Natural frequencies of prestressed concrete beams: theoretical prediction and numerical validation

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SUMMARY. The paper explores the effects of prestress forces on modal parameters of concrete beams. This ongoing research has the twofold purpose of deepening a task of structural identification and of investigating possible applications with significant technical relevance. Concerning the former aspect, it is still not clarified whether prestress forces should be included or not in standard identification-model-updating frameworks. On the other hand, a well founded theoretical model, able to catch the experimental evidence, would be desirable in order to explore the feasibility of dynamic methods for assessing prestress level and prestress losses in concrete beams.

1 INTRODUCTION

Structural identification methods for the damage detection in important structures and infrastructures are desirable tools since they are potentially able to put in evidence structural deficiencies thus reducing the risk of collapses. These methods can also be employed to better allocate the limited resources available for the maintenance, providing a list of priority of the damaged structures. Nevertheless, it must be pointed out that the reliability of these methods is strongly influenced by many parameters, among which the type of material employed for the construction of the structure.

Generally speaking, better estimates of the dynamic properties can be obtained for steel structures rather than for concrete ones. Indeed, the formers are mainly affected by uncertainties in the type of the external supports, temperature variation and non-structural masses, while it can be reasonably accepted for the in-service free-vibration analysis that the steel elastic modulus and the geometrical characteristics of the structural components are well defined. On the contrary, reinforced and prestressed concrete structures, alongside the uncertainties exhibited by steel structures, present the following features that can affect their dynamic properties:

- Non-linear behavior of the concrete in compression;
- Non-linear behavior of the concrete in tension that can cause cracking;
- Influence of amount and position of reinforcing steel;
- Influence of amount and position of prestressing steel;
- Influence of type of prestressing (bonded - unbonded, straight - draped - parabolic);
- Influence of prestressing forces.

The scientific community reported contradictory results on the dynamic behavior of prestressed structures. One of the oldest papers on this topic has been written by Saiidi et al. [1]. The Authors, recalling the well known ”compression softening” effect, reported the discrepancy between theoretical and experimental results. Several discussions followed this paper [2, 3, 4] and each of them pointed out that the “compression-softening” effect is due to externally applied forces and not to an internal action as is the case of bonded prestress. Nevertheless, the experimental results provided by Saiidi et al. [1] showed that the higher is the level of prestress the higher are the eigenfrequencies of the prestressed concrete beam. It is opinion of both, the Authors and the Discussers, that this increase in the natural frequency can be ascribed to the closure of the micro-cracks produced in the concrete by the shrinkage realized by the prestressing force.

An analytical study aimed at examining the influence on girders’ natural frequencies and mode shapes of vibration of many parameters among which the prestress force level and the shape of the bottom prestressing strands has been conducted in 1996 by Grace and Ross [5]. The results of the parametric study indicated that the prestress force level and the shape of the bottom prestressing strands have the most noticeable influence on the natural frequencies, while modal shapes are only slightly influenced by such parameters. All the girders modeled with parabolic bottom prestressing strands exhibited increases in the natural frequency as the prestress force level was increased while with straight bottom prestressing all the girders exhibited decreases in the natural frequencies as the prestress force level was increased.

Dall’Asta and Leoni [6] studied the free vibrations of a class of concrete beams prestressed by means of unbonded internal cables in order to test the influence of the elastic characteristic of the materials, as well as the influence of the path and stress of the prestressing cables on the beam free vibrations. The Authors, analyzing only the first bending modes (vertical and horizontal) and the first twisting mode, observed a significant difference between results concerning the bending vibration modes and those relative to the twisting vibration mode. They found that the natural periods of the bending modes are only slightly modified by variations of the cable stress and cable profile while much bigger variations have been observed for the twisting mode.

Miyamoto et al. [7] studied the effect of the external tendons on the frequency of vibration of prestressed concrete beams. Their analytical formulation foresees two counterbalancing effect: the ”compression-softening” effect due to the axial force and the stiffening effect due to the tendon profile. So, the eigenfrequencies of the externally prestressed beam increase if the effect due to the path of the cable prevails on the “compression-softening” effect, being also true the contrary. Experimental tests validated the proposed formulation. Some years later, Hamed and Frostig [8] presented a study dealing with the effect of the magnitude of the prestress force on the natural frequencies of prestressed beams with bonded and unbonded tendons. The results obtained using the proposed nonlinear analytical formulation of the dynamic behavior of the prestressed beams indicated that the magnitude of the prestress force does not affect the natural frequencies of either bonded or unbonded prestressed beams.

In summary, no complete agreement exists in the literature on the dynamic behavior of prestressed concrete beams. Indeed, experimental results generally agree that there is an influence of prestress forces on the modal features of concrete beams, depending on many parameters including the type of prestress and the prestress level. Nonetheless, analytical models are usually unable to explain such phenomena, which cannot be always ascribed to geometric stiffness but often depend on more complex nonlinear phenomena such as material nonlinearities, concrete cracking and so on. In order to investigate this topic, the authors have recently started a research program with the following schedule: literature review; numerical modeling; analytical modeling; experimental investigation. The results concerning the first two steps have been published in a recent work by the authors [9]. In this paper the earliest attempts to establish a theoretical formulation are presented,

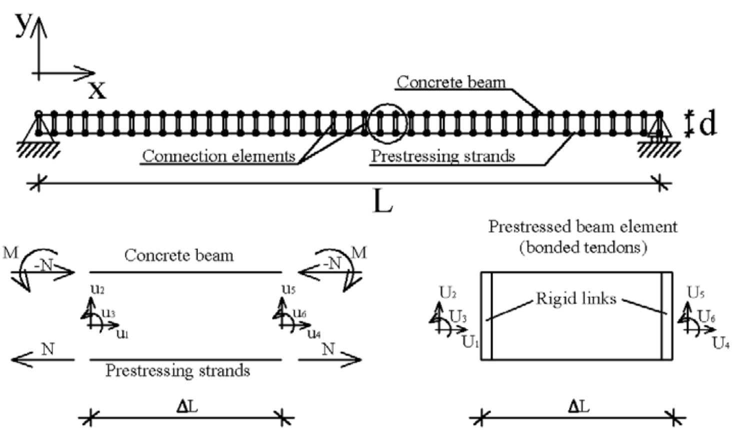

Figure 1: Sketch of a prestressed concrete beam.

along with a numerical validation based on refined finite element (FE) models.

2 MATHEMATICAL FORMULATION

Let us consider a prestressed concrete simply supported beam with span length L, which deflects in the vertical plane $x-y$. At a first stage of investigation, this simple structure can be idealized as a composite system whose elements are: the concrete beam, the straight eccentric prestressing strands and some connection elements between the two. In the following developments, the eccentricity between the concrete beam axis and the axis of the prestressing strands is denoted by d, the cross section and the relevant inertial moment of the concrete beam are denoted by A_c and J_c, respectively, while those of the prestressing strands are denoted by A_s and J_s. For the sake of simplicity, the concrete beam is also assumed to be in the uncracked state and fully compressed. A sketch of this simple mechanical system is shown in Fig. 1.

With the purpose of discussing prestress effects on natural frequencies, the stiffness characteristics of the prestressed beam must be considered and their sensitivities to variations of the prestress force must be evaluated. Here, this task is accomplished in the framework of standard matrix structural analysis and the equilibrium state is taken as reference to account for geometric and material nonlinearities. Letting N be the magnitude of the prestress force acting on the prestressing strands, the internal forces acting on a small finite element of the beam having length ΔL are: the tension force N on the prestressing strands, the compression force $-N$ and the bending moment $M = Nd$ on the concrete beam (see Fig. 1). At a first stage of investigation, material nonlinearities can simply be modeled as the dependence of the Young's modulus of concrete E_c and the Young's modulus of the prestressing steel E_s on the average normal stresses, i.e. on the magnitude of the prestress force:

$$\begin{aligned} E_c &= E_c(N) \\ E_s &= E_s(N) \end{aligned} \tag{1}$$

In the framework of large-displacements-small-strains elasticity the 6×6 stiffness matrices of the concrete beam element and the element representing the prestressing strands, denoted by $\mathbf{K}^c$ and

$\mathbf{K}^s$, respectively, can be written as:

$$\begin{array}{c}\mathbf{K}^c = \mathbf{K}^c_E + \mathbf{K}^c_G \\ \mathbf{K}^s = \mathbf{K}^s_E + \mathbf{K}^s_G\end{array} \tag{2}$$

where the generalized nodal displacements, in the local coordinate systems, are denoted by u_1, u_2, . . ., u_6 (see Fig. 1). In Eq. (2), the subscript E stands for elastic stiffness matrix while G stands for geometric stiffness matrix. The elastic stiffness matrices in Eq. (2) are the standard ones of Euler-Bernoulli beam elements while, on the basis of first order analysis, the non-zero terms in the geometric stiffness matrices are the following:

$$\begin{array}{c}K^c_{G22} = K^c_{G55} = -K^c_{G25} = -K^c_{G52} = -\frac{N}{\Delta L} \\ K^s_{G22} = K^s_{G55} = -K^s_{G25} = -K^s_{G52} = \frac{N}{\Delta L}\end{array} \tag{3}$$

Once the connection elements are introduced, an ideal finite element of prestressed beam is obtained whose global stiffness matrix $\mathbf{K}^{id}$ can be written as:

$$\mathbf{K}^{id} = \mathbf{K}^{id}_E + \mathbf{K}^{id}_G \tag{4}$$

Let, for instance, the case of bonded tendons be considered. In such a case, the connection elements can be regarded as rigid links that connect the prestressing strands to the upper beam. In this way, the prestressing strands are constrained to move rigidly with the beam in the vertical plane and no slip is allowed between the beam and the prestressing strands. This reduces the degrees of freedom of the prestressed beam element from 12 to 6, as 3 degrees of freedom U_1, U_2, U_3 characterize the left hand node of the composite beam while U_4, U_5 and U_6 characterize the right hand one (see Fig. 1). The elastic stiffness matrix $\mathbf{K}^{id}_E$ in Eq. (4) can then be written in terms of ideal sectional properties A_{id} and J_{id}, which are defined as:

$$\begin{array}{c}J_{id}(N) = J_c + A_s d^2 \frac{E_s(N)}{E_c(N)} + J_s \frac{E_s(N)}{E_c(N)} \\ A_{id}(N) = A_c + A_s \frac{E_s(N)}{E_c(N)}\end{array} \tag{5}$$

where the variation of the position of the centroid has been neglected for the sake of simplicity. On the other hand, the ideal geometric stiffness matrix reflects on a null 6×6 matrix. Indeed, since the concrete beam and the prestressing strands vertically deflect as two systems in parallel, it can be readily obtained that:

$$K^{id}_{G22} = K^c_{G22} + K^s_{G22} = -\frac{N}{\Delta L} + \frac{N}{\Delta L} = 0 \tag{6}$$

and, similarly, $K^{id}_{G55} = K^{id}_{G52} = K^{id}_{G25} = 0$. The sensitivity of the stiffness matrix of the prestressed beam with respect to variations of the prestress force magnitude, which is clearly related to the variation of natural frequencies, can then be evaluated as:

$$\frac{\partial \mathbf{K}^{id}}{\partial N} = \frac{\partial \mathbf{K}^{id}}{\partial E_c}\frac{\partial E_c}{\partial N} + \frac{\partial \mathbf{K}^{id}}{\partial E_s}\frac{\partial E_s}{\partial N} \neq 0 \tag{7}$$

From Eq. (7) it immediately turns out that the stiffness characteristics, and so the natural frequencies, of prestressed beams with bonded tendons depend upon the magnitude of the prestressing force N as a consequence of material nonlinearities characterizing both the concrete and the prestressing steel. On the contrary, geometric effects do not play any role as soon as the concrete beam and the reinforcement are vertically connected. Indeed, in such a case, the compression softening effect on the concrete beam is exactly compensated by the tension hardening effect on the prestressing strands. It is also noteworthy that Eq. (7) is still valid in the case of unbonded parabolic tendons. Indeed, in such a case, Eq. (6) holds as there is a vertical contact between the beam and the prestressing strands and the links connecting the two can be modeled as rigid truss element. On the contrary, in the case of straight unbonded tendons, geometric nonlinearities also play a role on the variation of natural frequencies from the initial to the equilibrium state if there is no continuous vertical connection between the beam and the prestressing strands. As already mentioned, these results are valid under the assumption that no concrete cracking occurs.

Now, having formally discussed the main aspects which affect natural frequencies of prestressed concrete beams, a refined numeric model is required in order to validate the above reported statements and to quantify the expected frequency variations in practical cases. To this end, a three-dimensional model is needed, in which material nonlinearities can be more realistically modeled as local stress-strain relationships. A similar model is presented in Section 3 while a parametric investigation for a sample case study is presented in Section 5.

3 REFINED FE MODEL OF PRESTRESSED CONCRETE BEAMS

The general features of the refined FE model are here briefly introduced. It has been developed using the commercial software ABAQUS 6.7. The concrete has been modeled with 8-node linear brick elements while two-nodes connectors with suitable axial stiffness and initial length have been used to model the ordinary and the prestressing steel. A nonlinear incremental static analysis permits the equilibrium solution to be obtained after the application of the prestress force. The modal analysis is then performed onto the system linearized around the equilibrium configuration. Thus, the following steps will be carried out in each analysis:

1. Undeformed reference state;
2. Application of the prestressing (when present) by means of connectors with proper axial stiffness and initial length;
3. Application of the girder's self weight;
4. Non-linear incremental static analysis.
5. Linear perturbation around the deformed equilibrium configuration, considering the tangent stiffness matrix updated to account for geometric nonlinear terms and geometric variations, to extract the eigenfrequencies and the modal shapes.

4 *THE CASE STUDY*

The numerical analyses have been developed assuming as a sample prestressed concrete beam the one investigated by Unger et al. [10] and by Jacobs et al. [11]. Its main characteristics are listed in the following sections.

4.1 *Geometry*

The prestressed girder has a total length of 17.6 m, a span of 14.56 m and a total depth of 0.8 m. The widths of the upper and lower flange are 0.8 m and 0.5 m, respectively. At the two ends of the beam the cross section becomes square with dimension 0.8 m. The layout of the ordinary reinforcement is made up of 4+4 ϕ10 on the upper and lower flanges and 4+4 ϕ8 along the web. The prestressing reinforcement is assumed to be composed by seven 0.6” strands.

4.2 *Concrete and reinforcement*

A class C50/60 has been considered for the concrete. According to EN 1992-1-1 [12] the following mechanical properties have been used:

- Secant modulus of elasticity for linear analysis:

$$E_{cm} = 22 \left[\frac{f_{cm}}{10} \right]^{0.3} \tag{8}$$

- Stress-strain relation for non-linear analysis (see Fig. 2):

$$\frac{\sigma_c}{f_{cm}} = \frac{k\eta - \eta^2}{1 + (k-2)\,\eta} \tag{9}$$

where $f_{cm} = f_{ck} + 8\ (MPa)$, $\eta = \varepsilon_c/\varepsilon_{c1}$, $k = 1.05\, E_{cm} \cdot |\varepsilon_{c1}| / f_{cm}$ and ε_{c1} is equal to 0.00245 for f_{ck} equal to 50 MPa.

Steel type S500 (characteristic yield strength f_{yk} of 500 MPa) has been used for the ordinary reinforcement. Linear elastic stress-strain relation has been assumed for the ordinary steel since no plastic deformation is expected for the service condition. The elastic module $E_s = 2.05 \cdot 10^8\, kN/m^2$ has been used in the analysis.

4.3 *Prestressing steel*

The harmonic steel has been assumed with characteristic tensile strength f_{ptk} of 1860 MPa. Several configurations of the prestressing steel layout have been investigated to cover different prestressing techniques (see Fig. 3). Linear elastic stress-strain relation has been assumed also for the prestressing steel, employing the elastic module $E_p = 1.95 \cdot 10^8\, kN/m^2$.

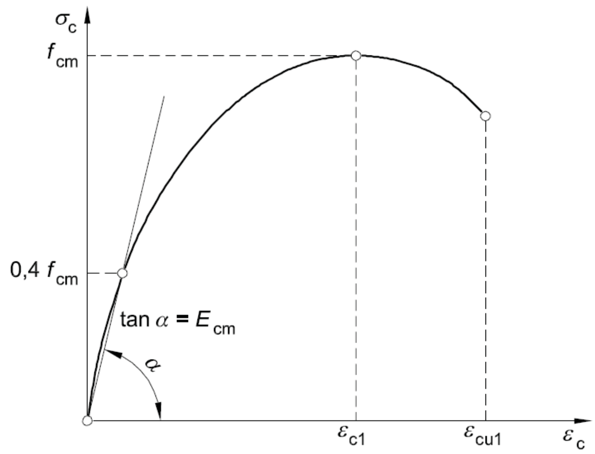

Figure 2: Stress-strain relationship for concrete taken from [12].

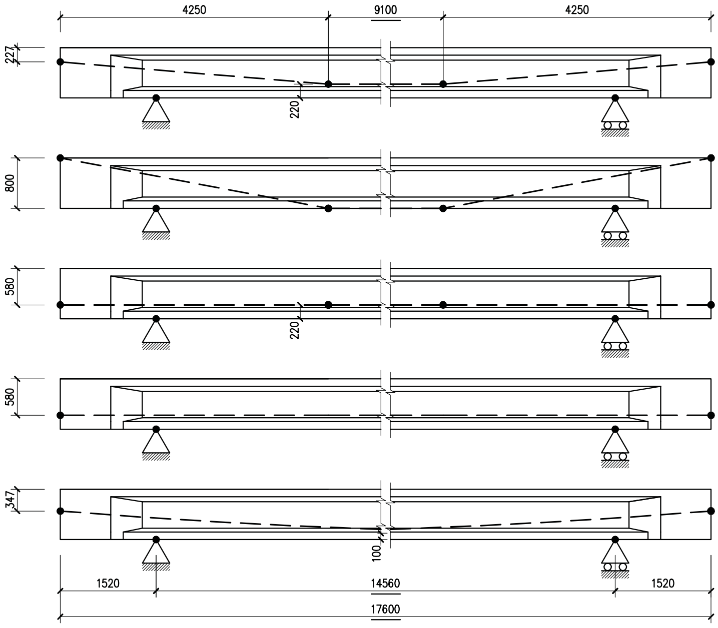

Figure 3: Longitudinal layout of the prestressing steel. From the top: a) draped D1, b) draped D2, c) draped D3, d) straight bonded and unbonded, e) parabolic bonded.

5 PARAMETRIC INVESTIGATION

The FE model used for the parametric investigations is shown in the Fig. 4. The results of these investigations are summarized in the Figs. 5 and 6 where the eigenfrequencies of the 1 st and 2nd vertical bending modes are depicted, respectively. In the figures the code XX-YYY-ZZ has been used to indicate the different cases, where:

- XX can assume the values NP for non-prestressed element and P for prestressed element;
- YYY can assume the values: 1st letter A, U or B for absent, unbonded or bonded prestressing steel, respectively; 2nd letter A, S, D or P for absent, straight, draped or parabolic prestressing reinforcement; 3rd letter A or R for absent or present ordinary reinforcing and 1, 2 or 3 for the layout type of the draped tendons;
- ZZ can assume the values L for linear concrete and NL for non-linear concrete.

Hereinafter the results of the analyses are presented in detail starting from those aimed at putting in evidence the influences of concrete and steel reinforcement modeling (ordinary and prestressing steel) on the dynamic properties of RC beams and proceeding with those related to the different prestressing techniques.

5.1 *Non-prestressed girders*

The first analysis investigated the effect of concrete modeling and amount of reinforcement on the frequency of vibration of the PC beams. The major differences between the numerical results are

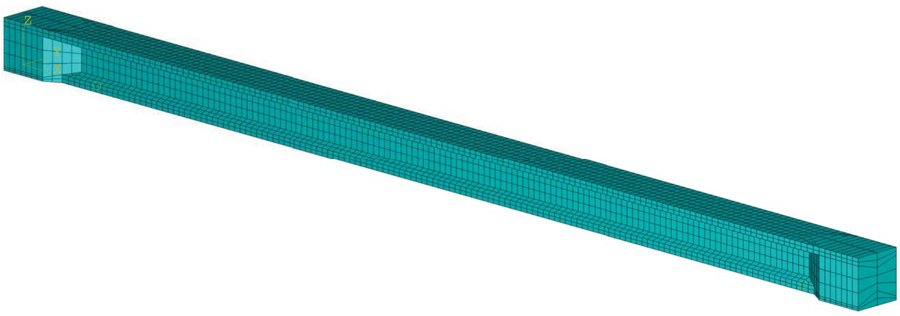

Figure 4: Three-dimensional FE model for the prestressed beam.

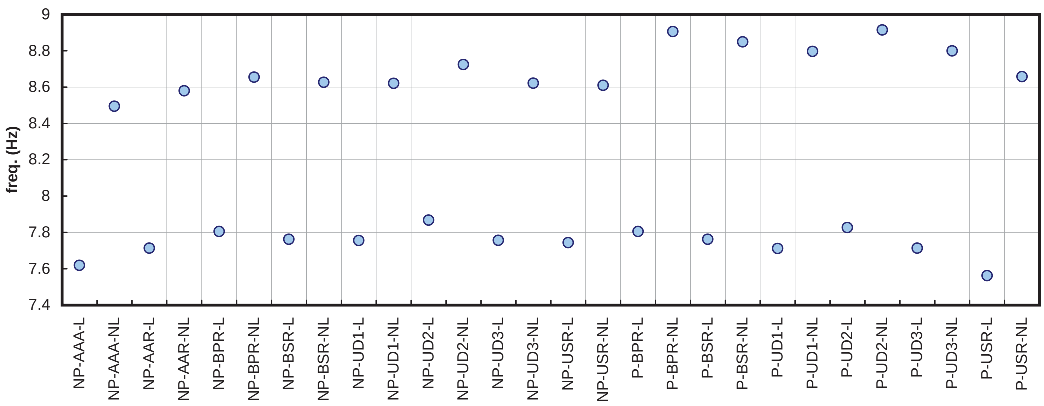

Figure 5: Variation of the eigenfrequency of the 1^{st} vertical bending mode.

caused by the mechanical behavior assumed for the concrete. A non-linear stress-strain relationship, according to EN 1992 (2004), produces an average increase of the eigenfrequencies of the linear model roughly equal to 12%. This variation has been detected by means of the linear perturbation analysis around the deformed shape of the beam that uses for a large volume of the concrete the tangent elastic modulus in the origin (being the beam without prestressing) rather than the secant modulus of elasticity commonly employed in linear analysis for structural design.

Further analyses allowed emphasizing the role of the reinforcement. As well known it is common practice to neglect the presence of the reinforcement in an elastic analysis. Nevertheless, more accurate analyses are required for structural identification and damage prognosis purposes. In fact, it turned out that considering or not the reinforcement in the FE model can vary the frequencies of the vertical bending modes by roughly 2%, variation that can be comparable to that caused by severe damage.

5.2 *Bonded vs Unbonded tendons*

The comparison of the results obtained for prestressed bonded and unbonded straight tendons (for the unbonded tendons relative movements in the vertical direction between PC beam and tendons are possible) allowed to put in evidence that bonded tendons provide an increase in frequencies of

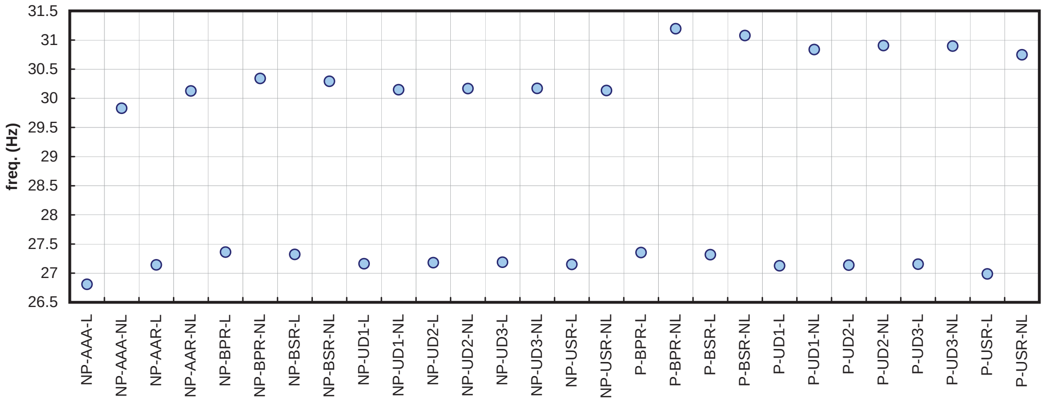

Figure 6: Variation of the eigenfrequency of the 2^{nd} vertical bending mode.

the bending vibration modes, leaving almost unaffected the torsional ones.

5.3 *Tendons' path*

The influence of the prestressing tendons' path for unbonded prestressing steel has been investigated analyzing the 5 configurations depicted in 3 with straight, draped and parabolic tendons' path where the draped configurations aim to take into account internally and externally prestressed elements. The results of the analyses showed that for both the linear and the non-linear cases, the draped configuration n. 2 produce a major increase in the eigenfrequency of the 1st vertical bending mode equal to 3.5% and 3.0%, while the draped configuration n.1 and 3 produce smaller increases equal to 2.0% and 1.6%. The influence of the tendons' path on the higher modes is much weaker.

5.4 *Prestressing force*

Finally, the influence of the prestressing force has been investigated for prestressed and non prestressed linear beams with bonded and unbonded straight tendons. From the results of the unbonded cases it can be observed the well known "compression-softening" effect. This effect disappears in the case of bonded prestressing steel where the small differences between the cases with and without prestressing force can be ascribed to the change in the deformed configuration of the beam caused by the prestressing action. Similar results have been obtained also for the cases with linear concrete and draped configuration, as happen for example in externally prestressed bridges [13], putting in evidence that when dealing with externally prestressed concrete elements, the effect of the prestressing force has to be taken into account to correctly model the structural dynamic behavior. The effect of the prestressing force is even larger if the non-linear behavior of the concrete is taken into account.

6 CONCLUSIONS

In this paper the results of a theoretical and numerical analysis involving many aspects related to the dynamic properties of prestressed concrete beams has been reported. The results shown that, when using dynamic methods for structural identification and damage health analysis, the effects of material non-linearity, position and amount of reinforcement, prestressing and prestressing technology should not be neglected, since their effect can be comparable to that produced by severe damages. For the cases where the hypothesis of linear materials and bonded tendons can be assumed the effect of prestressing can be neglected, since the variation in the deformed equilibrium configuration do not modify significantly the dynamic properties of the structure. On the contrary, the effect of the prestressing force should always be considered when dealing with externally prestressed beam, even when assuming for the concrete linear isotropic homogeneous elasticity. Finally, if the non-linear behavior of the materials is considered in the dynamic analyses, the influence of the prestressing action should be considered, even for bonded tendons, to obtain more accurate results. The prestressing action, in fact, modifies the tangent stiffness matrix and the tangent elastic modulus in the deformed equilibrium condition, thus affecting the dynamic properties of the structure.

References

[1] Saiidi, M. and Douglas, B., “Prestress force effect on vibration frequency of concrete bridges,” *J. Structural Engineering*, **120**(7), 2233-2241 (1994).

[2] Dall’Asta, A. and Dezi, L., “Discussion on ”Prestress force effect on vibration frequency of concrete bridges” by Saiidi M. et al.,” *J. Structural Engineering*, **121**, 458-459 (1996).

[3] Deák, G., “Discussion on ”Prestress force effect on vibration frequency of concrete bridges” by Saiidi M. et al.,” *J. Structural Engineering*, **121**, 458-459 (1996).

[4] Jain, S.K and Goel S.C., “Discussion on ”Prestress force effect on vibration frequency of concrete bridges” by Saiidi M. et al.,” *J. Structural Engineering*, **121**, 459-460 (1996).

[5] Grace, N.F. and Ross, B., “Dynamic characteristic of post-tensioned girders with web openings,” *J. Structural Engineering*, **122**(6), 643-650 (1996).

[6] Dall’Asta, A. and Leoni, G., “Vibration of beams prestressed by internal frictionless cables,” *J. Sound and Vibration*, **222**(1), 1-18 (1999).

[7] Miyamoto, A., Tei, K., Nakamura, H. and Bull, J.W., “Behavior of prestressed beam strengthened with external tendons,” *J. Structural Engineering*, **126**(9), 1033-1044 (2000).

[8] Hamed, E. and Frostig, Y., “Natural frequencies of bonded and unbonded prestressed beams-prestress force effects,” *J. Sound and Vibration*, **295**, 28-39 (2006).

[9] Materazzi, A.L., Breccolotti, M., Ubertini, F. and Venanzi, I., “Experimental modal analysis for assessing prestress forces in PC bridges: a sensitivity study,” in *Proc. III International Operational Modal Analysis Conference* (IOMAC Conference Proceedings vol. II, Brescia, 2009), Portonovo, Italy, May 4-6, 2009, 585-594 (2009).

[10] Unger, J.F., Teughels, A. and De Roeck, G., “System identification and damage detection of a prestressed concrete beam”, *J. Structural Engineering*, **132**(11), 1691-1698 (2006).

[11] Jacobs, S., Matthys, S., De Roeck, G., Taerwe, L., De Waele, W. and Degrieck, J., “Testing of a prestressed concrete girder to study the enhanced performance of monitoring by integrating optical fiber sensors”, *J. Structural Engineering*, **133**(4), 541-549 (2007).

[12] CEN, EN 1992-1-1 *Eurocode 2: Design of concrete structures - Part 1-1: General rules and rules for buildings* (2004).

[13] Burdet, O. and Badoux, M., I. “Comparison of Internal and external prestressing for typical highway bridges”, in *Proc. 16th Congress of IABSE*, Lucerne, Switzerland (2000).