An interface micro-mechanical approach for the masonry mechanisms analysis

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SUMMARY. In this paper a micromechanical numerical procedure based on the homogenization approach is proposed to assess the strength domain of masonry material, taking into account the effects of the interface phenomena through a specific contact model. The numerical analyses are performed in the elastic field with a generalized plane state formulation in axial deformation. In particular two different methods are used. The first considers an overall criterion based on the strength values of three phases assessed by their mean stress tensors: brick, mortar and interface. The numerical formulation allowes to apply the method to a masonry arranged both in a periodic and a non-periodic pattern. The latter method takes into account the arising of the local phenomena and the foundamental failure mechanisms are recovered. As a preliminary application, this method is applied to a periodic masonry. In particular a periodic unit cell, composed by different phases as brick, bed and head mortar joints, bed and head mortar interfaces, is analyzed with periodic boundary conditions. Then, the failure load for the homogenized cells is obtained when reaching the failure criteria of any of the components in terms of the mean stress tensor. The numerical results are in a good agreement with the experimental observations.

1 INTRODUCTION

In the last decades, many numerical models able to describe the behavior of masonry have been developed using tools drawn from different branches of mechanics, as for example the theory of elasticity and plasticity, fracture mechanics, micromechanics of the continuum media and from the homogenization techniques.

In particular, in the context of the homogenization approaches, many analysis methods relevant to the periodic masonry loaded in the one's own plane have been proposed both in the elastic field [1, 2, 3, 4] and in the plastic one. In the latter case, different modeling choices were done in order to obtain the strength domain [5, 6] or the overall response beyond the elastic limit of the heterogeneous material using both interface models [7] and damage constitutive laws [8, 9, 10].

For what concerns non-periodic masonry, Cluni and Gusella proposed a procedure to evaluate the elastic characteristics of the homogenized continuum using the "test windows method" [11, 12] to find the RVE. In Šejnoha et al. [13] a different procedure is proposed, in which the homogenized elastic-plastic characteristics are obtained on a statistically equivalent periodic unit cell (SEPUC), determined through the analyses of geometric stochastic parameters.

In this paper, in order to estimate the strength domain of periodic and non-periodic masonry material, two different approaches are proposed.

The first determines the strength domain by means of an overall failure criterion based on the strength values of three phases assessed by their mean stress tensors: brick, mortar and interface. The procedure is able to reach the principal failure modes with considerable advantages in terms of computational costs. At first the method is applied to a masonry arranged in a periodic pattern and then, owing to the universal form of the procedure, to an actual masonry arranged in a non-periodic pattern.

The latter approach presents a more detailed formulation and is able to describe the development of local failure mechanisms that involve the global collapse. As a preliminary application, this method is applied to a periodic masonry through the analysis of a Periodic Unit Cell subjected to periodic boundary conditions.

The numerical results have been compared with experimental ones found in the literature.

2 FAILURE CRITERIA FOR MASONRY MATERIAL AND ITS CONSTITUENTS

The definition of a failure criterion for masonry material is a problematic matter because, generally, all phases participate to the overall strength, each with a peculiar, different behavior.

The method proposed in this paper is based on the assumption that the overall strength limit of the specimen is reached when, with the increase of the boundary loads or of the boundary displacements, the mean stress tensor of one of the phases satisfies its own yield condition. It should be noted that, in this way, the local evolution of microcracks and/or plastic strains developed before the global collapse is taken into account in an overall way. On the other hand, the method allows to perform elastic analyses in generalized plane state of axial deformation with significant computational advantages and, at the same time, adequate results, as will be shown in the following.

Owing to the linearity of the method, which allows effects superposition, and since the problem is kinematically plane, each specimen has been studied under only six different kinds of boundary conditions, $\boldsymbol{\sigma}_\xi^0$ and $\boldsymbol{\varepsilon}_\xi^0$ (with $\xi = 1, 2, 3$):

$$\boldsymbol{\sigma}_1^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\sigma}_3^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

$$\boldsymbol{\varepsilon}_1^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_3^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

where $\boldsymbol{\sigma}^0$and $\boldsymbol{\varepsilon}^0$ indicate Neuman and Dirichlet boundary conditions respectively.

Then, the general state of stress $\langle\boldsymbol{\sigma}_\eta\rangle$, with $\eta = \sigma, \varepsilon$ according to the kind of boundary conditions applied, has been obtained from the variation of the a, b, c coefficients in the expression

$$\langle\boldsymbol{\sigma}_\eta\rangle = \kappa\big[a\langle\boldsymbol{\sigma}_{1\eta}^0\rangle + b\langle\boldsymbol{\sigma}_{2\eta}^0\rangle + c\langle\boldsymbol{\sigma}_{3\eta}^0\rangle\big] \tag{3}$$

where κ is a small factor that provides a state of stress far enough from the crisis and $\langle\boldsymbol{\sigma}_{\xi\eta}^0\rangle$ is the overall mean stress tensor given by the application of ξ type and η boundary conditions.

Being $\langle\boldsymbol{\sigma}_\eta^{(i)}\rangle$ the mean stress state of the i^{th} phase and $\lambda_\eta^{(i)}$ a stress state multiplicative factor by which the strength condition of the phase is satisfied

$$F^{(i)}(\lambda_\eta^{(i)}\langle\boldsymbol{\sigma}_\eta^{(i)}\rangle) = 0, \tag{4}$$

the ultimate strength value of the heterogeneous material is defined by the following overall mean stress tensor

$$\boldsymbol{\sigma}_\eta^f = \min_i\{\lambda_\eta^{(i)}\} \cdot \langle\boldsymbol{\sigma}_\eta\rangle. \tag{5}$$

For what concerns the choice of the strength condition for each material phase a failure criterion proposed by Lubliner et al. [14], in which both the isotropic and the deviatoric parts of the stress

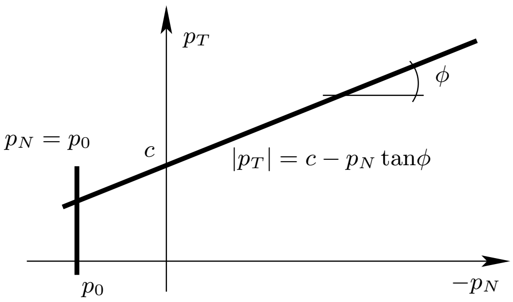


Figure 1: Mohr-Coulomb with tension cut-off criterion assigned to the interface.

tensor participate to the ultimate strength, has been used. The yield function calculated on the mean stress tensor of the i^{th} phase is

$$F(\langle\boldsymbol{\sigma}\rangle) = \frac{1}{1-\alpha}\left[\alpha I_1 + \sqrt{3J_2} + \beta\hat{\sigma}_{\max}\right] - \sigma_c \tag{6}$$

where I_1 and J_2 are the first invariant of the isotropic part and the second invariant of the deviatoric part of the stress tensor respectively, α and β are parameters which depend on uniaxial tension and compression (σ_t and σ_c) and equibiaxial compression (σ_b) of each single phase, as defined below

$$\alpha = \frac{\sigma_b - \sigma_c}{2\sigma_b - \sigma_c} \qquad\qquad \beta = \frac{\sigma_c}{\sigma_t}\big(1-\alpha\big) - \big(1+\alpha\big) \tag{7}$$

and $\hat{\sigma}_{\max}$ is the maximum value of the principal stress tensor.

Concerning the interface, let us consider the generic point $\mathbf{x}$, belonging to the contact surface between the phases, and the contact pressure $\mathbf{p}(\mathbf{x}) = \{p_N, p_T\}$, where p_N and p_T are the normal and tangential stress components. At the interface a Mohr-Coulomb criterion with tension cut-off is used

$$\begin{array}{ll} |\langle p_T\rangle| = c - \langle p_N\rangle\,\tan\phi & \qquad \langle p_N\rangle \geq p_0 \\ \langle p_T\rangle = 0 & \qquad \langle p_N\rangle < p_0 \end{array} \tag{8}$$

where c and ϕ are the coesion and the friction angle respectively (Fig. 1). It should be noted that the strength criterion (8) is expressed in terms of macroscopic values of contact stresses calculated over the interface.

3 OVERALL EVALUATION OF STRENGTH DOMAIN

3.1 *Periodic masonry*

The numerical procedure explained before has been applied to the periodic masonry. The analysis has been performed in ABAQUS environment using bidimensional elements with 3 and 4 nodes with a generalized plane state formulation according to two different approaches:

- in a first step the three models at different values of bed joints slope ($0°$, $22.5°$ and $45°$ from the vertical direction) have been considered (Fig. 3.1) and the results obtained by the application of boundary conditions $\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0$ and $\boldsymbol{\varepsilon}_1^0, \boldsymbol{\varepsilon}_2^0$ have been superposed;

- in a second step the boundary conditions $\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0$ and $\boldsymbol{\varepsilon}_1^0, \boldsymbol{\varepsilon}_2^0, \boldsymbol{\varepsilon}_3^0$ have been applied to define the failure surface in three-dimensional space σ_n, σ_p, τ, where σ_n ,σ_p and τ are the components of stress normal, parallel and tangential to the bed joints.

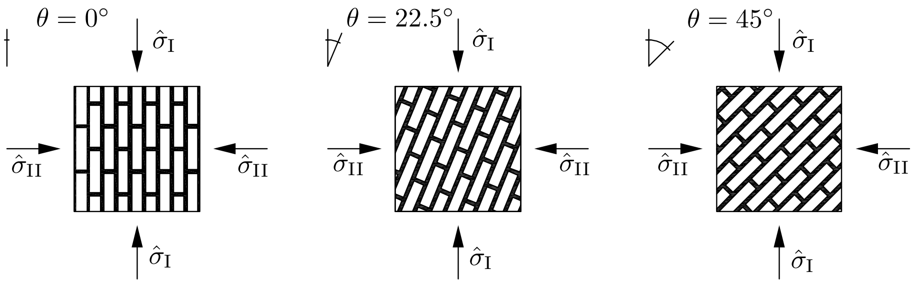


Figure 2: Specimen layout of the experimental test.

Phase	E [MPa]	ν	σ_c [MPa]	σ_t [MPa]	σ_b [MPa]	p_0 [MPa]	c [MPa]	ϕ
Brick	6740	0.167	15.41	1.5	17.0	-	-	-
Mortar	1700	0.20	5.08	0.5	6.0	-	-	-
Interface	-	-	-	-	-	0.13	0.35	30°

Table 1: Mechanical characteristics of the masonry constituents - E: Young's modulus; ν: Poisson's ratio; σ_c: failure stress in uniaxial compression; σ_t: failure stress in uniaxial tension; σ_b: failure stress in equibiaxial compression (see [14] for details); p_0: tensile strength at the interface; c: coesion; ϕ: friction angle.

The choice of the mechanical parameters has been based on the experimental data [15, 16] and the numerical elaborations of Shieh-Beygi and Pietruszczak [17] for the missing values (Tab. 1). Fig. 3 shows the comparison between the numerical results and the experimental ones obtained in the first step of analysis, where the zones corresponding to different failure mechanisms are indicated. The curves differ in the boundary conditions applied, natural (dashed line) and essential (continuous line). It should be noticed that the failure of the interface mainly occur under biaxial tension-tension and tension-compression states of stress, while the failure of brick prevails in biaxial compression, in agreement with the experimental observations.

Fig. 4 shows the comparison between the numerical results and the experimental ones obtained in the second step of the analysis. Fig. 4(a) displays the strength domain plotted on the σ_n, σ_p plane, for different values of τ. It should be put in evidence that the failure mechanisms obtained from the proposed procedure are comparable with the ones described in [18] (Fig. 4(b)): following the symbolism of that paper, in region A the failure occurs by splitting of the panel in a plane parallel to the faces of the panels, in region B the failure takes place along bed joint planes, with the panel capable of sustaining a shear stress on the bed joint after the initial failure; in region C the failure takes place along bed joint planes with separation of the bed joints; in region D the failure involves separation of the vertical joints.

3.2 *Non-periodic masonry*

The procedure is then applied to estimate the strength domain of a masonry arranged with a non-periodic pattern (Fig. 5(a)). The wall, having 2000 mm square size, has been subdivided using four partitions named P_1, P_2, P_3 and P_4 made of 25, 16, 9 and 4 portions respectively (Fig. 5); the term P_5 indicates the entire panel.

The test windows method has been used to obtain the elastic parameters and the strength domain

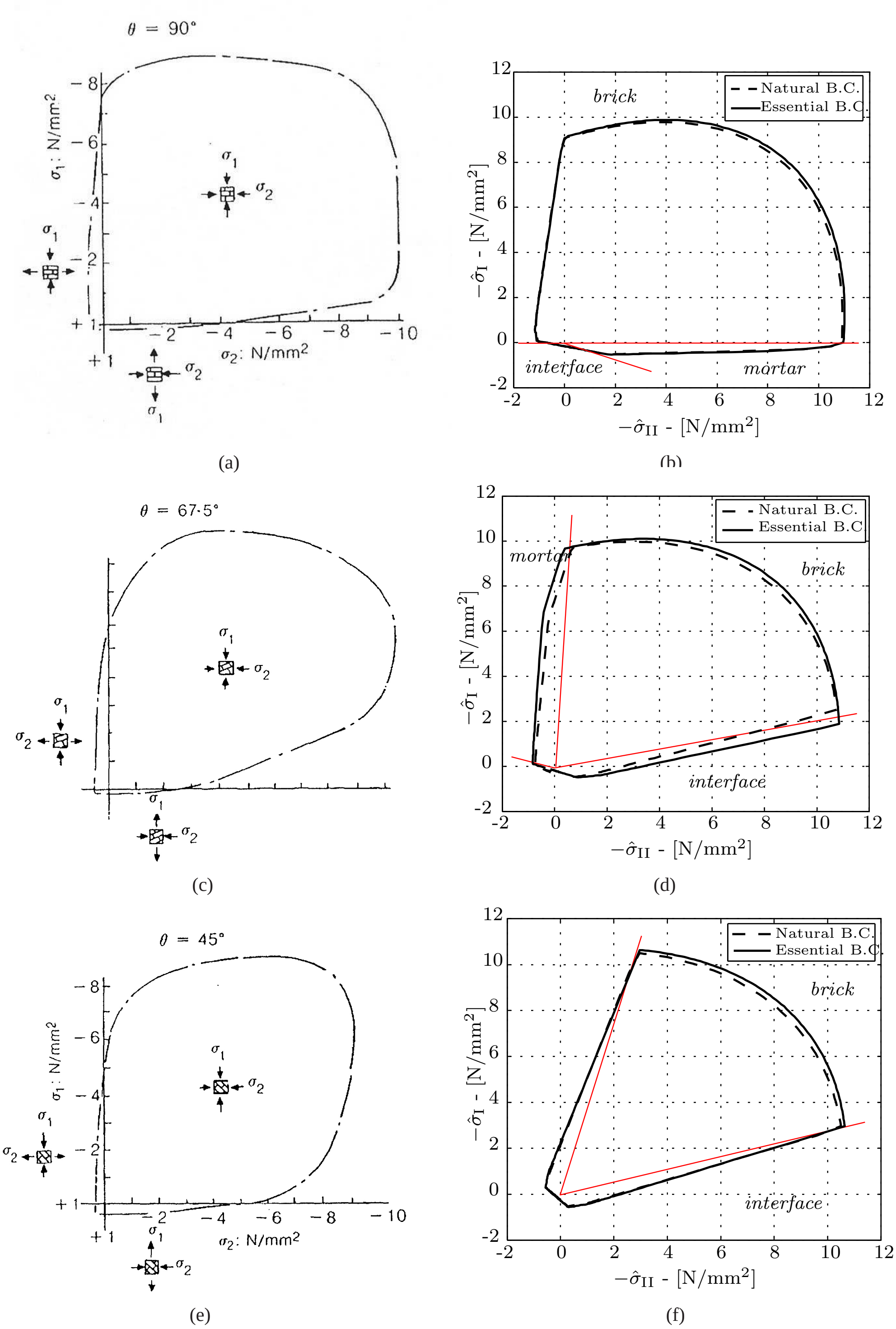
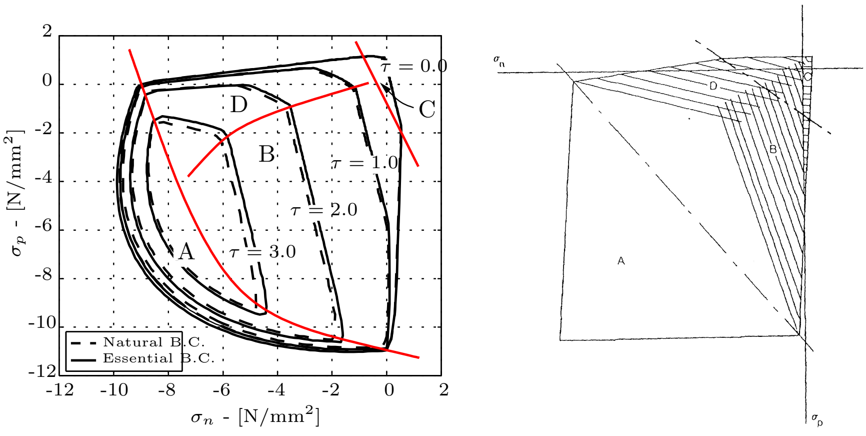
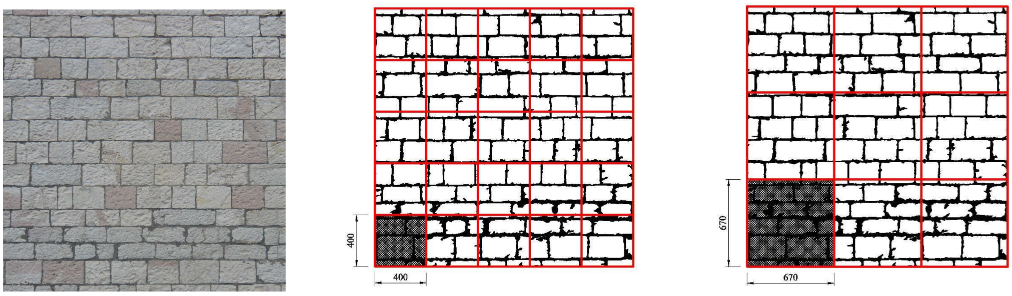


Figure 3: Strength domains of periodic masonry obtained by experimental tests [15] (a)-(c)-(e) and by numerical procedures (b)-(d)-(f).



(a) (b)

Figure 4: (a) Strength domain for periodic masonry plotted on the σ_n, σ_p plane, for different values of τ. (b) Failure mechanisms observed during experimental tests [18].



(a) (b) (c)

Figure 5: Photographic image of the non-periodic masonry wall analysed (a); examples of partitions of the masonry wall: P_1 with $n_1 = 25$ (b) and P_3 with $n_3 = 9$ (c).

of the homogenized continuum [19]. The hierarchy of strength domains (9) proposed by He [20] is respected

$$\bigcap_{\mathbf{x}\in\Omega} \mathscr{D}(\mathbf{x}) \subseteq \bigcap_{i=1}^{n} \mathscr{D}^{app}_{\sigma i} \subseteq \mathscr{D}^{app}_{\sigma} \subseteq \mathscr{D}^{app}_{\varepsilon} \subseteq \sum_{i=1}^{n} \gamma_i \mathscr{D}^{app}_{\varepsilon i} \subseteq \langle \mathscr{D} \rangle. \tag{9}$$

in which $\mathscr{D}^{app}_{\sigma i}$ and $\mathscr{D}^{app}_{\varepsilon i}$ are the apparent strength domains of the i^{th} portion for natural and essential boundary conditions respectively, $\mathscr{D}(\mathbf{x})$ is the local strength domain of a generic point and the two bounding values

$$\bigcap_{\mathbf{x}\in\Omega} \mathscr{D}(\mathbf{x}) \qquad \langle \mathscr{D} \rangle = \frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega} \mathscr{D}(\mathbf{x}) d\Omega \tag{10}$$

represent the limits of Sachs and Taylor respectively.

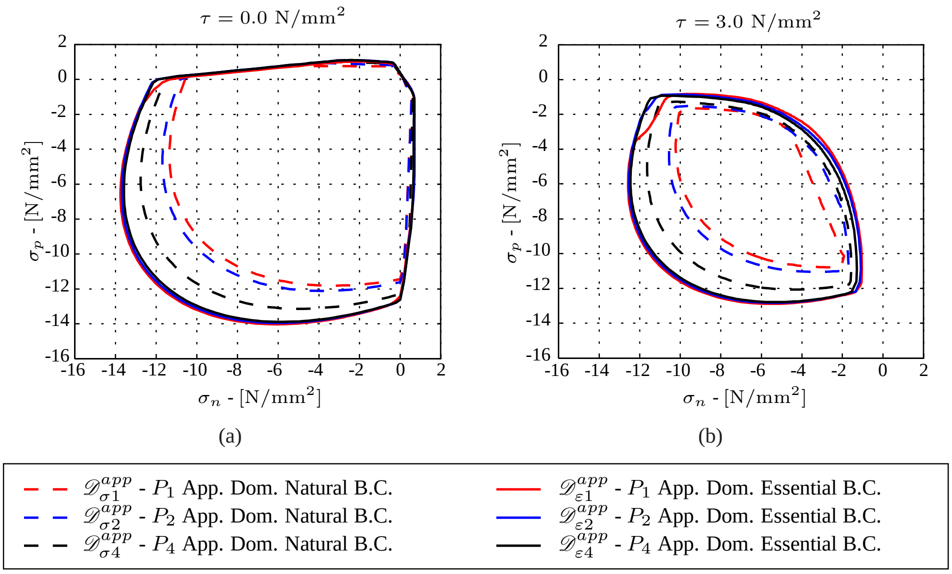


Figure 6: Strength domains: convergence of the homogenized failure surfaces of the partitions P_1, P_2 and P_4 on $\tau = 0.0$ (a) and $\tau = 3.0$ (b) planes.

Figure 6 graphically shows the convergence of the surfaces with the increase in size of portions. In particular, the surfaces of partitions P_1 (red), P_2 (blue) and P_4 (black) evaluated in natural (dashed line) and essential (continuous line) boundary conditions have been sectioned by planes with constant values of τ. The results differ from the ones obtained in absence of the contact model [19] mainly for the presence of p_N at the interface, while the effects concerning the p_T appear not relevant for portions of certain dimensions (greater or equal than those of P_3).

4 PERIODIC UNIT CELL AND FAILURE CRITERION

4.1 *Homogenization of composite materials with periodic microstructure: the case of masonry*

The theoretical discussion of the effective properties of heterogeneous materials with periodic microstructure is largely treated in the literature [21, 22]. For heterogeneous material with periodic microstructure the Representative Volume Element is defined by a Periodic Unit Cell and the effective properties of the composite are determined from the geometrical and the material properties of the PUC wich generates by periodic repetition the whole microstructure of the composite. Due to the repetition of the cell in all directions according to the prescribed translation vectors, periodic deformations and anti-periodic tractions have to be satisfied at each corresponding pair of boundary nodes. A homogeneous equivalent material can be determined if the displacement field is also assumed to exhibit some forms of local periodicity. These assumptions allow to identify the periodic unit cell on which the equilibrium problem can be formulated and solved [2].

Let us consider a periodic masonry arranged in a running bond pattern. The unit cell is not uniquely defined, however the effective behavior computed from different unit cells generating the same microstructure should coincide (Fig. 7). The choice of the unit cell is often motivated by the difference in geometrical symmetries which can be used to simplify the numerical solution in terms of the definition of the periodic boundary conditions.

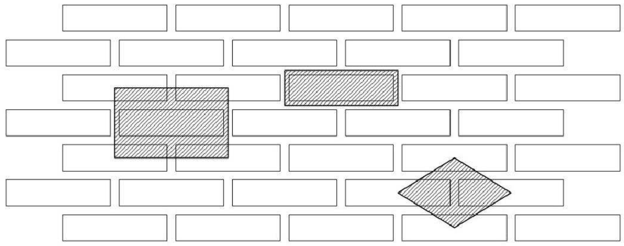


Figure 7: Examples of possible periodic unit cell for the running bond masonry.

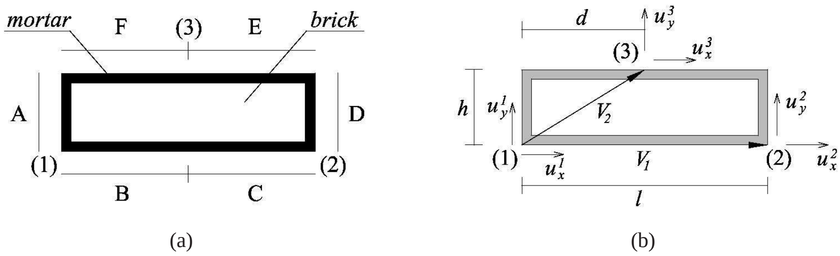


Figure 8: (a) Periodic unit cell used in the analysis. (b) Translation vectors and DOF of the PUC.

In the following the periodic unit cell shown in Fig. 8 will be considered. The periodicity of the displacement fluctuations at the boundary enforces the corresponding segment pairs (A-D, B-E, C-F) to exhibit the same shape in the deformed configuration (Fig. 8(a)). This is obtained by using the concept of controlling points introduced by Teply and Dvorak [23] and then imposing the following linear constraints between the points belonging to the respective boundary segments

$$\mathbf{u}^D - \mathbf{u}^A = \mathbf{u}^2 - \mathbf{u}^1 \tag{11}$$

$$\mathbf{u}^E - \mathbf{u}^B = \mathbf{u}^3 - \mathbf{u}^1 \tag{12}$$

$$\mathbf{u}^F - \mathbf{u}^C = \mathbf{u}^3 - \mathbf{u}^2. \tag{13}$$

Considering the framework refernce given by the translation vectors $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ (Fig. 8(b)), a specific set of boundary conditions can be applied to the six degrees of freedom $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$ to obtain a prescribed macroscopic tensor of strain $\langle \boldsymbol{\varepsilon} \rangle$ or stress $\langle \boldsymbol{\sigma} \rangle$ [24]. To inhibit any rigid displacements of the basic cell the following conditions are imposed

$$\begin{aligned} &u_x^1 = u_y^1 = 0 \\ &l u_x^3 = d u_x^2 + h u_y^2 \end{aligned} \tag{14}$$

and then a general macroscopic strain tensor $\langle \boldsymbol{\varepsilon} \rangle = (\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22})$ is attained by

$$\begin{aligned} &u_x^2 = l\varepsilon_{11} \\ &u_y^2 = l\varepsilon_{12} \\ &u_y^3 = d\varepsilon_{12} + h\varepsilon_{22} \end{aligned} \tag{15}$$

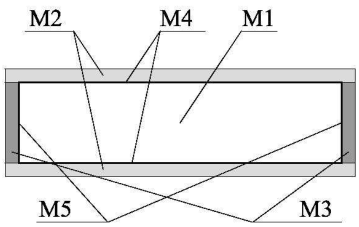


Figure 9: Identification of the five internal phases.

that characterize particular Dirichlet boundary conditions, while a general macroscopic stress tensor $\langle\boldsymbol{\sigma}\rangle = (\sigma_{11}, \sigma_{12}, \sigma_{22})$ is obtained by applying the master forces F_x^2, F_y^2 and F_y^3 on the corresponding degrees of freedom

$$\begin{aligned} F_x^2 &= h\sigma_{11} \\ F_y^2 &= 2h\sigma_{12} - d\sigma_{22} \\ F_y^3 &= l\sigma_{22} \end{aligned} \tag{16}$$

that determine Neuman boundary conditions. In the analyses the conditions (14) and (16) have been used.

According to the geometry and the peculiar arrangement of the masonry, five internal phases are distinguished to properly describe the development of the failure mechanisms: brick (M1), mortar bed joint (M2), mortar head joint (M3), bed joint interface (M4) and head joint interface (M5) (Fig. 9).

The overall failure criterion described in the section 2 is modified in this way: the interface is splitted in two parts, the bed joints (M4) and the head joints (M5), but only the satisfaction of the brick (M1) or mortar (M2+M3) criteria determines the overall failure; if one of the two mean contact pressures belonging to M4 and M5 satisfy its own strength condition the beginning of the development of a local failure mechanism is considered. At this point the cell analyzed is replaced with another one that presents a strength downgrade of the contact formulation in the corresponding interfaces, by which penetration and sliding between the two components (brick and mortar) are still prevented, but the separation is allowed. The strength downgrade of the contact formulation is then reproposed until the failure of M1 or M2 or M3 is attained. In Fig. 10 a scheme of the procedure is illustrated.

As a preliminary application, the Fig. 11 shows the results obtained in absence of macroscopic tangential components. The comparison with the experimental results obtained by Page [15, 16] is encouraging because the foundamental developments of failure mechanisms are recovered.

5 CONCLUSIONS

In this paper two different approaches for the strength domain evaluation of periodic and non-periodic masonry has been proposed.

The numerical analyses are based on several peculiar aspects, as the use of the generalized plane state formulation, the characterization of a specific contact model for the evaluation of the contact pressures at the interface, the definition of a consistent overall failure condition for the composite material and the assignement of a failure criterion to each phase participant to the overall strength.

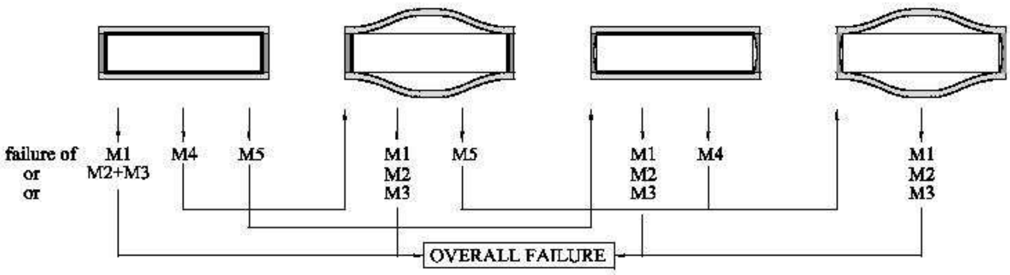


Figure 10: Scheme of the numerical procedure.

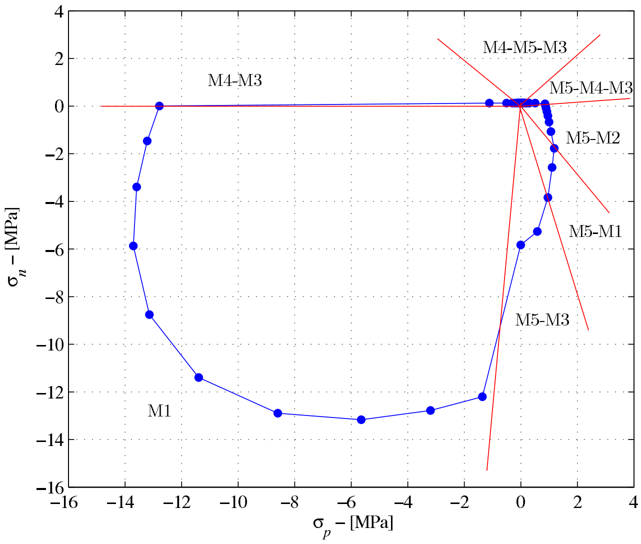


Figure 11: Strength domain for periodic masonry considering local phenomena.

With the first approach both the periodic and non-periodic masonry have been studied. The results obtained with periodic masonry have been compared with the exeprimental data available in the literature verifying the agreement for the principal failure modes. In a second step, an actual non-periodic masonry has been analyzed and the results compared with those obtained in absence of the interface modeling.

With the latter approach the basic concepts for the evaluation of the effective properties of heterogeneous materials with periodic microstructure have been used. In this way, only periodic masonry has been analyzed. Periodic boundary conditions have been assigned to a prescribed Periodic Unit Cell and the development of the foundamental failure mechanisms are recovered. The preliminary results are in agreement with those obtained by the first approach and with the experimental data available in the literature.

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