}} = q_{\hat{n}}, \quad -\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{b}} = \boldsymbol{q} \cdot \hat{\boldsymbol{b}} = q_{\hat{b}}. \tag{6}$$

Using Frenet's formula and considering that $\boldsymbol{\tau} = \tau\hat{\boldsymbol{t}}$ the component of $grad_{\|}(\boldsymbol{\tau})$ are

$$\partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{t}} = \partial_s \tau, \quad \partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} = \tau\,\chi, \quad \partial_s \boldsymbol{\tau} \cdot \hat{\boldsymbol{b}} = 0, \tag{7}$$

where $\chi = \|\partial_s \boldsymbol{\tau}\|$ is the curvature of the funicular curve.

Finally the intrinsic representation of the equilibrium equations (3) are

$$\begin{aligned} -\partial_s \tau(s) &= q_{\hat{t}}(s) \\ -\tau(s)\chi(s) &= q_{\hat{n}}(s) \\ q_{\hat{b}}(s) &= 0, \end{aligned} \tag{8}$$

with the boundary conditions

$$\begin{aligned} \tau(0) = -\boldsymbol{\tau}(0) \cdot \hat{\boldsymbol{t}} \qquad & \boldsymbol{v}(0) = \boldsymbol{v}_0 \\ \tau(L) = \boldsymbol{\tau}(L) \cdot \hat{\boldsymbol{t}} \qquad & \boldsymbol{v}(L) = \boldsymbol{v}_L. \end{aligned} \tag{9}$$

2.3 *Cartesian representation of the equilibrium equations*

Projecting the equations (3) on the Euclidean spatial frame we obtain, (noting that $\partial_s \boldsymbol{\tau} \cdot \boldsymbol{e}_i = \partial_s(\boldsymbol{\tau} \cdot \boldsymbol{e}_i) \quad \forall i = 1, 2, 3.$)

$$-\partial_s(\tau\hat{\mathbf{t}} \cdot \mathbf{e}_x) = \boldsymbol{q} \cdot \mathbf{e}_x, \quad -\partial_s(\tau\hat{\mathbf{t}} \cdot \mathbf{e}_y) = \boldsymbol{q} \cdot \mathbf{e}_y, \quad -\partial_s(\tau\hat{\mathbf{t}} \cdot \mathbf{e}_z) = \boldsymbol{q} \cdot \mathbf{e}_z, \tag{10}$$

and remembering the definition of $\hat{\mathbf{t}} = \partial_s x \mathbf{e}_x + \partial_s y \mathbf{e}_y + \partial_s z \mathbf{e}_z$ we obtain

$$-\frac{\partial}{\partial s}\left(\tau(s)\frac{\partial x}{\partial s}\right) = q_x(s), \quad -\frac{\partial}{\partial s}\left(\tau(s)\frac{\partial y}{\partial s}\right) = q_y(s), \quad -\frac{\partial}{\partial s}\left(\tau(s)\frac{\partial z}{\partial s}\right) = q_z(s). \tag{11}$$

The projection of the internal traction stress resultant $\boldsymbol{\tau}$ along the cartesian directions are usually called thrust and shear

$$\mathcal{H}(s) = \boldsymbol{\tau} \cdot \mathbf{e}_x = \tau(s)\frac{\partial x}{\partial s}(s), \quad \mathcal{K}(s) = \boldsymbol{\tau} \cdot \mathbf{e}_y = \tau(s)\frac{\partial y}{\partial s}(s), \quad \mathcal{V}(s) = \boldsymbol{\tau} \cdot \mathbf{e}_z = \tau(s)\frac{\partial z}{\partial s}(s). \tag{12}$$

The cartesian equilibrium equations assume the compact form

$$-\partial_s \mathcal{H}(s) = q_x(s), \quad -\partial_s \mathcal{K}(s) = q_y(s), \quad -\partial_s \mathcal{V}(s) = q_z(s). \tag{13}$$

By a first integration along s we have

$$\mathcal{H}(s) = \mathcal{H}(0) - \int_0^s q_x(s)ds, \quad \mathcal{K}(s) = \mathcal{K}(0) - \int_0^s q_y(s)ds, \quad \mathcal{V}(s) = \mathcal{V}(0) - \int_0^s q_z(s)ds. \tag{14}$$

A new integration along s yelds the parametric representation of the funicular configuration

$$x(s) = \int_0^s \frac{\mathcal{H}(0) - \int_0^s q_x(s)ds}{\tau(s)}\, ds - x(0), \quad y(s) = \int_0^s \frac{\mathcal{K}(0) - \int_0^s q_y(s)ds}{\tau(s)}\, ds - y(0),$$
$$z(s) = \int_0^s \frac{\mathcal{V}(0) - \int_0^s q_z(s)ds}{\tau(s)}\, ds - z(0), \tag{15}$$

where the resultant stress traction is defined by

$$\tau(s) = \sqrt{\left(\mathcal{H}(0) - \int_0^s q_x ds\right)^2 + \left(\mathcal{K}(0) - \int_0^s q_y ds\right)^2 + \left(\mathcal{V}(0) - \int_0^s q_z ds\right)^2} \tag{16}$$

3 FORMULATION OF AN ELASTIC CATENARY ELEMENT

In this section it is shown the particularization of the equilibrium equation to the case of an elastic catenary obeying Hooke's law, suspended at its ends and subjected only to its own weight.

A discussion on a wide variety of elastic catenaries can be found in [6], [3] and [4].

3.1 *Assumptions*

The limitations of the present formulation are:

1. Small deformation only are considered (but large displacements).
2. Linear-elastic constitutive behavior is only considered ($\tau = EA_0\, \varepsilon$).

3. Only self-weight acts on the cable, and it is assumed conservation of mass of the cable element during the deformation process, i.e. the value of the weight per unit-length varies in agreement with the mass conservation.

4. Bending stiffness is neglected.

5. Plane funicular geometries, only are considered for simplicity ($\hat{\boldsymbol{b}}$ defines the plane of the funicular curve).

3.2 *Equations of the elastic cable element*

A total Lagrangian approach is used. As reference configuration we adopt the inextensible catenary configuration of the cable and denote with $s_0 \in [0, L_0]$ the arc-length coordinate, with L_0 the length of the non-deformed cable.

Since we consider that the only external action is the self weight along the z-direction we have from equations (14)

$$\mathcal{H}(s) = \mathcal{H}(0), \quad -\partial_s \mathcal{V}(s) = q_z(s). \tag{17}$$

Equation (13) reduces to

$$\tau(s)\frac{\partial x}{\partial s} = \mathcal{H}(0), \quad -\frac{\partial}{\partial s}\left(\tau(s)\frac{\partial z}{\partial s}\right) = q_z(s), \tag{18}$$

with

$$\tau(s_0) = \sqrt{\mathcal{H}_0^2 + \left(\mathcal{V}_0 - \int_0^{s_0} q_z \frac{\partial s}{\partial s_0} ds_0\right)^2} = \sqrt{\mathcal{H}_0^2 + \left(\mathcal{V}_0 - \frac{W}{L_0}s_0\right)^2} \tag{19}$$

where mass conservation $q_{z,0}ds_0 = q_z ds$, have been used, and W denotes the total weight of the cable. Integrating the preceding equations on the Lagrangean configuration we have

$$\begin{aligned} x(s_0) - x_0 &= \int_0^{s_0} \frac{\mathcal{H}_0}{\tau(s)} \left.\frac{\partial s}{\partial s_0}\right|_{s(s_0)} ds_0 \\ z(s_0) - z_0 &= \int_0^{s_0} \frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\tau(s)} \left.\frac{\partial s}{\partial s_0}\right|_{s(s_0)} ds_0. \end{aligned} \tag{20}$$

Considering that $\varepsilon = (\frac{ds}{ds_0} - 1)$ and assuming a linear constitutive relation $\tau = A_0 E\left(\frac{ds}{ds_0} - 1\right)$ the previous equations become

$$\begin{aligned} x(s_0) &= \int_0^{s_0} \left(\frac{\mathcal{H}_0}{EA_0} + \frac{\mathcal{H}_0}{\tau(s_0)}\right) ds_0 + x_0 \\ z(s_0) &= \int_0^{s_0} \frac{V_0 - \frac{W}{L_0}s_0}{\tau(s_0)} \left(\frac{\tau(s_0)}{EA_0} + 1\right) ds_0 + z_0 \end{aligned} \tag{21}$$

and integrating we have

$$x(s_0) - x_0 = \frac{\mathcal{H}_0 s_0}{EA_0} + \frac{\mathcal{H}_0 L_0}{W}\left(\mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0}{\mathcal{H}_0}\right] - \mathrm{Sinh}^{-1}\left[\frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\mathcal{H}_0}\right]\right) \tag{22}$$

$$
\begin{aligned}
z(s_0) - z_0 = \frac{W s_0}{EA_0}\left(\frac{\mathcal{V}_0}{W} - \frac{s_0}{2L_0}\right) + \\
+ \frac{\mathcal{H}_0 L_0}{W}\left[\sqrt{1+\left(\frac{\mathcal{V}_0}{\mathcal{H}_0}\right)^2} - \left(\sqrt{1+\left(\frac{\mathcal{V}_0 - \frac{W}{L_0}s_0}{\mathcal{H}_0}\right)^2}\right)\right].
\end{aligned}
\tag{23}
$$

The kinematic of the elastic catenary element is described by equation (22) and (23), which are resumed as

$$
x(L_0) - x_0 = l(L_0) = f(\mathcal{H}_0, \mathcal{V}_0, L_0), \quad z(L_0) - z_0 = h(L_0) = g(\mathcal{H}_0, \mathcal{V}_0, L_0). \tag{24}
$$

The function f and g are non linear functions of $\mathcal{H}_0$, $\mathcal{V}_0$ and L_0. We note that for a *i*-th cable of length $L_{0,i}$ and weight W_i assigned, the *i*-th funicular configuration is defined by six parameter, four geometric $\boldsymbol{p}_0^i = \{x_0^i, z_0^i\}$ and $\boldsymbol{p}^i(L_0) = \{x^i(L_0), z^i(L_0)\}$, and two static $\boldsymbol{\tau}_0^i = \{\mathcal{H}_0^i, \mathcal{V}_0^i\}$. If a net of catenary elements is cut at the connecting nodes, each isolated element will be in equilibrium via non linear equations. The conditions of global equilibrium and kinematic compatibility are used to derive the global equations of the entire net of cables. Overall equilibrium requires the balance of all the forces appearing at the ends of catenary elements connected to a node with the external loads applied on the node.

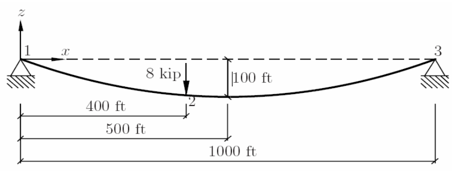
Enforcing the compatibility and equilibrium equations at each node the solution of the elastic cable net is determined. A Newtow-Raphson numerical scheme is adopted to solve equations (24). The initialization of this procedure is made linearizing the set of equations using as initial trial configuration the parabolic inextensible cable element. This strategy provides a method with large numerical robustness permitting a very accurate treatment of the equilibrium in the final configuration of the cable net, as will be show by the examples in the next section.

4 NUMERICAL EXAMPLES

In this section first we show simple benchmarks and then analyze different more complex typologies of cable nets.

4.1 *Example 1*

The present example, taken from Tibert [6], is a benchmark already considered by other authors to validate different methods for simulating cables,(O'Brien [8], and Jayaraman [9]). The initial



(a)

Description		Magnitude
Cross-sectional area of cable		0.85 in^2
Equivalent modulus of elasticity		19000 kips/in^2
Self weight of cable		3.16 lb/ft
Unstrained length	Segment 1–2	412.8837 ft
	Segment 2–3	613.0422 ft

(b)

Figure 1: 1(a) Geometry description and 1(b) data,(from A. Andreu, L. Gil, P. Roca.).

configuration is a suspended elastic catenary subjected only to self weight, and data can be found in figure 1. Successively a non symmetric point force is applied.

In the figure 2 we show the initial inextensible configuration of the catenary subjected to self weight (in blue), the initializing inextensible parabolic configuration (in purple) and the final elastic catenary configuration.

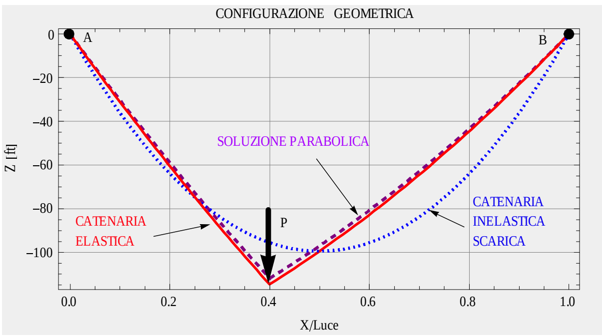


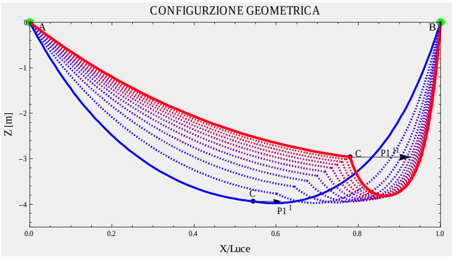
Figure 2: Inextensible initial catenary configuration, initializing inextensible parabolic configuration and elastic catenary configuration.

The horizontal and vertical displacements of the point of the application of the load agree with those given by of O'Brien [8] and Roca [2].

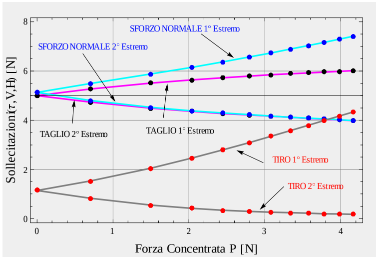
4.2 *Example 2*

In this example we study the behavior of a self-weight cable subjected to an horizontal point force applied on the centre-line. The distance between the ends is $l = 5m$, the length is $L_0 = 10m$.

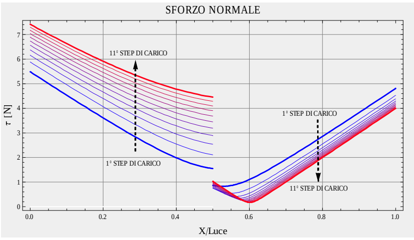
and the self-weight for unity of inextensible length is $q = 1N/m$. In figure-3 the incremental configurations are plotted. As the horizontal force increases, the thrust in the right segment of the



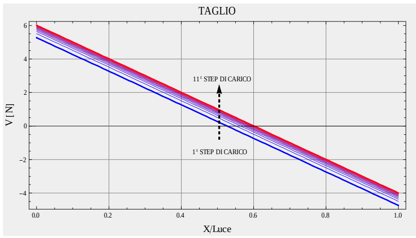
(a)



(b)



(c)



(d)

Figure 3: 3(a) Incremental configurations, 3(b) reactions at the ends, 3(c) stress resultant traction and 3(d) shear component.

cable tends to a constant value, corresponding to the reaction of the isolated segment subjected to its self weight only (see figure 3(b)). If the cables are inextensible, this is a limiting configuration no matter how large is the horizontal load whose increments are completely balanced by the left segment (see figure 3(c)).

4.3 *Example 3*

In this example we study the behavior of a self-weight cable. The distance between the ends is $l = 5m$, the length is $L_0 = 10m$, and the self-weight for unity of inextensible length is $q = 0.5N/m$.

A point load is applied to the cable by a pulley. In the incremental load-process we the vertical component of this force is kept constant and only the horizontal component increases.

From the figure-4 we note that the point of application of the load during the incremental process describes an ellipse having as foci the suspension end-points. The centre of weight of the system describes an ellipse rotated of $\pi/2$. In figure 5 are plotted the incremental horizontal and vertical end-reactions.

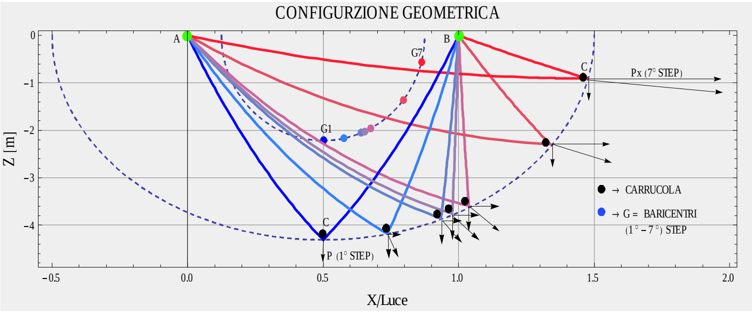
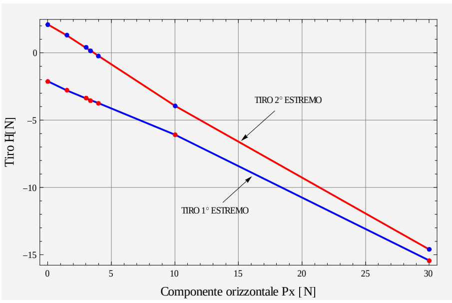
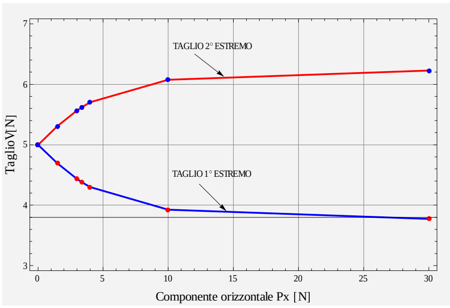


Figure 4: Incremental configuration and ellipse of application point and weight-centre.



(a)



(b)

Figure 5: 5(a) Horizontal and 5(b) vertical end-reactions components.

4.4 *Example 4*

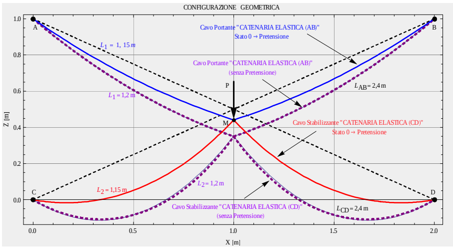
In this example we study the behavior of a self-weight cable net constituted by four cable suspended at the ends and subjected to a point force applied to the junction of cables.

The distance between the ends is $l = 2m$, the initial length of cables is $L_1 = L_2 = 2.4m$, and the self-weight for unity of inextensible length is $q = 1N/m$, $EA = 10^3 N$.

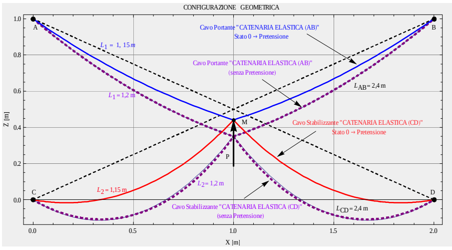
The 0-state of the net is associated to a pretension given by a reduction of the initial lengths of the cables.

The 1-state and 2-state are associated to the different direction of the load (simulating the wind effects).

Form figure 6(c) and 6(d) it appears a different behavior of the net, in fact in the 1-state the upper cable balance the load, while in the 2-state, after the upper cable has released the elastic deformation energy associated to pretension, is the lower cable that balances the load.



(a)



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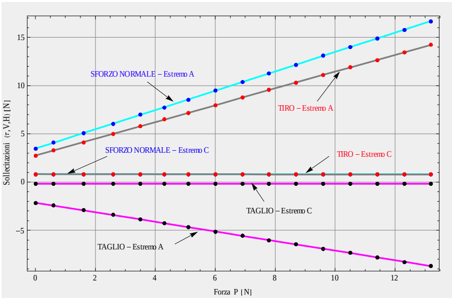

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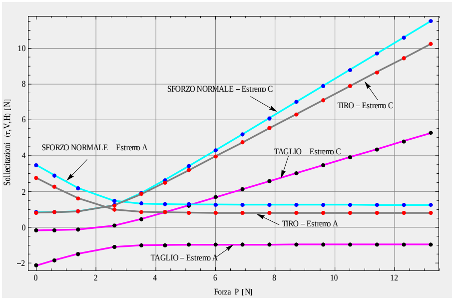

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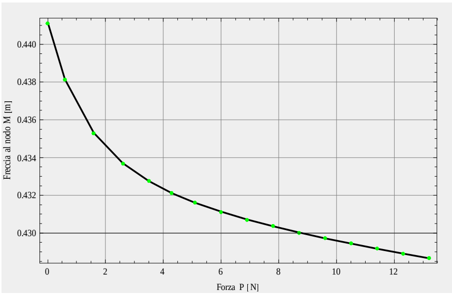
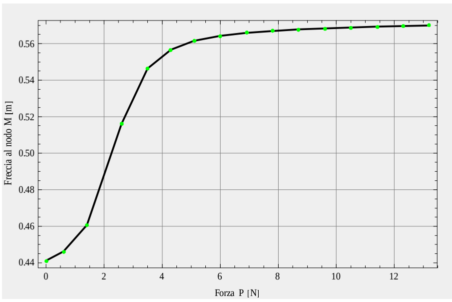


(e)

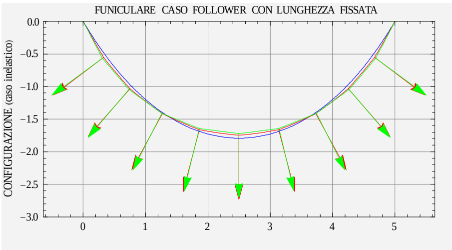


(f)

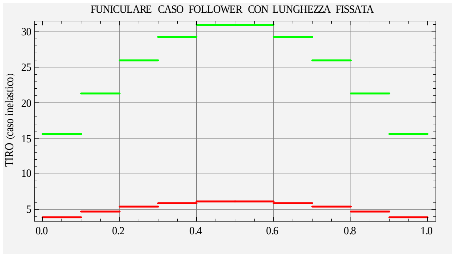
Figure 6: In figure 6(a) and 6(b) are plotted the geometry and loads, in figure 6(c) and 6(d) are plotted the component of reactions and the traction stress at the ends and in figure 6(e) and 6(f) are plotted the vertical coordinate of the loaded point.

4.5 *Example 5*

In this example we analyze a follower load. We consider an inextensible cable initially subjected to only self-weight, in a second step a further distribution of pressure directed along the normal of the funicular curve is applied on the cable. We note that this load is follower only in the direction of application and not in the module of load-pressure, because the cable is considered inextensible. As showed in the figure the initially catenary tends to a circular funicular configuration and the traction stress resultant becomes constant (see figure 7(d)).



(a)



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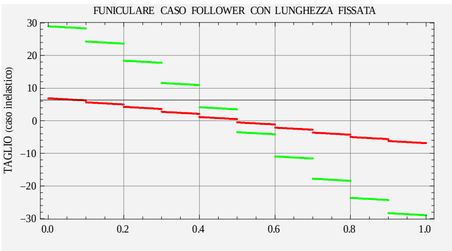

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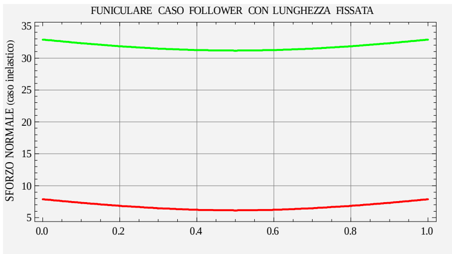

(d)

Figure 7: In figure 7(a) is plotted the change of configuration during the incremental load-process, in figure 7(b) is plotted the horizontal component of traction stress, in figure 7(c) is plotted the vertical component of traction stress and in figure 7(d) is plotted the traction stress.

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