Nonlinear Normal Modes of 2-DOF piecewise linear mechanical systems

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Keywords: Nonlinear normal modes, piecewise linear systems, bifurcation diagrams, internal resonance.

SUMMARY. Nonlinear modal properties of a vibrating system with piecewise linear restoring forces typical of a beam with a breathing crack are investigated. The system is non linearizable and exhibits the peculiar feature of a number of nonlinear normal modes (NNMs) greater than the degrees of freedom; since the nonlinearity is concentrated at the origin, its nonlinear frequencies are independent of the energy level and uniquely depend on the damage parameters. The influence of damage on the nonlinear frequencies has been investigated and bifurcation scenarios characterized by the onset of superabundant modes in internal resonance have been analyzed. Bifurcated modes exhibit significantly different shape than that of modes on fundamental branch. Finally the forced response of the system under base excitation has been studied. It has been found that additional nonlinear resonances occur in the neighborhood of the frequencies of the superabundant NNMs.

1 INTRODUCTION

The classical modal analysis in linear dynamics can be extended to nonlinear systems by introducing the concept of nonlinear normal modes (NNMs). According to Rosenberg [1,2], a NNM of an undamped system is defined as a synchronous periodic oscillation where all generalized coordinates of the system reach their extreme values or pass through the zeros simultaneously. To make this definition suitable for nonsmooth systems, it is also necessary to include the periodic motions in which all generalized coordinates vibrate equiperiodically without passing through the zeros simultaneously [3, 4]. The NNMs of a system are important because, in analogy to linear theory, resonance in forced systems typically occurs in the neighborhood of NNM frequencies. Hence, knowledge of the normal modes of a nonlinear system can provide valuable insight regarding the position of its resonances, a feature of considerable engineering importance. Moreover, since the number of normal modes of a nonlinear system may exceed its degrees of freedom (*superabundant NNMs*), certain forced resonances are essentially nonlinear and have no analogies in linear theory; in such cases a linearization of the system might not be possible, or might not provide all the possible resonances that can be experienced.

In a recent previous work [4], a discrete model of a beam with a breathing crack, an example of non-linearizable system, has been dealt with using the asymptotic method of Lindstedt-Poincaré, and limiting the analysis to the fundamental branch solutions and their stability. In [5] a similar system has been then numerically investigated by means of the Poincaré map: particular attention has been focused to the onset of superabundant modes besides the fundamental branches.

The present paper analyzes the nonlinear modal characteristics of a more general 2-DOF piecewise-smooth mechanical system with two damage parameters. The system is non-linearizable and thus can exhibit an unusual feature that is the number of NNMs is greater than the degrees of freedom. Since the nonlinearity is concentrated at the origin of the restoring force, its frequencies are independent of the energy level or oscillation amplitude; they depend uniquely on the two damage parameters.

2 SYSTEM MODEL

The oscillator under investigation consists of two masses connected by two piecewise linear springs, Fig. 1a, with damage parameters ε_1 and ε_2: the relevant restoring forces exhibit the bilinear behaviour shown in Fig. 1b. The phase space of this dynamical system can be divided into four regions delimited by two discontinuity boundaries, Fig. 1c, in each region the system having a different smooth functional form of the vector field. Since the vector field is the same in the adjacent regions, whereas its Jacobian changes due to the bilinear stiffness, the problem is governed by a *continuous* PSS, according to the definition widely used in the literature. Depending on the values of masses and undamaged stiffnesses, this oscillator can model the dynamics of a damaged shear-type frame as well as an asymmetrically cracked cantilever beam vibrating in bending and hence exhibiting a bilinear stiffness, depending on whether the crack is open or closed. For ε_1=0 or ε_2=0, the oscillator under investigation turns into the particular cases investigated in [3,4,5].

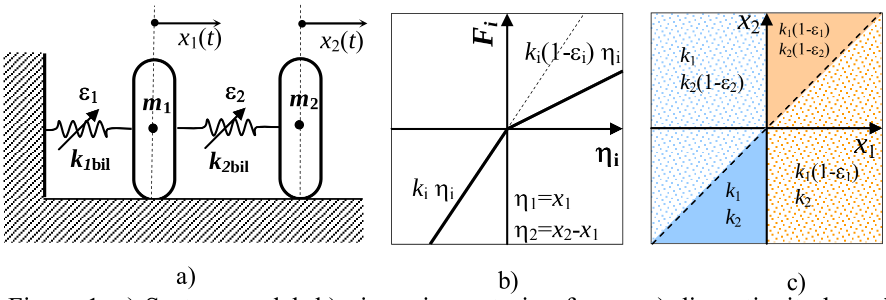


Figure 1. a) System model; b) piecewise restoring forces; c) discontinuity boundaries in the physical plane.

2.1 *Equations of motion*

By assuming the displacements x_1 and x_2 as Lagrangian coordinates, the stiffnesses of the nonlinear springs read, Fig. 1b:

$$k_{bil,i} = k_i(1 - H(\eta_i)\varepsilon_i), \quad H(\eta_i) = \begin{cases} 1 & \eta_i \geq 0 \\ 0 & \eta_i < 0 \end{cases} \quad i = 1,2 \tag{1}$$

where ε_1 and ε_2 are the damage parameters, H is the Heaviside function, $\eta_1=x_1$ and $\eta_2=x_2-x_1$.
With reference to Fig. 1, the following dimensional equations of undamped free motion in time domain are found:

$$\begin{cases} m_1\ddot{x}_1 + [k_{bil,1} + k_{bil,2}]x_1 - k_{bil,2}x_2 = 0 \\ m_2\ddot{x}_2 + k_{bil,2}\, x_2 - k_{bil,2}x_1 = 0 \end{cases} \tag{2}$$

2.2 *Linear Normal Modes*

For $\varepsilon_1=\varepsilon_2=0$ the system is linear and exhibits the two LNMs $\mathbf{u_{01}}$ and $\mathbf{u_{02}}$ the modal lines of which are straight line passing through the origin with frequencies:

$$\omega^2_{01,02}=\frac{(\alpha+\beta+\alpha\beta)\mp\sqrt{(\alpha+\beta+\alpha\beta)^2-4\alpha\beta}}{2\alpha}\frac{k_1}{m_1} \tag{3}$$

where $\alpha=m_2/m_1$, $\beta=k_2/k_1$.
The frequency ratio ω_{02}/ω_{01} for the undamaged oscillator will be denoted r_0 and, according to Eq. (3), is given by:

$$r_0=\frac{\omega_{02}}{\omega_{01}}=\sqrt{\frac{(\alpha+\beta+\alpha\beta)+\sqrt{(\alpha+\beta+\alpha\beta)^2-4\alpha\beta}}{2\sqrt{\alpha\beta}}} \tag{4}$$

Parameter r_0 uniquely depends on nondimensional parameters α and β.

As it will be shown, the dynamic behaviour exhibited by the system when $\varepsilon_1\neq0$ and/or $\varepsilon_2\neq0$ is strongly affected by r_0; for instance are of interest the particular cases: r_0=1.95, below the 1:2 internal resonance, r_0=2.62, typical of a shear-type frame with equal masses and stiffnesses and below the 1:3 internal resonance, r_0=6.27, typical of a cantilever beam already addressed in [4]. These values will be therefore taken into account in the subsequent analyses.

2.3 *Nonlinear Normal Modes*

In presence of damage an analytical estimate of the frequency ratio is given by [6]:

$$r(\alpha,\beta,\varepsilon_1,\varepsilon_2)=\frac{\omega_2}{\omega_1}\cong\frac{\omega_{eq2}}{\omega_{eq1}} \qquad r(\alpha,\beta,0,0)=r_0 \tag{5}$$

where:

$$\omega_{eq1}=\frac{2\omega_{01}\omega_{\varepsilon1}}{\omega_{01}+\omega_{\varepsilon1}},\quad \omega_{eq2}=\frac{2\omega_{02}\omega_{\varepsilon2}}{\omega_{02}+\omega_{\varepsilon2}} \tag{6}$$

The frequencies ω_{0i}, Eq. (3), and $\omega_{\varepsilon i}$ can be interpreted as the eigenfrequencies before and after the stiffness changes exhibited by the springs, assuming that these changes occur at the same time.

The nonlinear frequency ratio r is independent of m_1 and k_1, and is considered as a distinctive smooth function of damage ε_1 and ε_2 for piecewise linear oscillators characterized by parameters α and β; furthermore when r attains an integer or a rational value an internal resonance occurs and it can be the origin of a bifurcation. When ε_1 and ε_2 vary, first and second NNM may interact generating (*n:m*) internal resonances: this occurs when the nonlinear frequencies ω_1 and ω_2 are nearly commensurate i.e. $n\omega_1\cong m\omega_2$.

3 FREE RESPONSE

An extensive analysis of the system's NNMs is performed in [6], here only some significant cases will be reported. Figures 2 and 3 refer to bifurcations occurring in (*n:1*) internal resonance for a system with r_0=2.62 and ε_2=0.

Figure 2 reports the period-damage plot with the various branches of (*n:1*) periodic solutions: the period of Mode 2 (black curve) is almost constant, whereas the period of Mode 1 significantly increases with ε_1. There is a sequence of higher periodic NNMs bifurcating from the backbone of Mode 1, called tongues. Each tongue takes place in the neighborhood of a (*n:1*) internal resonance. Enlargement in Fig. 2 and Figures 3 refer to the case (3:1): at $\varepsilon_1=\varepsilon_{3:1}$ one stable and one unstable superabundant NNM (C) generated by a cyclic-fold bifurcation appear. In particular, Figures 3 report the Poincaré maps with the relevant modal line in the configuration plane (x_1, x_2).

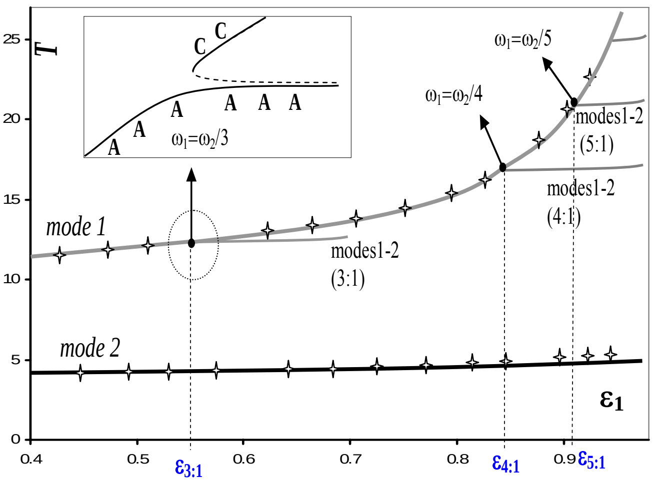


Figure 2 Period-damage plot of NLMs 1 and 2: (n:1) internal resonance.

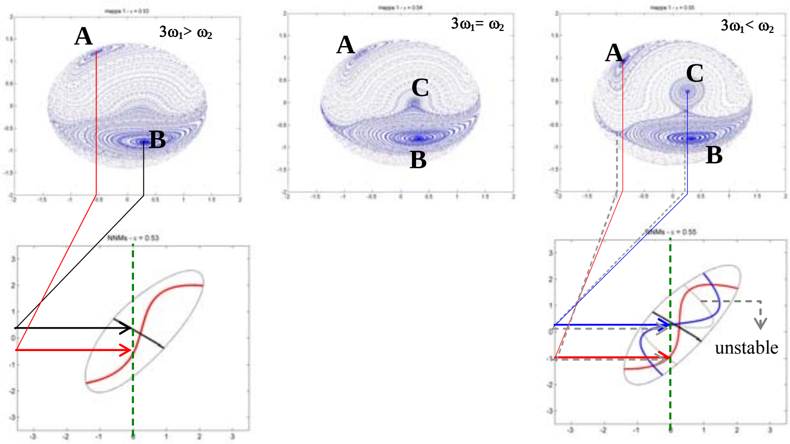


Figure 3. Cyclic-fold bifurcation caused by a (3:1) internal resonance around $\varepsilon_{3:1}$=0.54

Figures 4 refer to the bifurcation in the neighborhood of the (7:2) internal resonance: the first mode (A) loses its stability and bifurcates in a NNM (B) with period doubling. Furthermore a second NNM (C) appears and approaches the unstable NNM as long as the first mode recovers the stability. Qualitatively similar behaviour is exhibited by the other (n:2) bifurcations: however for larger values of ε_1, the bifurcated modal curves are more complicated and the windows around $\varepsilon_{n:2}$ becomes narrower.

Qualitatively different changes in the Poincaré maps are produced by (n:3,4,5…) resonances. For instance, Figures 5 refer to the case (8:3): unlike the case (n:2), the first mode (A) is always stable but two pairs of stable and unstable NNMs (B, C) appear and disappear; the frequency content in B and C is characterized by two main frequencies the ratio of which is exactly 8-to-3.

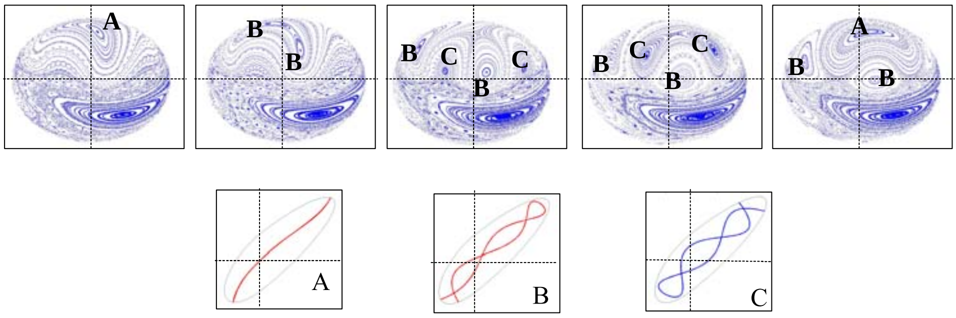


Figure 4. (7:2) internal resonance near $\varepsilon_{7:2}$=0.7585: period-doubling.

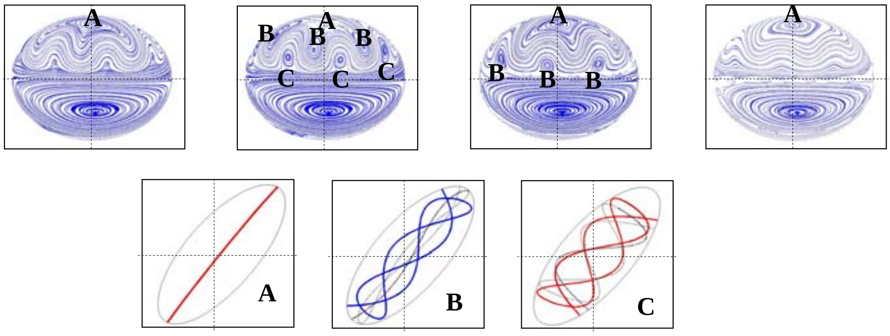


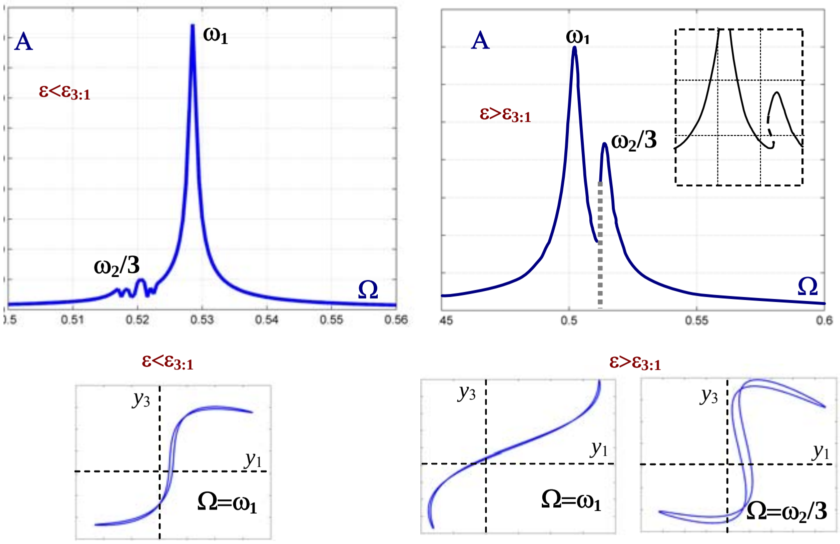
Figure 5. (8:3) internal resonance near $\varepsilon_{8:3}$=0.138.

4 FORCED RESPONSE

An harmonic base excitation $a(t)=a_g \sin(\Omega t)$ is then considered and both low and high damping is assumed to investigate the counterpart of the illustrated phenomena within the forced response.

As an example the forced response for damage parameter near $\varepsilon_{3:1}$ is here considered. In Figs. 6 the frequency response curves are reported by assuming low damping and two cases have been analysed: *i*) ε_1 slightly smaller than $\varepsilon_{3:1}$, Fig. 6a, and *ii*) ε_1 slightly larger than $\varepsilon_{3:1}$, Fig. 6b; in the same figures, the steady-state response corresponding to $\Omega=\omega_1$ and $\Omega=\omega_2/3$ are also shown in the x_1-x_2 plane. In the case $\varepsilon < \varepsilon_{3:1}$ the amplitude-frequency plot shows a significant peak corresponding to the resonance $\Omega=\omega_1$: the relevant steady-state response exhibits the same shape exhibited by the first nonlinear normal mode (A) of the autonomous system, Fig. 3 (red line).

As soon as ε exceeds $\varepsilon_{3:1}$, two peaks appear in the frequency response plot: the shape of the forced response for $\Omega=\omega_1$ changes its curvatures and becomes similar to the nonlinear normal mode (C), Fig. 3 (blue line) whereas the response corresponding to $\Omega=\omega_2/3$ is similar to nonlinear normal mode A.



a) b)

Figure 6. Frequency response curves and steady-state responses for $\Omega=\omega_1$ and $\Omega=\omega_2/3$ near $\varepsilon_{3:1}=0.54$. Low damping. a) $\varepsilon=0.52$; b) $\varepsilon=0.60$.

Depending on the initial conditions, these two different periodic responses can coexist for values of Ω corresponding to the dotted gray line as better illustrated in the zoom of Fig. 6b.

Figure 7 refers to the same case of Fig. 6b but with high damping. The frequency response plot becomes rounded and the steady-state responses corresponding to $\Omega=\omega_1$ and $\Omega=\omega_2/3$ exhibit not clearly distinguishable shapes in the x_1-x_2 plane.

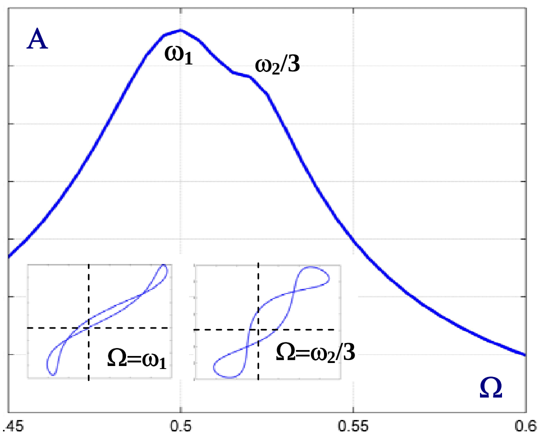


Figure 7. Frequency response curves and steady-state responses corresponding to $\Omega=\omega_1$ and $\Omega=\omega_2/3$ at $\varepsilon=0.60$. High damping.

5 CONCLUSIONS

A 2-DOF piecewise linear oscillator, representative of a cracked beam, has been studied in free oscillations and under harmonic base excitation. A parametric analysis of the NNMs has been performed for a wide range of the damage parameter: the influence of damage on the nonlinear frequencies has been investigated and bifurcations characterized by the onset of superabundant modes have been revealed. The fundamental branches of the two modes, and their stability are then evaluated. The bifurcated branches are followed by a numerical procedure based on continuation method and the stable superabundant modes are determined via direct integration. Particular attention has been devoted to the study nonlinear modal interaction producing global changes in the Poincarè maps. The influence on the forced response has been then investigated. By considering an harmonic base excitation it has been found that the NNMs of the free motion play a key role in the system forced response. In particular, additional nonlinear resonances, not predicted by linear theory, occur in the neighborhood of the frequencies of the superabundant NNMs of the free system.

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