On boundary conditions for Ladevèze's Plate Theory

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SUMMARY. A new theory for linearly elastic plates has been recently proposed which relies on the classical superposition of short and long wavelength solutions. The feature of this approach is that the long wavelength solution, namely Saint Venant's solution, is derived in exact way depending only on two generalized 2D variables. Due to difficulties in properly defining the boundary conditions associated to this (arbitrarily thick) plate theory starting from local considerations, a variational framework has been now formulated allowing a consistent definition of the associated boundary conditions which converges to the classical ones at Kirchhoff limit.

1 INTRODUCTION

The recently formulated Exact Theory of Plates (ETP) [1-4] relies on a long wavelength Saint Venant's (SV) solution of the isotropic and homogeneous elastic plate problem that is exact, therefore independent from any plate thickness assumption, introducing a point of view that substantially departs from the classical thickness dependent plate formulation framework [5]. In particular the solution depends upon a couple of generalized variables $(\tilde{w}, \tilde{m})$ with values on the plate reference (middle-)plane, which exhibit filtering properties with respect to the localized contributions which both characterizes the 3D exact problem and classical first order plate formulations (e.g. Reissner and Mindlin plates [6-7]). Edge effects, load variations into the domain, as well as boundary layers which are controlled by the transversal spin component, are therefore the separate object of a localized, uncoupled, correction of the long wavelength solution field.

The two second order PDE governing SV solution have been initially derived by exploiting local and global governing equations recalling Marcus formulation for classical plate theories [8]. Despite an easy definition of continuity conditions (in $[H^1(\Sigma_m)]^2$) for the couple of generalized variables can be used for example to enforce the connection with an elastic boundary, the direct approach seems to fail in properly defining consistent boundary conditions, except for the simple definition of the generalized simply support condition.

This work therefore aims to give a contribution toward the completion of the framework of SV solution in ETP formulation by introducing a consistent definitions of boundary condition, also permitting the natural recovery of the classical plate formulation (Kirchhoff), as well of recently proposed first order theories [10], only by reducing plate thickness up to thin plates limit case. To obtain the searched result a variational formulation of the problem is firstly derived on the basis of a proper Hellinger-Reissner principle. Stationary conditions of this governing functional produce the aforementioned set of equations as well as the corresponding consistent boundary conditions in compact weak form.

Despite the quite easy interpretation of their mechanical meaning, the obtained conditions actually appears to be not trivially derivable by direct kinematics and static consideration, and this is especially true for the free edge case. The structure of the boundary conditions also reveals that, without any approximation for plates that are completely supported along the boundary and by assuming moderately thick plates hypothesis in the case of free edges, a new generalized variable $\tilde{\varphi} \equiv \tilde{w} - k_t^{-1}\tilde{m}$ ($\kappa_t = \frac{5}{3}\mu h$ is the shear stiffness) can be naturally introduced which has a clear mechanical meaning. In this way a first order irrotational theory (local rotation admit a potential and the normal spin is zero), that is basically equivalent to a Reissner's like plate theory [9] except for the absence of boundary layers, is readily derived and compared with some formulations recently appeared in literature [10].

1 THE PLATE MODEL

The ETP model has been already widely and deeply described in some previous works [1-4]. This contribution therefore will only focus on the SV solution; for a sake of simplicity, however, basic hypotheses and main results related to the field model will be first briefly summarized.

1.1 *General problem*

The small strain equilibrium of a linearly elastic, homogeneous and isotropic solid cylinder bounded by the domain $\Omega = \Sigma_m \times 2h =$ middle plane $\times$ thickness is herein investigated. The body is acted upon volume forces $\boldsymbol{F}$ in Ω and surface tractions $\boldsymbol{f}$ and $\boldsymbol{h}$ applied, respectively, on the two bases Σ^+ , Σ^- and on the free portion Σ^σ of the lateral surface Σ described by the (continuous) normal outward vector $\boldsymbol{n}$ (a counter clock-wise tangential vector $\boldsymbol{t}$ is also introduced).

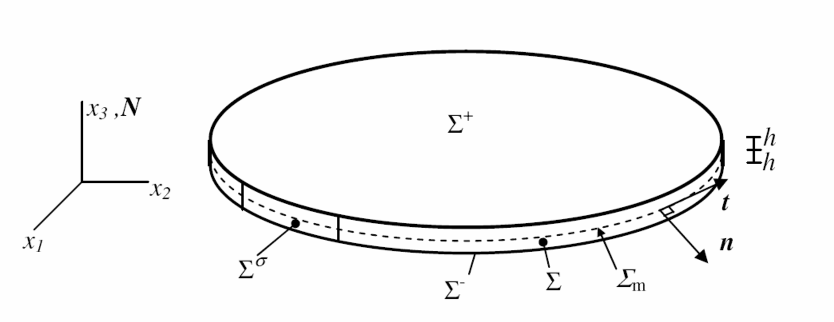

Figure 1: Geometry of the reference problem

Displacements are therefore supposed known over the remaining part Σ^u of the lateral surface. Under these hypotheses the 3-D elastic problem can be fully described by the classical set of equations governing equilibrium, compatibility and elastic law [4], whose solution can be splitted into a membrane and a flexural contribution that are assumed corresponding, respectively, to the even and odd decomposition of the displacement field solution with respect to the middle plane Σ_m. An a-priori selection of a purely flexural solution in Ω enforces the following restriction on the nature of volume forces and surface tractions

$$\boldsymbol{F} = \boldsymbol{N}p \text{ in } \Omega;\ \boldsymbol{f}^+ = \boldsymbol{N}P \,\text{su}\, \Sigma^+;\ \boldsymbol{f}^- = -\boldsymbol{N}P \,\text{su}\, \Sigma^- \tag{1a,b}$$

where N is the vector orthogonal to the middle plane in the verse of the coordinate axis $-h \le x_3 \le h$, p is an even function with respect to Σ_m and, both p and P, are supposed independent on the coordinate $\boldsymbol{x}$ in Σ_m (i.e. P = const, $p = p(x_3)$).
The *plate problem* is identified with the flexural solution in the previously defined sense.
Let us now introduce the projection operator

$$\boldsymbol{\Pi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{2a}$$

such that

$$\boldsymbol{V} = \boldsymbol{\Pi}\,\boldsymbol{U};\; w = \boldsymbol{N}\,\boldsymbol{U} \tag{3a,b}$$

where $\boldsymbol{U}$ is the displacement field of component $\boldsymbol{V}$ (in plane) and w (out of plane). By adopting a coherent splitting the equilibrium equations can be written in the following form

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} + \boldsymbol{\tau}_{,3} = \boldsymbol{0};\, \boldsymbol{\tau}\,|_{x_3=\pm h} = \boldsymbol{0} \tag{4a}$$

$$\nabla \cdot \boldsymbol{\tau} + \sigma_{33,3} + p = 0;\; \sigma_{33}\,|_{x_3=\pm h} = \pm P \tag{4b}$$

with the positions

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\Pi\sigma\Pi}^t = \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix},\; \boldsymbol{\tau} = \boldsymbol{\Pi\sigma N} = \begin{bmatrix} \tau_{13} \\ \tau_{23} \end{bmatrix} \tag{4c}$$

and an obvious meaning of symbols (the ∇ operator is in 2D).
By exploiting the elastic law through the adoption of Lame's constants λ, μ and introducing the scalar stress variable a defined as

$$a = \lambda(\nabla \cdot \boldsymbol{U}) + 2\mu\nabla \cdot (\boldsymbol{\Pi}\,\boldsymbol{U}) \tag{5}$$

the following relation follows

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = \nabla a + 2\mu(\boldsymbol{N} \wedge \nabla\omega) \tag{6}$$

or $\operatorname{div}\tilde{\boldsymbol{\sigma}} = \operatorname{grad} a + 2\mu \operatorname{curl}\omega$, with $\omega = -\frac{1}{2}\nabla \cdot (\boldsymbol{N} \wedge \boldsymbol{V}) = \frac{1}{2}(V_{2,1} - V_{1,2})$ (i.e. the component of spin tensor along $\boldsymbol{N}$) and $\operatorname{curl}\omega = [-\omega_{,2}, \omega_{,1}]^t$.

1.2 *Saint Venant solution*

The SV solution of the reference problem is simply defined [1-4] as the displacement field $(\boldsymbol{V}, w)$ satisfying the kinematic constraint

$$\omega = -\tfrac{1}{2}\nabla \cdot (\boldsymbol{N} \wedge \boldsymbol{V}) = 0 \tag{7}$$

Condition (7) is a smoothness constraint and it has been already demonstrated [11,12] that boundary layers, if present, are governed by amplitude parameters controlled by the spin ω. From a purely kinematic point of view a solution $\boldsymbol{V}$ complying with (7) has the following potential structure

$$\boldsymbol{V} = \nabla \phi \tag{8}$$

By considering the dual stress point of view, (7) also implies the relation

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = \nabla a \tag{9}$$

In the frame of ETP, the SV solution is expressed as a function of two generalized variable $(\tilde{w}, \tilde{m})$ defined as

$$\tilde{m}(\boldsymbol{x}) = \int_{-h}^{h} a(\boldsymbol{x}, x_3)\, x_3\, dx_3 \tag{10a}$$

$$\tilde{w}(\boldsymbol{x}) = \tfrac{3}{4h^3}\left\{ \int_{-h}^{h} w(\boldsymbol{x}, x_3)(h^2 - x_3^2)dx_3 - \tfrac{(1+\nu)(1-\frac{\nu}{2})}{E} \int_{-h}^{h} a(\boldsymbol{x}, x_3)(\tfrac{h^2}{5}x_3 - \tfrac{1}{3}x_3^3)dx_3 \right\} \tag{10b}$$

that are identically null in case of localized (w, a) fields.
In particular, it is relatively an easy task [4] to derive the following solution for stress quantities

$$a(\boldsymbol{x}, x_3) = \tfrac{3}{2h^3}\tilde{m}(\boldsymbol{x})x_3 + a'(x_3) \tag{11a}$$

$$\boldsymbol{\tau}(\boldsymbol{x}, x_3) = \tfrac{3}{4h^3}\nabla\tilde{m}(\boldsymbol{x})(h^2 - x_3^2) \tag{11b}$$

$$\sigma_{33}(x_3) = \tfrac{3}{4h^3}P_{tot}(h^2 x_3 - \tfrac{1}{3}x_3^3) - \int_0^{x_3} p(x_3)\,dx_3\,;\, P_{tot} = 2P + \int_{-h}^{h} p(x_3)dx_3 \tag{11c,d}$$

$$a'(x_3) = -\tfrac{3}{2h^3}P_{tot}\left(\tfrac{1}{5}h^2 x_3 - \tfrac{1}{3}x_3^3\right) - \tfrac{\nu}{1-\nu}\int_0^{x_3} p dx_3 - \tfrac{3x_3}{4h^3}\tfrac{\nu}{1-\nu}\int_{-h}^{h} p(h^2 - x_3^2)dx_3 \tag{11e}$$

and, through the elastic law, the correspondent kinematical relations are

$$\boldsymbol{V}(\boldsymbol{x}, x_3) = -x_3\nabla\tilde{w}(\boldsymbol{x}) + \tfrac{3}{4\mu h}\left(\tfrac{10-\nu}{10}x_3 - \tfrac{2-\nu}{6}\tfrac{x_3^3}{h^2}\right)\nabla\tilde{m}(\boldsymbol{x}) \tag{11f}$$

$$w(\boldsymbol{x}, x_3) = \tilde{w}(\boldsymbol{x}) + \tfrac{3\nu}{40\mu h}(1 - \tfrac{5x_3^2}{h^2})\,\tilde{m}(\boldsymbol{x}) + w'(x_3) \tag{11g}$$

$$w'(x_3) = \tfrac{3(1-\nu^2)}{8h^3 E}P_{tot}(h^2 x_3^2 - \tfrac{1}{6}x_3^4) - \tfrac{1-\nu^2}{E}\int_0^{x_3}\int_0^{x_3} p(x_3)dx_3 dx_3 - \tfrac{\nu(1+\nu)}{E}\int_0^{x_3} a'(x_3)dx_3 \tag{11h}$$

2 VARIATIONAL FORMULATION AND BOUNDARY CONDITIONS

In order to obtain a proper variational formulation, owing to the presence of mixed statical and kinematical fields, the following Complimentary Energy associated to SV solution with homogeneous displacement boundary conditions, is initially derived by means of some simple calculations

$$E_c(a,\sigma_{33}) = \int_V \left(\tfrac{1-\nu^2}{2E} a^2 - \tfrac{\nu(1+\nu)}{E} \sigma_{33} a \right) dx_3 d\boldsymbol{x} \tag{12}$$

The latter expression is the basis for the definition of a semi-inverted Reissner-like functional for SV plate formulation in a 3-D framework

$$\begin{gathered} R = \int_V \left(a \nabla \cdot \boldsymbol{V} - \tfrac{(1-\nu^2)}{2E} a^2 + \tfrac{1}{2\mu} |\boldsymbol{\tau}|^2 \right) dx_3 d\boldsymbol{x} + \int_{\partial V} (\tilde{\boldsymbol{\sigma}}\boldsymbol{n} - a\boldsymbol{n})^{\mathrm{t}} \boldsymbol{V}\, dx_3 d\boldsymbol{x} + \\ -L_{ext}(w; P, p) - \int_{\partial V_\sigma} (\tilde{\boldsymbol{\sigma}}\boldsymbol{n})^{\mathrm{t}} \boldsymbol{V}\, dx_3 d\boldsymbol{x} - \int_{\partial V_\tau} (\boldsymbol{\tau}\boldsymbol{n})^{\mathrm{t}} w\, dx_3 d\boldsymbol{x} \end{gathered} \tag{13}$$

where the work associated to field constant forces (P,p) is explicitly derived by means of solution (11a-f) as

$$L_{ext}(\tilde{w},\tilde{m}) = P_{tot}\tilde{w} - \tfrac{\nu}{2\kappa_t} \left[P_{tot} - \tfrac{5}{4} \int_{-h}^{h} p(x_3)(1 - \tfrac{x_3^2}{h^2})\, dx_3 \right] \tilde{m} \tag{14}$$

and the second term represent the (higher order) contribution due to transversal deformation. Coefficients $\kappa_f = \frac{2}{3} E h^3 / (1 - \nu^2)$ and $\kappa_t = \frac{5}{3} \mu h$ define flexural and shear stiffness, respectively. An expression of the functional referred to middle plane and generalized variable (except for constant terms and referring to unloaded free edge) is readily obtained through substitution of (11) in (13) and reads

$$R^*(\tilde{w},\tilde{m}) = \int_{\Sigma_m} \left\{ \tilde{m}\left(\tfrac{1}{k_t}\Delta\tilde{m} - \Delta\tilde{w}\right) - \tfrac{1}{2k_f}\tilde{m}^2 + \tfrac{1}{2k_t}|\tilde{m}|^2 \right\} d\boldsymbol{x} - L_{ext}(\tilde{w},\tilde{m}) + \int_{\partial\Sigma_m} \int_{-h}^{h} (\tilde{\boldsymbol{\sigma}}\boldsymbol{n} - a\boldsymbol{n})^{\mathrm{t}} \boldsymbol{V}\, dx_3 d\boldsymbol{x} \tag{15a}$$

where

$$\tilde{\boldsymbol{\sigma}}\boldsymbol{n} - a\boldsymbol{n} = -2\mu \left[(\tilde{w}_{,nt} x_3 - \tilde{m}_{,nt} f(x_3))\boldsymbol{t} - (\tilde{w}_{,tt} x_3 - \tilde{m}_{,tt} f(x_3))\boldsymbol{n} \right] \tag{15b}$$

$$f(x_3) = \tfrac{3}{4\mu h} \left(\tfrac{10-\nu}{10} x_3 - \tfrac{2-\nu}{6} \tfrac{x_3^3}{h^2} \right) \tag{15c}$$

The field equations of ETP [1] are obtained as stationarity conditions of (15)

$$\begin{cases} \Delta\tilde{m}(\boldsymbol{x}) + P_{tot} = 0 \\ k_f \Delta\tilde{w}(\boldsymbol{x}) + \tilde{m}(\boldsymbol{x}) + \tilde{m}' = 0 \end{cases} \tag{16a}$$

with the position

$$\tilde{m}' = \tfrac{2-\nu}{5} P_{tot} h^2 + \tfrac{\nu}{2(1-\nu)} \int_{-h}^{h} p(x_3)(h^2 - x_3^2)\, dx_3 \tag{16b}$$

and the boundary terms (expressed in weak form for the case of smooth boundary)

$$\int_{\partial\Sigma_m} \left[\tilde{m}_{,n}\delta\tilde{w} - \tilde{m}(\delta\tilde{w}_{,n} - \tfrac{1}{k_t}\delta\tilde{m}_{,n}) + \int_{-h}^{h} (\tilde{\boldsymbol{\sigma}}\boldsymbol{n} - a\boldsymbol{n})^{\mathrm{t}} \delta\boldsymbol{V}\, dx_3 \right] d\boldsymbol{x} = 0 \tag{16c}$$

The latter expression let a natural definition of the consistent boundary conditions for the problem at hand (the condition $\delta(\tilde{\boldsymbol{\sigma}}\boldsymbol{n}-a\boldsymbol{n})^t\boldsymbol{V}=0$ has been taken into account being satisfied for both free and $\boldsymbol{V}$-constrained edge).
The following four conventional alternatives arise:

1) Simply supported edge (which implies $\tilde{\boldsymbol{\sigma}}\boldsymbol{n}-a\boldsymbol{n}=0$)

$$\tilde{w}=0;\tilde{m}=0 \tag{17a}$$

2) Simply clamped edge (i.e. with $\delta\boldsymbol{V}=0$)

$$\tilde{w}_{,n}=0;\tilde{m}_{,n}=0 \tag{17b}$$

3) Clamped supported edge (i.e. with $\delta\boldsymbol{V}=0$)

$$\tilde{w}=0,\tilde{w}_{,n}-\tfrac{1}{k_t}\tilde{m}_{,n}=0 \tag{17c}$$

4) Free edge (taking into account that $\delta\tilde{m}\neq 0$ is inadmissible because it implies $\delta a\neq 0$)

$$\int_{\partial\Sigma_m}\left\{[\tilde{m}_{,n}-\tfrac{4\mu h^3}{3}(\tilde{w}-\tfrac{1}{k_t}\tilde{m})_{,ttn}]\delta\tilde{w}-[\tilde{m}+\tfrac{4\mu h^3}{3}(\tilde{w}-\tfrac{1}{k_t}\tilde{m})_{,tt}]\delta\tilde{w}_{,n}+\tfrac{1}{k_t}\tilde{m}\delta\tilde{m}_{,n}\right\}d\boldsymbol{x}=0 \tag{17d}$$

As previously discussed in [4], where a comparisons with a well known generalized 1D model has been made, note that the second of (17c) can be interpreted as a constraint condition for the generalized flexural rotation component alone.
It is also worth noticing that the derived free edge condition is not variationally consistent. As stressed by (15b), this boundary condition in fact involves second derivatives of the generalized functions along the boundary and this is incompatible with the second order structure of the functional. The discrepancy is an obvious consequence of the nature of the stress (strain) field, that is a function of the second derivative of the generalized variable and therefore, for a general edge condition, it implies a formulation based on total potential energy principle constructed on a strain field derived from (11f).
Now, by introducing the auxiliary quantity $\tilde{\varphi}\equiv\tilde{w}-k_t^{-1}\tilde{m}$, which has the meaning of a generalized displacement associate to the pure flexural behavior, the general boundary condition for a free edge can be rewritten as

$$\int_{\partial\Sigma_m}\left\{[\Delta\tilde{w}_{,n}+(1-\nu)\tilde{\varphi}_{,ttn}]\delta\tilde{w}-(1-\nu)\tilde{\varphi}_{,tt}\delta\tilde{w}_{,n}+(\Delta\tilde{w}+\tfrac{1}{k_f}\tilde{m}')\delta\tilde{\varphi}_{,n}\right\}d\boldsymbol{x}=0 \tag{18}$$

As a proof of consistency Clebsch conditions for Kirchhoff's plate are readily obtained from (18) as limit case by assuming $\tilde{w}=\tilde{\varphi}=w(x_3=0)=\overline{w}$ and $\tilde{m}'=0$

$$\int_{\partial\Sigma_m}\left\{[\overline{w}_{,nnn}+(2-\nu)\overline{w}_{,ttn}]\delta\overline{w}-[\overline{w}_{,nn}+\nu\overline{w}_{,tt}]\delta\overline{w}_{,n}\right\}d\boldsymbol{x}=0 \tag{19}$$

3 SOME REMARKS

The SV solution of ETP shows strong similarities with classical first order shear deformable plate theories framed in a potential formulation for rotation vector [10].
By considering, for example, the case of a completely clamped plate acted upon by volume forces only, distributed with the law $p(x_3) = Q\frac{3}{4h^3}(h^2 - x_3^2)$ and referring to the couple of variable ($\tilde{w},\tilde{\varphi}$), the governing system read

$$\begin{cases} k_t\Delta(\tilde{w}-\tilde{\varphi})+Q=0 \\ k_f\Delta\tilde{w}+k_t(\tilde{w}-\tilde{\varphi})+\frac{k_f}{k_t}Q=0 \end{cases} \quad (23a)$$

$$\tilde{w}=0, \tilde{\varphi}_{,n}=0 \quad (23b)$$

that is formally the same problem can be obtained by adopting a potential function for the rotation vector in Reissner theory of plates [9,10]. However once $\tilde{\varphi},\tilde{w}$, and hence $\tilde{m}$, are derived as the solution of system (23), the EPT gives internal stresses and displacements governed by relations (11) which differs from the Reissner's assumption of plane stress linearly varying in the thickness.

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