Design strategies for FRP reinforcement of no-tension structures

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SUMMARY. In the paper some preliminary results are shown about the elaboration of a behaviour model of a structure reinforced with composites. For modelling the structure the NRT material is used. The aim of the research is to design the best distribution of reinforcement over the existing structure and some basics of Limit Analysis and of Topology Optimization are used. Finally a first implementation of a search procedure has been performed, with reference to the location and shaping of the steel bars.

1 INTRODUCTION

The complexity in the behavior of a masonry fabric and the relatively recent application of composites in the refurbishment of ancient fabrics in the last tens of years is so that many technical characteristics of the materials are well known, but their application as reinforcement intervention is often casual. The absence of a basic project for the application of a FRP reinforcement (Fiber-Reinforced-Polymers) or a much wide intervention often breaks off the sense in the composites useful as removing materials, in some cases beginning a reinforcement opera much more invasive of the classical one.

Since some years a research about the optimization of the FRP reinforcement applied on a masonry fabric is developed (see e.g. [1]) and in the paper are shown some results about this study. The problem is approached with reference to a two-dimensional continuum Not Resisting Tension (NRT, for details see e.g. [2]) body, that is intended to be assimilated to a masonry wall or to a reinforced concrete beam and/or panel. The structure is modelled through the assumption that the basic material (e.g. masonry or concrete pr anything else) is an elastic material not resisting tension (NRT material) and the reinforcement is an indefinitely elastic sheet, possibly with variable thickness, to be glued over the panel. Moreover by means of the Structural Topology Optimization it is looking for the optimal distribution of resisting material in the interior of a given domain, able to resist given loads, subject to some constraints (e.g. the quantity of material involved, and/or the maximum stress/strain in the material) and aiming at optimizing some performance index or some design objectives. In the topology optimization approach for discrete structures, or for structures conceived as the assemblage of a number of connected components, the problem is set with the objective to identify the number, the dimensions and the arrangement of the members. For continuum systems, essentially the shape of the body able to resist the loads is optimized, and the solution consists in deciding whether any point in the domain Ω where the structure must be included is filled with material or not. This last approach leads to a 0-1 optimization, that may be not always tractable from the point of view of mathematical programming; the problem can be regularized by making recourse to a *density function* $\rho(\mathbf{x})$, that distributes the material with continuity over the domain, despite the fact that in solution the optimal layout may nevertheless be of a well-defined 0-1 type (for more details see e.g. [3, 4, 5]).

In the optimization process, some additional difficulties may arise from the fact that at some steps of the process the absence of material at some key points in the domain may produce singularities in the constitutive equations of the system, thus blocking the solution process. For the case of the reinforcement that is the object of the present paper, this problem should be mitigated, in that the structure to be reinforced is basically existent, and optimization only regards the addition of material. For the case of no-tension panels, anyway, the applied load pattern might be out the load-carrying capacity of the unreinforced body, or even at some intermediate stage of the procedure, and the optimization should be carried on following admissible paths. Similar problems are met with other instances of topology optimization when the structural response is governed by nonlinear relationships (see e.g. [6]).

In the following the fundamentals for reinforced NRT panels will be outlined, and the extension to reinforcement will be treated. The optimum problem for the distribution of the reinforcement will be set up, and some preliminary solution strategies will be outlined.

2 BASICS OF THE NRT PANEL WITH REINFORCEMENT

2.1 *NRT model basics and reinforcement condition*

Let consider the domain Ω, that is occupied by the considered NRT material, subject to surface tractions $\mathbf{p}$ and body forces $\mathbf{f}$, and the contour Γ of the domain Ω, that is subdivided in the constrained part Γ_u, where displacements are imposed and forces correspond to the reactions, and the part Γ_p, where displacements are free and forces are data.

Moreover, let $\mathbf{u}(\mathbf{x})$ and $\varepsilon(\mathbf{x})$ be respectively the displacement field and the total strain field relevant to the current point $\mathbf{x}$ in Ω, coincident both in the basic body and in the reinforcement, and denote by the suffix "b" the basic structural body and the relevant material, and by the suffix "r" the reinforcing material. After the reinforcement has been added, possibly with some graduation, and the forces are applied again, the response of the system is ruled by the following usual compatibility and equilibrium equations.

If $\boldsymbol{\varepsilon}_b(\mathbf{x})$ and $\boldsymbol{\varepsilon}_r(\mathbf{x})$ are the total strain field in the basic body and in the reinforcement, $\boldsymbol{\varepsilon}_{be}(\mathbf{x})$ and $\boldsymbol{\varepsilon}_{bf}(\mathbf{x})$ are the elastic strain field in the basic body and the fracture strain field in the basic body, $\boldsymbol{\sigma}(\mathbf{x})$ is the total stress in the reinforced body, $\boldsymbol{\sigma}_b(\mathbf{x})$ and $\boldsymbol{\sigma}_r(\mathbf{x})$ are the stress in the basic body and the stress in the reinforcement, $\rho(\mathbf{x})$ is the density function of the reinforcement which can be zero in absence of reinforcement and equal to 1 with the reinforcement, so one can write

$$\begin{aligned} &\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_r(\mathbf{x}) \\ &\boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_{be}(\mathbf{x}) + \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \\ &\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x}) \end{aligned} \tag{1}$$

where the symbols $\boldsymbol{\sigma}_b \leq \mathbf{0}$ or $\boldsymbol{\varepsilon}_{bf} \geq \mathbf{0}$ simply mean that the relevant tensors possess non positive or non-negative eigen-values.

Being $\mathbf{u}_t(\mathbf{x})$ the displacements of the constrained part, and $\mathbf{D}_b$, $\mathbf{D}_r$ the elastic tensors of the basic material and of the reinforcement, the basic conditions are

1) the compatibility condition:

$$\begin{aligned} &\boldsymbol{\varepsilon}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega \\ &\mathbf{u}(\mathbf{x}) = \mathbf{u}_t(\mathbf{x}) \qquad \forall \mathbf{x} \in \Gamma_u \end{aligned} \tag{2}$$

2) the stress and strain admissibility conditions: $\left.\begin{array}{r}\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq \mathbf{0} \\ \boldsymbol{\sigma}_b(\mathbf{x}) \leq \mathbf{0}\end{array}\right\} \quad \forall \mathbf{x} \in \Omega$ (3)

3) the constitutive relations: $\left.\begin{array}{l}\boldsymbol{\sigma}_b(\mathbf{x}) = \mathbf{D}_b \boldsymbol{\varepsilon}_{be}(\mathbf{x}) \leq \mathbf{0} \\ \boldsymbol{\sigma}_r(\mathbf{x}) = \mathbf{D}_r \boldsymbol{\varepsilon}_r(\mathbf{x}) = \mathbf{D}_r \boldsymbol{\varepsilon}(\mathbf{x}) \\ \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \mathbf{0}\end{array}\right\} \quad \forall \mathbf{x} \in \Omega$ (4)

For the equilibrium

$$\begin{cases}\mathrm{Div}\,\boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0} & \forall \mathbf{x} \in \Omega \\ \boldsymbol{\sigma}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) & \forall \mathbf{x} \in \Gamma_p\end{cases} \quad \rightarrow \quad \begin{cases}\mathrm{Div}\,\boldsymbol{\sigma}_b(\mathbf{x}) + \mathrm{Div}\left[\rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x})\right] + \mathbf{f}(\mathbf{x}) = \mathbf{0} & \forall \mathbf{x} \in \Omega \\ \left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) & \forall \mathbf{x} \in \Gamma_p\end{cases} \tag{5}$$

with the strength constraint $\boldsymbol{\sigma}_b(\mathbf{x}) \leq \mathbf{0}$.

Generally some statements can be enunciated: i) *compatible displacement fields* are the fields $\mathbf{u}(\mathbf{x})$ obeying the constraint boundary conditions, i.e. $\mathbf{u}(\mathbf{x}) = \mathbf{u}_t(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_u$; ii) *statically admissible* stress fields are defined such that equilibrium and admissibility stress conditions are fully verified; iii) *kinematically compatible* strain fields are defined such that compatibility (i.e. $\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_{be}(\mathbf{x}) + \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$) and admissibility (i.e. $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq \mathbf{0}$) strain conditions are fully verified.

For any couple of admissible stress and fracture fields $\boldsymbol{\sigma}_b(\mathbf{x})$ and $\boldsymbol{\varepsilon}_{bf}(\mathbf{x})$ the inequality $\boldsymbol{\sigma}_b(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \leq \mathbf{0}$ holds.

2.2 *Limit Analysis approach for the solution of a reinforced NRT panel problem*

On the basis of the relations governing the problem, the conditions for equilibrium or collapse under given forces **p**, **f** can treated by an extension of the basic theorems for Limit Analysis of reinforced NRT panels.

Consider basically that any displacement field $\mathbf{u}(\mathbf{x})$, associated to a compatible strain field $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq \mathbf{0}$, is a potential collapse mechanism. So

1) a *kinematically possible* mechanism exists if the external work is positive

$$L_e(\mathbf{u}) = \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) ds + \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) dA > 0 \tag{6}$$

2) a *statically possible* mechanism exists if the external work is non-positive

$$L_e(\mathbf{u}) = \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) ds + \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) dA \leq 0 \tag{7}$$

or considering any compatible mechanism ($\mathbf{u}$, $\boldsymbol{\varepsilon}_{bf}$) with $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$ and $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \Omega$, for the Principle of Virtual Work

1’) $L_e(\mathbf{u}) = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) dV > 0$ (8)

2’) $L_e(\mathbf{u}) = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) dA \leq 0$ (9)

In synthesis, two criteria are stated as basis for deciding if collapse occurs or not:

1) if no statically admissible stress field exists under the given forces, collapse occurs;

2) if no kinematically possible displacement field exists, collapse cannot occur.

Moreover, consider that the reinforcement is assumed to enjoy very large strength, infinite in the limit. Therefore fractures cannot occur where the reinforcement is applied and the equality $\rho(\mathbf{x})\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \mathbf{0}$ holds everywhere in the solid.

3 L.A. SOLUTION FOR THE OPTIMAL REINFORCEMENT OF A NRT PANEL

The problem consists in finding the optimal distribution of $\rho(\mathbf{x})$ such that all above conditions are satisfied and can be approached in very different ways.

Basically the objective function is considered as the quantity of reinforcement that is applied to the basic material void volume produced by fractures in the basic body, so that its minimum value is searched.

3.1 *The problem layout by the static approach*

Being $\rho(\mathbf{x})$ the local density of the reinforcement with values in (0, 1), the problem consists on finding the optimal distribution of $\rho(\mathbf{x})$ as

$$F(\rho) = \int_{\Omega} \rho(\mathbf{x}) dA = \min \tag{10}$$

By static approach the constraints are:

a) Equilibrium is to be supplied by a stress field

$$\begin{aligned} &\mathrm{Div}\,\boldsymbol{\sigma}_b(\mathbf{x}) + \mathrm{Div}[\rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x})] + \mathbf{f}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \Omega \\ &[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x})]\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_p \end{aligned} \tag{11}$$

b) The reaction on the constrained contour Γ_u is equal to

$$\mathbf{r}(\mathbf{x}) = [\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x})]\mathbf{n}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_p \tag{12}$$

c) The stress field must be admissible (i.e. $\sigma_b(\mathbf{x}) \leq 0$ in Ω)

$$\begin{cases} I_{1b}(\mathbf{x}) \leq 0 \\ I_{2b}(\mathbf{x}) \geq 0 \end{cases} \quad \forall \mathbf{x} \in \Omega \tag{13}$$

with I_{1b} and I_{2b} the first and second invariant of the stresses $\sigma_b(\mathbf{x})$.

d) The reinforcement density $\rho(\mathbf{x})$ is everywhere a function in (0,1)

$$\begin{cases} \rho(\mathbf{x}) \geq 0 \\ \rho(\mathbf{x}) \leq 1 \end{cases} \quad \forall \mathbf{x} \in \Omega \tag{14}$$

The problem has been dealt with in detail in Ref. [1], yielding the following results with reference to the optimal solution:

a) In correspondence of the optimal reinforcement, a admissible displacement field exists such that the fracture strain is zero where the reinforcement is applied;

b) The reinforcement is zero, $\rho(\mathbf{x}) = 0$, everywhere the product $\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_f(\mathbf{x}) < 0$.

C) The reinforcement $\rho(\mathbf{x})$ turns out to be 0 or 1.

The method has been applied to the reinforcement of a concrete beam, yielding as output the well known reinforcement with lower and folded steel bars and stirrups.

Anyway, the same results suggest that the problem can be also approached from the kinematical point of view, as follows.

3.2 *The problem layout by the kinematic approach*

The problem can be approached in different ways, one of these is to consider the objective function as the maximum quantity of reinforcement that can be applied to the basic material volume produced by fractures in the basic body still leaving the structure prone to collapse. So, being $\rho(\mathbf{x})$ the local density of the reinforcement with 0-1 values

$$F(\rho) = \int_{\Omega} \rho(\mathbf{x}) dA = \max \tag{15}$$

By the kinematic approach the constraints are:

a) the compatibility conditions of the strain field, which is required to be compatible with the displacements field

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_{be}(\mathbf{x}) + \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x}) \tag{16}$$

b) the strain field must be admissible (i.e. $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq 0$ in Ω) and such that a collapse mechanism can be activated

$$\begin{cases} \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) ds + \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) dA \geq 1 \\ \nabla \mathbf{u}(\mathbf{x}) \geq 0 \quad \rightarrow \quad \begin{cases} J_{1b}(\mathbf{x}) \geq 0 \\ J_{2b}(\mathbf{x}) \geq 0 \end{cases} \\ \rho(\mathbf{x}) \nabla \mathbf{u}(\mathbf{x}) = 0 \quad \rightarrow \quad \begin{cases} \rho(\mathbf{x}) J_{1b}(\mathbf{x}) = 0 \\ \rho(\mathbf{x}) J_{2b}(\mathbf{x}) = 0 \end{cases} \end{cases} \tag{17}$$

with J_{1b} and J_{2b} the first and second invariant of the displacements in the basic material

$$J_{1b} = \varepsilon_{bx} + \varepsilon_{by} \quad ; \quad J_{2b} = \varepsilon_{bx}\varepsilon_{by} - \frac{1}{4}\gamma_{bxy}^2 \qquad \text{with} \qquad \boldsymbol{\varepsilon}_b = \begin{bmatrix} \varepsilon_{bx} & \frac{1}{2}\gamma_{bxy} \\ \frac{1}{2}\gamma_{bxy} & \varepsilon_{by} \end{bmatrix}$$

$$J_{1b} = \boldsymbol{\varepsilon}_b \cdot \boldsymbol{\Delta} = \nabla \mathbf{u} \cdot \boldsymbol{\Delta} \quad ; \qquad J_{2b} = \frac{1}{2}\left(\mathbf{R}^T \boldsymbol{\varepsilon}_b \mathbf{R}\right) \cdot \boldsymbol{\varepsilon}_b = \frac{1}{2}\boldsymbol{\varepsilon}_t \cdot \boldsymbol{\varepsilon}_b = \frac{1}{2}\boldsymbol{\varepsilon}_t \cdot \nabla \mathbf{u}$$

$$\boldsymbol{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \; \mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ; \; \boldsymbol{\varepsilon}_b = \begin{bmatrix} \varepsilon_{bx} & \frac{1}{2}\gamma_b \\ \frac{1}{2}\gamma_b & \varepsilon_{by} \end{bmatrix} ; \; \boldsymbol{\varepsilon}_t = \mathbf{R}^T \boldsymbol{\varepsilon}_b \mathbf{R} = \begin{bmatrix} \varepsilon_{by} & -\frac{1}{2}\gamma_b \\ -\frac{1}{2}\gamma_b & \varepsilon_{bx} \end{bmatrix} \tag{18}$$

c) The reinforcement density $\rho(\mathbf{x})$ is invariant with respect to the static approach and is everywhere not smaller than 0 and not larger than 1

$$\rho(\mathbf{x})[\rho(\mathbf{x}) - 1] = 0 \qquad \forall \mathbf{x} \in \Omega \tag{14'}$$

The Lagrange functional of the problem set up in the previous section, with the introduction of suitable multipliers, can be written down as follows

$$\mathscr{L}[\rho(\mathbf{x}),\mathbf{u}(\mathbf{x});\eta(\mathbf{x}),\chi(\mathbf{x}),\nu(\mathbf{x}),\upsilon(\mathbf{x}),\omega(\mathbf{x})]=$$

$$=\int_\Omega \rho(\mathbf{x})dA + k\left[\int_{\Gamma_p}\mathbf{p}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})ds+\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dA-1\right]+\int_\Omega \eta(\mathbf{x})J_{1b}(\mathbf{x})dA+\int_\Omega \chi(\mathbf{x})J_{2b}(\mathbf{x})dA+ \tag{19}$$

$$+\int_\Omega \nu(\mathbf{x})\rho(\mathbf{x})J_{1b}(\mathbf{x})dA+\int_\Omega \upsilon(\mathbf{x})\rho(\mathbf{x})J_{2b}(\mathbf{x})dA+\int_\Omega \omega(\mathbf{x})\rho(\mathbf{x})[\rho(\mathbf{x})-1]dA$$

where it is intended that $\nu(\mathbf{x})$ and $\upsilon(\mathbf{x})$ and $\omega(\mathbf{x})$ are Lagrange multipliers without any constraint on their sign, whilst k is a non-negative constant being $\int_{\Gamma_p}\mathbf{p}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})ds+\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dA$ a scalar quantity, and $\eta(\mathbf{x})$ and $\chi(\mathbf{x})$ are scalar non-negative multipliers, since they are related to the inequality constraints of the problem in Eqs (15) to (18).

Therefore one has

$$k\left[\int_{\Gamma_p}\mathbf{p}(x)\cdot\mathbf{u}(x)ds+\int_\Omega \mathbf{f}(x)\cdot\mathbf{u}(x)dA\right]=0\ ;\ \int_{\Gamma_p}\mathbf{p}(x)\cdot\mathbf{u}(x)ds+\int_\Omega \mathbf{f}(x)\cdot\mathbf{u}(x)dA-1\geq 0$$

$$\left.\begin{aligned}\eta(\mathbf{x})J_{1b}(\mathbf{x})&=0\\ \chi(\mathbf{x})J_{2b}(\mathbf{x})&=0\\ k&\geq \mathbf{0}\\ \eta(\mathbf{x})&\geq 0\\ \chi(\mathbf{x})&\geq 0\end{aligned}\right\}\quad \left.\begin{aligned}J_{1b}(\mathbf{x})&\geq 0\\ J_{2b}(\mathbf{x})&\geq 0\\ \rho(\mathbf{x})(\rho(\mathbf{x})-1)&=0\\ \rho(\mathbf{x})J_{1b}(\mathbf{x})&=0\\ \rho(\mathbf{x})J_{2b}(\mathbf{x})&=0\end{aligned}\right\} \tag{20}$$

Moreover, the variational conditions must be fulfilled in solution.

1) For independent variation of the displacement field $\mathbf{u}(\mathbf{x})$:

$$\delta_{\mathbf{u}}\mathscr{L}[\rho(\mathbf{x}),\mathbf{u}(\mathbf{x});\eta(\mathbf{x}),\chi(\mathbf{x}),\nu(\mathbf{x}),\upsilon(\mathbf{x}),\omega(\mathbf{x})]=0 \qquad \forall\delta\mathbf{u}(\mathbf{x})$$

$$\Downarrow \tag{21}$$

$$\left.\begin{aligned}k\left[\int_{\Gamma_p}\mathbf{p}(\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{x})ds+\int_\Omega \mathbf{f}(\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{x})dA\right]+\int_\Omega \eta(\mathbf{x})\delta J_{1b}(\mathbf{x})dA+\int_\Omega \chi(\mathbf{x})\delta J_{2b}(\mathbf{x})dA+\\ +\int_\Omega \nu(\mathbf{x})\rho(\mathbf{x})\delta J_{1b}(\mathbf{x})dA+\int_\Omega \upsilon(\mathbf{x})\rho(\mathbf{x})\delta J_{2b}(\mathbf{x})dA=0\end{aligned}\right\}\quad \forall\delta\mathbf{u}(\mathbf{x})$$

After some algebra and by the conditions in Eq, (18), one gets

$$k\left[\int_{\Gamma_p}\mathbf{p}(\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{x})ds+\int_\Omega \mathbf{f}(\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{x})dA\right]=-\int_\Omega \{\rho(\mathbf{x})[\nu(\mathbf{x})\mathbf{\Delta}+\upsilon(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})]+[\eta(\mathbf{x})\mathbf{\Delta}+\chi(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})]\}\cdot\nabla\delta\mathbf{u}(\mathbf{x})dA \tag{22}$$

If $S_b(\mathbf{x})$ is a no-tension stress field, because of the non-negative character of $\eta(\mathbf{x})$ and $\chi(\mathbf{x})$ and of the tensors Δ and $\varepsilon_t(\mathbf{x})$, and is expressed as

$$\mathbf{S}(\mathbf{x})=\rho(\mathbf{x})\mathbf{S}_r(\mathbf{x})+\mathbf{S}_b(\mathbf{x});\ \ \mathbf{S}_r(\mathbf{x})=-[\nu(\mathbf{x})\mathbf{\Delta}+\upsilon(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})];\ \ \mathbf{S}_b(\mathbf{x})=-[\eta(\mathbf{x})\mathbf{\Delta}+\chi(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})] \tag{23}$$

the Eq, (21) becomes

$$k\left[\int_{\Gamma_p} \mathbf{p}(x)\cdot\delta\mathbf{u}(x)ds + \int_{\Omega} \mathbf{f}(x)\cdot\delta\mathbf{u}(x)dA\right] = \int_{\Omega}\{\rho(\mathbf{x})S_r + S_b(\mathbf{x})\}\cdot\nabla\delta\mathbf{u}(\mathbf{x})dA \qquad \forall\delta\mathbf{u}(\mathbf{x}) \tag{24}$$

which, by the PVW, means that the external forces are in equilibrium with a stress field that is non-negative semi-definite where $\rho(\mathbf{x}) = 0$.

2) For independent variation of the reinforcement distribution:

$$\delta_\rho \mathscr{L}[\rho(\mathbf{x}), \mathbf{u}(\mathbf{x}); \eta(\mathbf{x}), \chi(\mathbf{x}), \nu(\mathbf{x}), \upsilon(\mathbf{x}), \omega(\mathbf{x})] = 0 \qquad \forall\delta\rho(\mathbf{x})$$

$$\int_\Omega \delta\rho(\mathbf{x})dA + \int_\Omega \nu(\mathbf{x})\delta\rho(\mathbf{x})J_{1b}(\mathbf{x})dA + \int_\Omega \upsilon(\mathbf{x})\delta\rho(\mathbf{x})J_{2b}(\mathbf{x})dA + \int_\Omega \omega(\mathbf{x})[2\rho(\mathbf{x})-1]\delta\rho(\mathbf{x})dA = 0 \qquad \forall\delta\rho(\mathbf{x}) \tag{25}$$

and after some algebra

$$\int_\Omega \left\{1 + \omega(\mathbf{x})[2\rho(\mathbf{x})-1] + \left[\nu(\mathbf{x})\mathbf{\Delta} + \frac{1}{2}\upsilon(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})\right]\cdot\nabla\mathbf{u}(\mathbf{x})\right\}\delta\rho(\mathbf{x})dA = 0 \tag{26}$$

The third addend in the coefficient of $\delta\rho(\mathbf{x})$ in the integrals is always positive. So $\omega(\mathbf{x})$ shall be positive where $\rho(\mathbf{x}) = 0$ and we tend to add the reinforcement, while it shall be negative where the reinforcement exists and we want to drop it out.

4 EXAMPLE: APPLICATION OF THE LIMIT ANALYSIS OPTIMIZATION APPROACH

As an example, the problem of the reinforcement of a panel with a rectangular hole is considered. The panel is complemented by a horizontal girder on the top and by a steel platband on the hole (Fig. 1). The load pattern is given by the self-weight of the panel and by a horizontal force at the left end of the girder, that cannot be resisted by the simple unreinforced system.

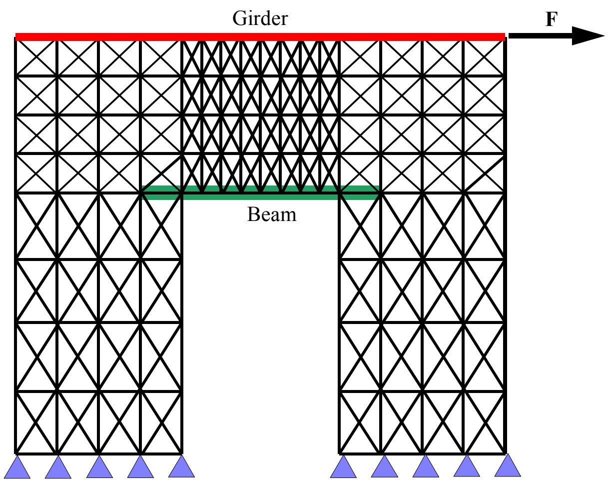

Figure 1: The FEM model of the considered panel

The procedure has been implemented by means of a Basic program built ad-hoc for the problem and the optimal design of the reinforcement is shown (Fig. 2) with the proceeding of the calculations.

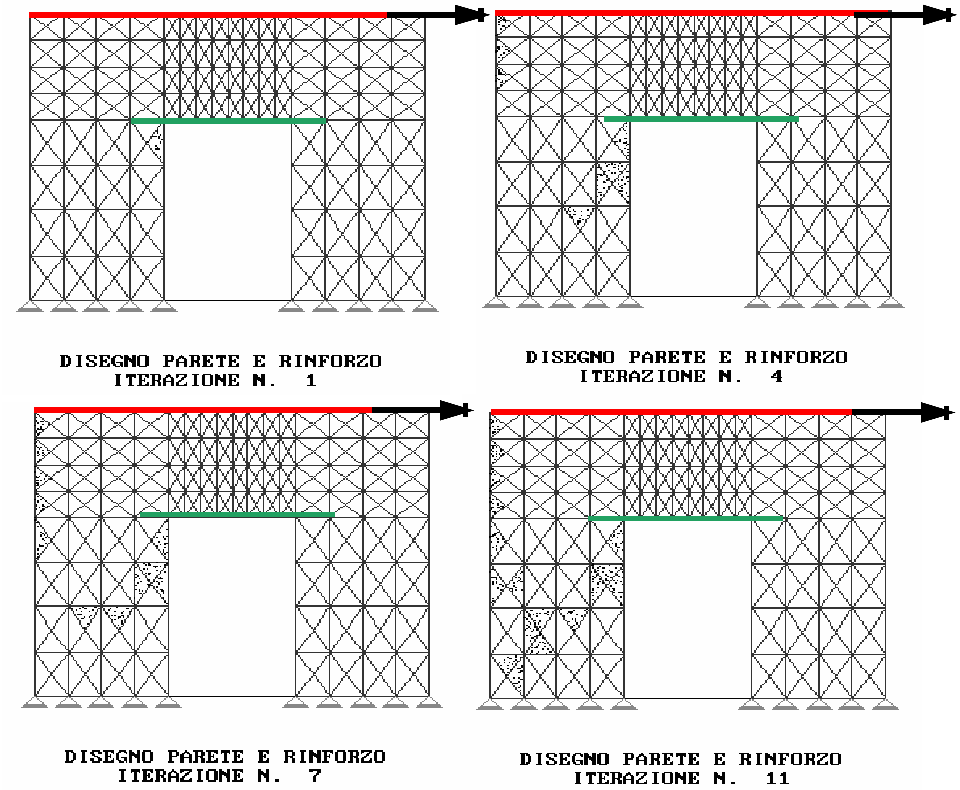

Figure 2: Samples of optimal reinforcement distribution (signed by the black points) during four phases of the proceeding.

It is easy to understand that the procedure yields a resistant mechanism, able to neutralize the collapsing action of the horizontal force as in Fig. 3, where the reinforcement acts as a tie-rod.

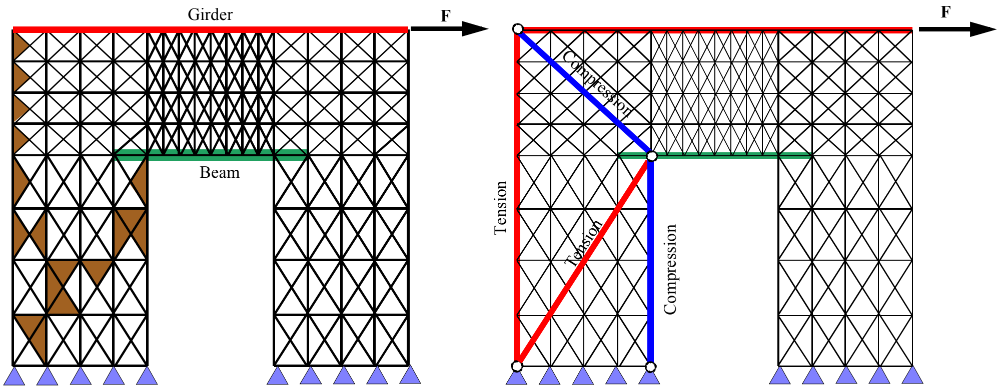

Figure 3: Resistant mechanism after the reinforcement

This interpretation is confirmed also looking at the results of the FEM-NRT reinforced panel, and in particular to the isostatic compression lines (Fig. 4).

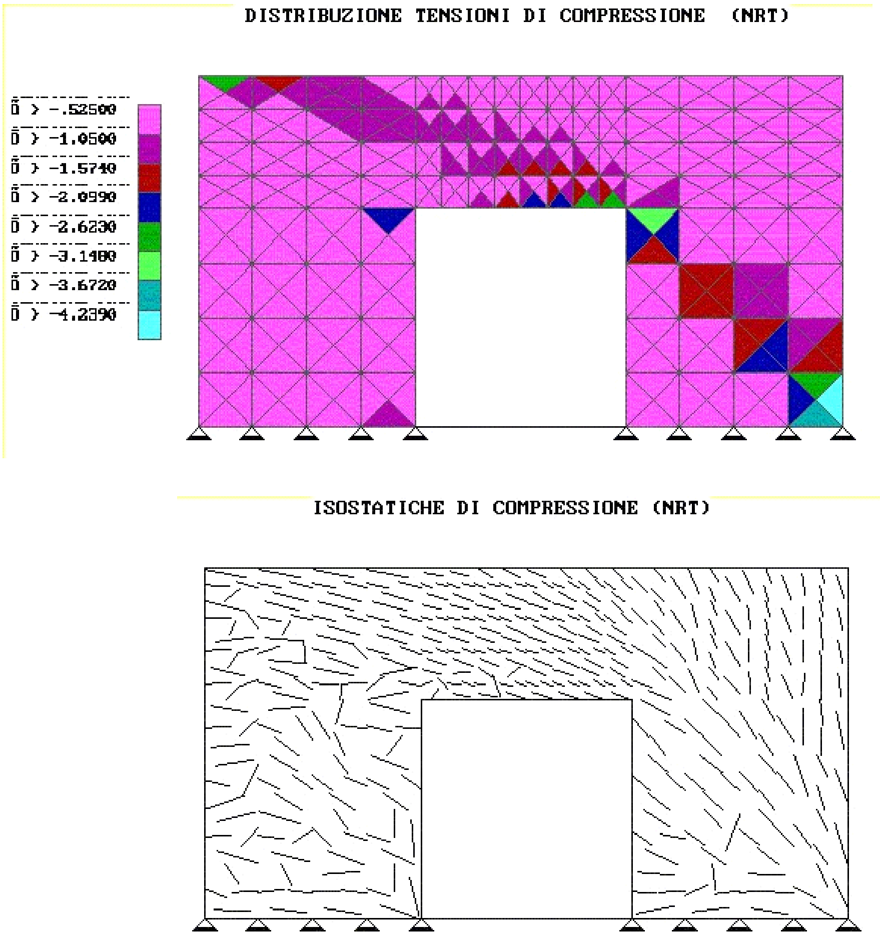

Figure 4: Results from FEM analysis of the reinforced panel.
Compressive stresses in the basic NRT material

5 CONCLUSIONS

In the paper some results are shown about the elaboration of a design strategy for reinforcement of a structure. For modelling the structure the NRT material is used. NRT models are recognized as an effective tool for analyzing a large class of Civil Engineering Structures (say e.g. masonry and reinforced concrete members), so that the problem is strongly felt also from the point of view of practical technique. The reinforcement of NRT panels by the application of superposed high-strength sheets or insertion of tensile bars, has been considered, with the purpose to set up a design path aiming to distribute the new material according to some optimal criterion. This aim has been taken applying the basics of Limit Analysis, with the static and kinematic theorems, and of Topology Optimization. The optimal reinforcement is approached by stating one possible criterion that would allow to design the best distribution of reinforcement over the existing body structure, and the relevant objective function and constraints have been formulated. An implementation of a search procedure on the basis of the Kinematical Approach has been performed, aiming at finding the most efficient distribution of the reinforcement.

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