From continuum masonry mechanics to macroelements: scaling and homogenization

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SUMMARY. Some considerations are proposed with reference to different approaches adopted when the material investigated is heterogeneous and micro-structured and the structure assemblage is complex. In particular, in the case of masonry, an extensive literature exists on micro and macro approaches, but the theories investigating a link between the two scales of analysis is still to be developed. With this aim, the authors describe the different approaches at the different scales, and try to investigate some possible strategies to overcome the critical length problem.

1 INTRODUCTION

When complex structures composed by heterogeneous materials are investigated, a problem is the characteristic size analysis. Hence the attention may be focused on the scale analysis. Three scale of analysis may be pointed out: micro-scale – masonry component; meso-scale: elementary portion of masonry structures; macro-scale: complex assemblage of structural elements.

At the micro-scale level the characteristic size is represented by blocks and joints modelled as a micro structured skeleton, hence for computational problem only limited portion of masonry structures –i.e. piers, spanned beam, lintels…-; can be studied [1]. At the macro-scale level the characteristic size is the macro-element scale, a portion of structures that may be studied with simple equivalent system (i.e., vaults, pediment, walls…) [2]. A possible link between micro and macro scale, the former very detailed, but limited in the size applications, the latter manageable at the structural scale, but rough at the constitutive level, may be found in the homogenisation-identification procedures [3]. Unfortunately these procedures, used to define an equivalent continuum, propose constitutive functions for the material (in linear, non linear and limit field) but their application at the structural level are very poor, also for the difficulty to use standard codes and procedures. The structural heterogeneity and the internal microstructure of masonry may not be disjointed from the size effect, that depends on the relation between a characteristic length l_{ch} and the size of structure.

For the homogenization procedure l_{ch} is the least portion of structure, i.e. the so called REV (representative elementary volume) that contains in a small scale all the geometric and mechanical properties to describe the body as a whole. This assumption is acceptable under the hypothesis that the masonry keeps its properties constant for any global dimension. But, as shown by experimental tests [4] and by dynamic analyses, the masonry structure behaviour is strictly connected to the scale. When the masonry global size increases the ultimate strength decreases. Hence approaches based on different characteristic lengths l_{ch} may produce incomparable results. With this aim, some considerations on the hypotheses of different approaches are here proposed.

2 BASIC ASSUMPTION IN HOMOGENIZATION PROBLEMS

As is well known, the homogenization procedure is based on the definition of a homogeneous material, equivalent to the heterogeneous one in its geometry and in the properties of its constituent materials. This approach may be used under the assumption of "periodic" body structure. This implies that the body is obtained by regular repetition of REV (Representative Elementary Volume) [1]. Consider a body in a reference configuration $\mathfrak{I}$ and identify in its interior a REV that is small compared to the dimensions of the body and able to reconstruct the body itself by its repetitions (Fig. 1). Define as ε the ratio between the scale of representation of $\mathfrak{I}$ and that of the REV. When ε tends to zero, the body tends to become homogeneous and therefore ε measures the degree of its heterogeneity. Given a position p within $\mathfrak{I}$ and calling x and y its representations in two different systems of coordinates, the following relation holds:

$$y = \boldsymbol{\varepsilon}^{-1} x \tag{1}$$

This assumption is crucial in the homogenization methods and implies that all fields are periodic over the REV. The field problem is obtained as superposition on a mean field (at order zero of ε) of strongly oscillating fields (of higher orders of ε). Hence the first step is the solution of a field problem on the REV starting from the geometry of the constituent REV and its constitutive function. Then it is possible to obtain macroscopic global constitutive functions used in structural analysis. Here a 2D plane model is developed and the panel exhibits a periodic structure both in directions 1 and 2. The following notations are used: $\mathbf{N}=(N)_{\alpha\beta}$ is the macroscopic in-plane (membrane) stress field for the homogenized panel; $\mathbf{E}=(E)_{\alpha\beta}$ is the macroscopic in-plane strain tensor; $\mathbf{D}=(D)_{\alpha\beta}$ is the corresponding in-plane strain rate field; $\mathbf{u}=(u)_{\alpha\beta}$ is the displacement field; $\mathbf{V}=(V)_{\alpha\beta}$ is a virtual velocity field.

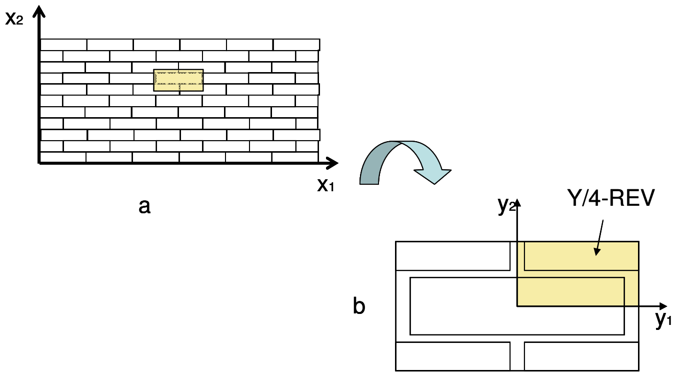


Figure 1-a) running bond masonry panel; b) REV

2.1 *Homogenisation problem in linear elasticity*

The input data are the geometry of Y-REV and the constitutive functions of its constituent –mortar and blocks – here assumed as isotropic linear elastic [1]. If $\boldsymbol{\sigma}$ is the Cauchy stress tensor; $\boldsymbol{\varepsilon}$ is the microscopic strain tensor, $\mathbf{E}$ is the macroscopic in-plane strain tensor; $(E)_{\alpha\beta}=(u_{\alpha,\beta}+u_{\beta,\alpha})$, e_α is the unit vector in the in-plane directions; $\mathbf{u}^{per}$ is a periodic displacement field, $\mathbf{a}$ is the constitutive function defined as: $\mathbf{a}^B$ for $y \in$ block and $\mathbf{a}^M$ for $y \in$ joint. Then, the following auxiliary problem can be solved on the Y-REV:

$$
\begin{cases}
div\sigma = 0 \quad \text{micro-equilibrium} \\
\sigma = \mathbf{a}(y)\,\varepsilon \quad \text{local constitutive function} \\
\varepsilon = \mathbf{E} + sym(grad\,\mathbf{u}^{per}) \quad \text{compatibility} \\
\sigma\,\mathbf{e}_{\alpha} \quad anti-periodic \text{ boundary conditions } on\ \partial Y \\
\mathbf{u}^{per} periodic\ on\ \partial Y
\end{cases}
\tag{2}
$$

The solution of the field problem (2) provides the homogenised constitutive function of a panel subjected to in-plane loading:

$$
\mathbf{N} = \langle \boldsymbol{\sigma} \rangle = \mathbf{A}^{H}\mathbf{E} \tag{3}
$$

where $\mathbf{N}$ is the in-plane membrane tensor, $\mathbf{A}^{H}$, is the constitutive homogenised moduli and $\langle\cdot\rangle$ is the averaging operator. On the basis of these considerations, the macroscopic field problem at the structural level may be built.

2.2 *Homogenisation problem at collapse*

The input data are the geometry of Y-REV and the yield criterion for the two materials that constitute the masonry: rigid and infinitely resistant blocks; mortar joints assumed as Mohr-Coulomb interfaces [4]. For every ($\mathbf{N}$), the set of statically admissible (S.A.) stress fields $\boldsymbol{\sigma}=(\sigma_{\alpha\beta})$ of a unit cell is defined by:

$$
\begin{gathered}
\text{S.A.}(\mathbf{N}) = \{\boldsymbol{\sigma}\ |\mathrm{N}_{\alpha\beta} = \mathrm{t}\langle \sigma_{\alpha\beta}\rangle, \mathrm{div}\boldsymbol{\sigma} = 0 \quad \text{o}\mathbf{n}\ \mathrm{Y}, \\
\boldsymbol{\sigma}\cdot\mathbf{n}\ \text{antiperiodic} \quad \text{o}\mathbf{n} \quad \partial\mathrm{Y}
\end{gathered}
\tag{4}
$$

For the materials constituting the heterogeneous panel the convex domain G($\mathbf{y}$) characterizing their strength capacities is assigned at every point $y\in$ Y-REV. The strength domain of the homogenized panel is defined as:

$$
\mathrm{G}_{\mathrm{p}}^{\mathbf{hom}} = \{(\mathbf{N})|\ \exists\,\boldsymbol{\sigma}\in \text{S.A.}(\mathbf{N}),\ \boldsymbol{\sigma}(\mathbf{y})\in \mathrm{G}(\mathbf{y}),\ \ \forall \mathbf{y}\in \mathrm{Y}\} \tag{5}
$$

i.e., as a function of the strength capacities of each material. In the same way, for every ($\mathbf{D}$), the set of kinetically admissible (K.A.) velocity fields, $\mathbf{v}=(v_i(\mathbf{y}))$ is defined:

$$
\text{K.A.}(\mathbf{D}) = \{\mathbf{v}\ |\mathrm{sym}(\mathrm{grad}\mathbf{v}) = \tilde{\mathbf{D}} + \mathrm{sym}(\mathrm{grad}\mathbf{u}^{\mathrm{per}}),\ \mathbf{u}^{\mathrm{per}} \quad \text{Y-periodic}\} \tag{6}
$$

where $\tilde{\mathrm{D}}_{\alpha\beta} = \mathrm{D}_{\alpha\beta}$. It is well-known that every closed convex domain is uniquely determined by a corresponding positively homogeneous function of degree one called the support function. The support function $\langle\pi(\mathbf{d})\rangle$ of G($\mathbf{y}$) is defined as:

$$
\pi(\mathbf{d}) = \sup\{\boldsymbol{\sigma}\cdot\mathbf{d};\ \boldsymbol{\sigma}\in \mathrm{G}(\mathbf{y})\};\ \mathrm{G}(\mathbf{y}) = \{\boldsymbol{\sigma}\,|\,\boldsymbol{\sigma}\cdot\mathbf{d} \le \pi(\mathbf{d})\ \forall\mathbf{d}\}, \tag{7}
$$

where $\mathbf{d}=(d_{\alpha\beta})$ is the strain rate tensor. Hence using (4), (5) and (7) the macroscopic G_p^{hom} domain is obtained:

$$
\mathrm{G}_{\mathrm{p}}^{\mathbf{hom}} = \{(\mathbf{N})|\ \mathbf{N}\cdot\mathbf{D} \le \pi_{\mathrm{p}}^{\mathbf{hom}}(\mathbf{D},\chi),\quad \forall(\mathbf{D})\} \tag{8}
$$

with π_p^{hom}:

$$
\pi_{\mathrm{p}}^{\mathbf{hom}}(\mathbf{D}) = \mathbf{inf}\{\ \mathrm{t}\langle\pi[\mathrm{sym}(\mathrm{grad}(\mathbf{v}))]\rangle\quad \forall\mathbf{v}\in \text{K.A.}(\mathbf{D})\}, \tag{9}
$$

that represents the equivalence between the power on the Y-REV and the power on the homogenized panel. The homogenised domain may be used at the structural level, for instance as the input of a FEM simulation of the collapse of macroelements (see Section 6).

3 REV SENSITIVITY IN THE HOMOGENIZATION MODELS

As shown in the previous paragraphs 2.1 and 2.2, the hypothesis that the masonry keeps its properties constant for any global dimension, is the basic assumption of the homogenisation procedure. Here some considerations are proposed with reference to the linear field analysis (the same considerations could be extended to the limit analysis). The crucial assumption is that all the fields are periodic over the REV. The determination of elastic homogenised moduli is based on the imposition of a homogeneous displacement field on the REV. In what follows, the shear case is proposed, to obtain the A_{1212}^{H} homogenised shear modulus. The mesh, due to the symmetry of the REV may be referred to a quarter of REV. Figure 2 is relative to $[2(a+t)]\times(b+t)]$ REV-area, called $\tilde{Y}$; where b=width of block, a=height of block, t=thickness of mortar joint. Figure 3 is relative to 2x$\tilde{Y}$ REV-area. Figure 4 is relative to 4x$\tilde{Y}$ REV-area.

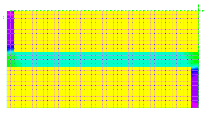


Figure 2 – $\tilde{Y}$-REV

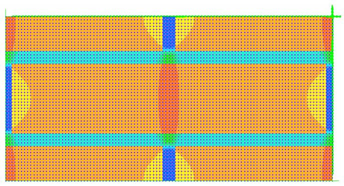


Figure 3 – 2x{$\tilde{Y}$-REV}

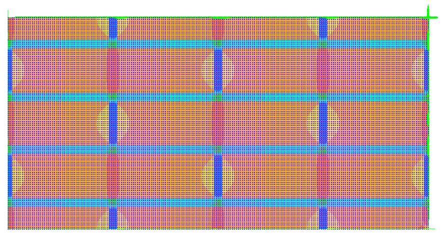


Figure 4- 4x{$\tilde{Y}$-REV}

Due to the homogeneity of the displacement field and to the periodicity of the structure, the same stress and strain distributions on the REV are obtained, therefore the same value of the homogenised shear modulus A_{1212}^{H} is obtained. Hence all the REVs are acceptable.

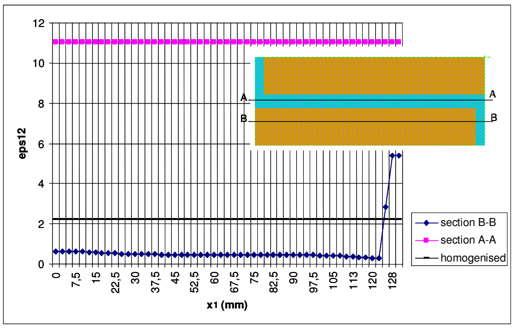


Figure 5 -ε_{12} trend in the cross sections A-A and B-B

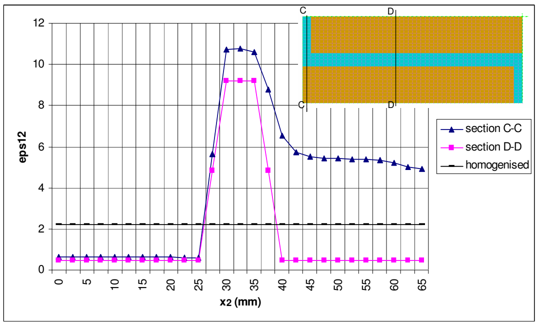
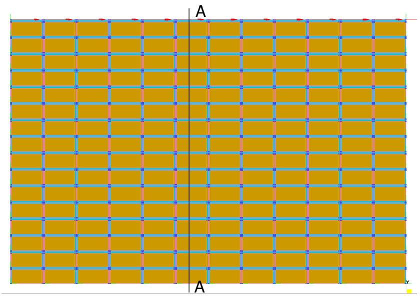


Figure 6 -ε_{12} trend in the cross sections C-C and D-D

A comparison between heterogeneous and homogenised solution is proposed. Hence on the REV of Figure 2 two horizontal and two vertical sections are analysed, reporting the local distribution of the ε_{12} strain. As shown in Figures 5 and 6, the homogenised solution is a mean field solution with respect to the effective distribution in the REV. After the verification of relation between the heterogeneous and homogenised solutions at the REV scale, a structural analysis is performed. A comparison between a 2D heterogeneous model (Fig. 7) and 2D homogenised model (in which the constitutive function are obtained according to section 2.1) has been carried out.

The panel dimension are: base B=1560mm, height H=955mm. The panel is subject to the following boundary conditions: for $x_2=0$ $u_1=0$, for $x_2=H$ $F_1= F_h>0$, for $x_1=0$ and $x_1=B$ $u_2=0$. The block characteristics are $E^b=E^m\times 20$MPa –Young modulus-; $\nu^b=0.2$ -Poisson ratio-. The mortar characteristics are $E^m=1000$MPa -Young modulus-; $\nu^m=0.2$ -Poisson ratio-. Figure 8 shows the trends of u_1 displacement in the heterogeneous and homogenised model for a vertical section. As expected, the homogenised model represents the mean field. The good agreement between the two models is strongly connected to the regularity of the panel geometry and the uniformity of load.



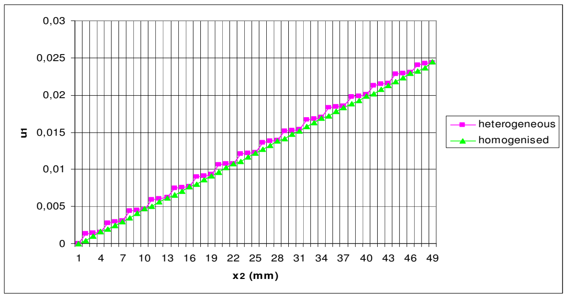


Figure 7 - 2D heterogeneous panel

Figure 8 - u_1 trend in the vertical cross section A-A

4. SIZE SCALE EFFECTS AND TRANSITION TO DAMAGE

As explained in the previous sections, even the description of the elastic behaviour of masonry is affected by scale dependence. Homogenisation procedures show that the elastic response is not sensitive to the choice of the REV size when homogeneous conditions are present, depending on the mesostructure.

When the onset of damage, in the form of cracks and large displacements, is considered, the role of the size scale of the masonry elements becomes more crucial. Structural size effects, in fact, are clearly evidenced, and in particular the apparent strength σ_u decreases while the apparent fracture energy G_f increases, with structural size [5]. This peculiar behaviour is controlled by a characteristic length l_{ch}, whose value is depending on the mesostructure of the masonry, and provides the so-called multifractal scaling behaviour [6], as shown in Figure 9.

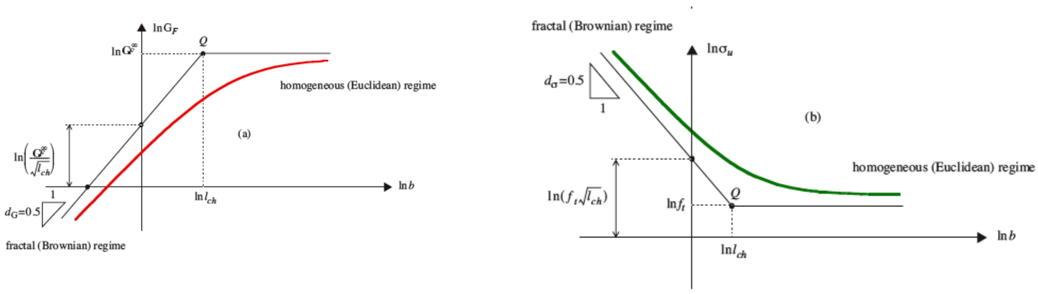


Figure 9. Multifractal size effect laws on toughness (a) and strength (b) [6]

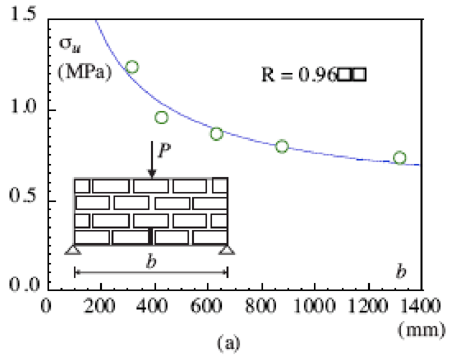


Figure els [5]

Tests on notched brick llows the above universal laws, like all granular disc) do. Of course, the value of the characteristic length l_{ch} for masonry is quite larger than in the case of concrete. For instance, in the tests shown in Figure 10, where the linear size of the bricks was 45x110x250mm and the mortar beds were about 0,8cm thick, the value of l_{ch} was equal to 1672mm, implying that an adequate REV size for the analysis of damage should be larger than 1,6 mt.

In the following paragraphs, some theories on elastic-plastic behaviour (limit state of damage) and on the ultimate behaviour (limit state of collapse) of masonry structures will be discussed. The models adopted for those limit states are normally disjointed from the elastic models, and especially in the case of collapse behaviour, engineers normally neglect the elastic behaviour and only look to the mechanisms activated for macro-elements.

5. PLASTIC (DAMAGE) LIMIT STATE

As is wel known, a wide literature deals with the theory of the state of damage, and of the material plastic phase. The point when the material becomes plastic and damage arises, interests several fields. One of these studies about structures composed by rigid-perfectly-plastic material, was originated from the basic premise presented in [7]. The application feasible due to this assumption consists in an approximate solution for the response of rigid-plastic structures subject to dynamic load, i.e., short duration of load and sign defined.

The approximate solution is expressed by the product of a modal form by a function of time response; the error due to this approximation is assessed through a particular index, which can be restricted depending on computed approximate solutions only. The only "load" applied on the structure is a velocity field at the initial time; the approximation consists in substituting this field with an approximate one, satisfying the requirements of kinematic admissibility, the limit conditions and the initial ones, in the way that the response is reduced to a single degree of freedom.

Baratta et al. [8] showed that, since the validity of this method is not strictly conditioned by the assumptions about the forces, the procedure for the evaluation of errors remains valid even for a generic forced oscillation. The theoretical results provide the possibility to study the structure behaviour during the transition from the elastic phase to the plastic one by modal approximations. The results are independent of the ground acceleration $\ddot{u}_g(t)$ and of the time t, and show that plastic deformations stop at a prescribed abscissa and cannot propagate indefinitely, due to scale confinement. The results originally developed for beams in the plastic phase [8] can be easily extended to masonry panels, through continuous models. Normally, nonlinear finite elements calculations are carried out within this framework [9, 10].

The possibility to calculate the upper limit of the error made by replacing the true velocity field with the approximate one, enables the process to disregard the knowledge of the real solution, and allows the application of the method even in cases where it is unknown. The numerical examples indicate that the "mode approximation" for rigid-plastic structures can lead to satisfying results for what concerns damage, but is not efficient for the assessment of the ultimate state of collapse.

6. COLLAPSE LIMIT STATE

In the following, the macro-elements modeling strategy for the collapse of masonry structures is proposed. An interesting comparison can be made with and a method proposed by [11]. It is interesting to note the correlation of results in both cases, extending the second method, originally proposed for arch structures, to other types of structural elements and to verify the consequences of the approximations in the two cases.

6.1 The macro-elements strategy

As is well known, modelling through macro-elements takes into account the limit state of collapse in terms of linear and non-linear dynamic analysis. A "portion" of the structure having characteristics recognizable by the constructive point of view, which may or may not coincide with the architectural and functional ones, is identified. The existence of fracture or weak lines is assumed, which represent the sliding planes or hinges that could lead to the collapse of a single portion with respect to the rest of the structure.

The hypothesis is that the material is not resistant to traction, with infinite resistance to compression and, conservatively, the approach neglects the energy of fracture and friction between the blocks. The blocks are subject to the weight P_i and to the seismic forces αP_i, in addition to the seismic forces transmitted from the rest of the structure αP_j, and to the generic horizontal loads F_h (Figure 11 a).

The kinematics of the system is governed by the generalized displacement d_k of a point K, usually chosen at the center of mass of the system (Figure 11 c).

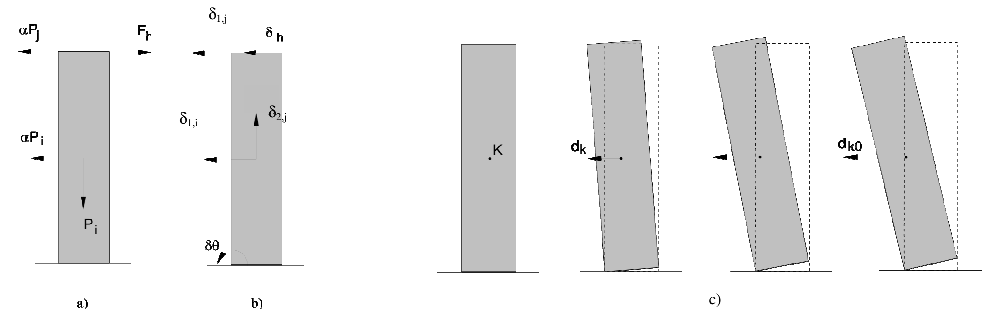


Figure 11. The rigid block. a) System of forces; b) system of displacements; c) subsequent configurations of the rigid block for NLKA

The relationship between d_k and the unknown multiplier a is obtained by writing the equilibrium in the current configuration, by means of the principle of virtual works

$$\alpha\left(\sum_{i=1}^{n} P_i\delta_{1,i} + \sum_{j=n+1}^{n+m} P_j\delta_{1,j}\right) - \sum_{i=1}^{n} P_i\delta_{2,i} - \sum_{h=1}^{o} F_h\delta_h - L_{if} = 0 \tag{10}$$

where δ_1 is the horizontal virtual displacement, δ_2 is the vertical virtual displacement, and L_{if} represents the work done by the internal forces, which are supposed to be null, since the fracture energy G_F is neglected.

In LKA (Linear Kinematic Analysis) the initial configuration is studied (Figure 11a). In NLKA (Nonlinear Kinematic Analysis), increasing d_k step by step, subsequent configurations are investigated until the failure displacement d_{k0} is attained, which is characterized by α =0 (Figure 11 c). The solution permits to plot the capacity curve, which is piecewise nonlinear if the forces display a variation (e.g. yielding or rupture of a tie).

The results are used to define the behavior of an equivalent SDOF system, which is expressed in terms of acceleration a^* and displacement d^*. The seismic spectral acceleration a^* is

$$a^* = \alpha \frac{\sum_{i=1}^{n+m} P_i}{M^*} \tag{11}$$

being the mass M^* of the SDOF system

$$M^* = \frac{\left(\sum_{i=1}^{n+m} P_i \delta_{x,i}\right)^2}{g \sum_{i=1}^{n+m} P_i \delta_{x,i}^2} \tag{12}$$

where g is the gravitational acceleration.

In LKA, the ultimate limit state is verified if $a^* > a_d$ where

$$a_d = \frac{a_g S}{q}\left(1 + 1.5\frac{Z}{H}\right) \tag{13}$$

with S is the soil factor and q is the behavior factor. Moreover, Z is the height of the mass center of the system, H is the height of the structure, both with respect to the foundations.

In NLKA the ultimate displacement d_u^* , i.e. the capacity of the system, is defined conventionally for $d_k = \min(d_{k1}, d_{k2})$ where $d_{k,1} = 0.4 d_{u,k}$ is computed considering only the forces which do exist up to failure, and d_{k2} is the displacement which is incompatible with the stability of the structural elements (e.g. lack of beam support).

The nonlinear system is replaced by a secant linear system defined conventionally for $d_s^* = 0.4 d_u^*$. The intersection between its capacity line and the demand curve identifies the demand displacement Δ_d. The structure is finally verified if $\Delta_d \le d_u^*$

An application of the macroelement strategy was carried out by the authors, according to the Italian Guidelines for Seismic Assessment of Historical Buildings [12], for the structural analysis of a theatre in Bari, [2]. The issue was to simplify the complex structure in simpler and more regular parts, i.e. the “macroelements” and to model them at collapse. In this way, the continuum behavior of the building under small loads was missed (e.g. the damage limit state was not modeled), but the ultimate (collapse) limit state was correctly modeled. As is well known, when fractures and detachments are evident, individuation of the macroelements is trivial. Instead, when they are not present (or not visible due to previous retrofit), the choice of the macroelements must be carried out with great accuracy, taking into account not only the original architectural and functional destination, but also the most probable patterns of weakness where fractures will appear. An example of two macroelements that were studied is reported in Figure 12.

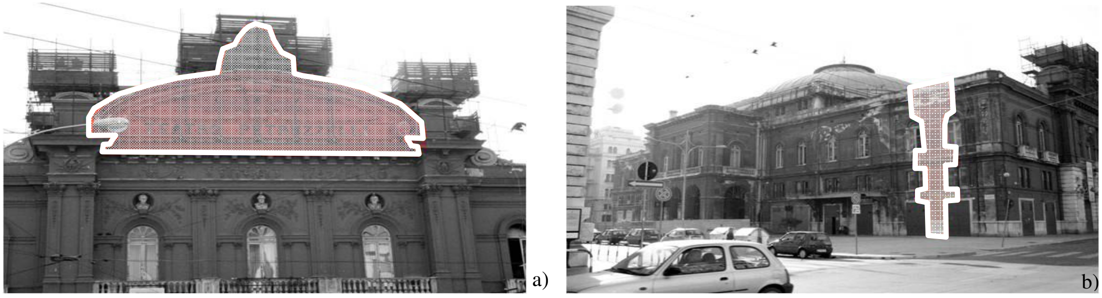


Figure 12. Petruzzelli theatre (Bari, Italy). a) Overturning of the pediment on the frontispiece; b) overturning of the walls of the foyer.

6.2 The "mode approximation" applied to the collapse state

The "mode approximation" [7] provides the solution even for the response of structural systems consisting of traction-free material, that are subject to pseudo-impulsive or impulsive dynamic load. As already mentioned in cap. 5, the approximation consists in replacing the initial velocity field with an approximate one, and in the reduction of the response into a SDOF system, on the basis of a predetermined modal form. The equations of motion are written by the principle of virtual work, considering the actual forces acting on the structure, and the virtual displacements δu_{1i} and δu_{2i}, read on the kinematic system. At a certain time instant, the field of real velocity coincides, except for a factor, with the virtual deformed shape of the structure and, through appropriate substitutions, the instantaneous displacement of the structure with a step by step procedure can be obtained.

This method, like the one described in the par. 6.1, starts by individuating a structural portion where a mechanism of collapse is supposed to be activated, e.g., with the formation of unilateral hinges. The structure becomes a rigid-kinematic failing system to which the hinges confer one (or possibly more) degree of freedom.

In the simple case of arch scheme, collapse occurs when the rotation centres of the various portions are aligned (see Figure 13 d)

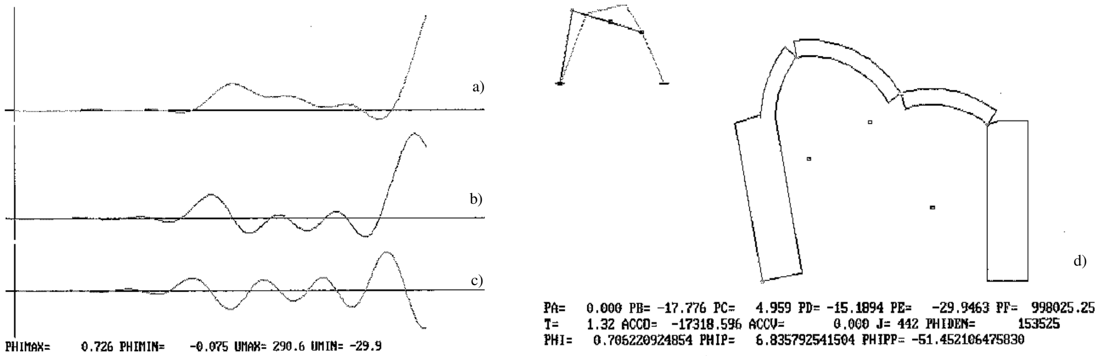


Figure 13. Un-damped motion. a) Rotations diagram b) velocity diagram c) forcing; d) the arch collapse configuration

Differently, in the macro-elements method described in par. 6.1, the principle of virtual work is written by means of the real forces and the virtual deformation, that is supposed to coincide with the real one, at least when calculated with a sufficiently small time step. Therefore, although the macro-elements approach appears more direct and less computationally expensive, it is certainly less accurate.

7. CONCLUSIONS

In this paper, a brief description of the theories usually adopted, at the various scales, to model masonry structures, has been put forward. It can be noticed that a link between the different scales is absent, and micro-scale theories, mainly based on homogenization procedures, are totally neglected at the macroscale of a real masonry building. On the other hand, while the ultimate state of collapse is adequately modeled by rigid-plastic behavior (macroelements or mode approximation), the crucial missing point is the onset of damage, which is clearly scale dependent, as shown by the experiments.

Starting from this preliminary paper, we are planning to set a true *multiscale theory of masonry*, as general as possible, taking into account the different constructive typologies, that could be put into a unified numerical description of the structures.

REFERENCES

[1] Cecchi A., Sab K., (2004), “A comparison between a 3D discrete model and two homogenised plate models for periodic elastic brickwork”, International Journal of Solids and Structures, vol.41/9-10, pp.2259-2276.

[2] Cennamo C., Chiaia B., D’Angelo S., Ferretti D. (2009),“A macro-element strategy for the seismic assessment and rehabilitation of a historical theatre in southern Italy”, International Journal of Architectural Heritage, in press.

[3] Trovalusci P., Masiani R., (1996), "Cosserat and Cauchy materials as continuum models of brick masonry", Meccanica, 31(4), pp. 421-432.

[4] Cecchi A., Milani G., Tralli A. (2007), A Reissner–Mindlin limit analysis model for out-of-plane loaded running bond masonry walls, International Journal of Solids and Structures, 44, pp.1438-1460.

[5] Carpinteri, A., Chiaia, B. and Bocca P. (1997) “Size dependence of the strength and fracture properties of brick masonry walls” *ASCE Journal of Engineering Mechanics*, **123**, 816-822.

[6] Carpinteri, A., (1994) “Scaling laws and renormalization groups for strength and toughness of disordered materials”, *International Journal of Solids and Structures* **31**, 291–302.

[7] Augusti, G., (1970) “Mode approximations for rigid-plastic structures supported by an elastic medium”, Rhode Island, Vol. 6, pp. 809-827.

[8] Baratta, A., Cennamo, C., Voiello, G., (1995) “Approssimazione modale della risposta di strutture rigido-plastiche sotto carico dinamico”, *XII Congresso Nazionale dell’Associazione Italiana di Meccanica Teorica e Applicata (AIMETA‘95)*, Napoli.

[9] Lourenço P.B., (2001), “Analysis of historical constructions: from trust-lines to advanced simulations”. *III Structural Analysis of Historical Constructions*, Guimares.

[10] Rots, J., (1997), *Structural Masonry, an Experimental/ Numerical Basis for Practical Design Rules*, Taylor & Francis, London.

[11] Baratta A., Cennamo C., Corbi O., (2000), “A*pplication of control techniques to masonry and monumental constructions”*. **LCPC 2000**, **3rd Int. Workshop on Structural Control**, Champs-sur-Marne.

[12] AA.VV. (2007), *Guidelines for evaluation and mitigation of seismic risk to cultural heritage*, Gangemi, Roma.