A simple structural model for a masonry arch-wall system subjected to dead vertical loads

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SUMMARY. Arches of various sizes and shapes can be found bearing the weight of vertical walls in almost all ancient masonry buildings. The most common arch shapes are round, or semicircular, and pointed, the so-called ogival arch, typical of Romanesque and Gothic architecture. The aim of this work is to study such structural systems by considering them composite systems made up of an arch and an overlying wall, both made of masonry. The study applies a simple mechanical model, in which the arch and wall are schematized as one-dimensional elements, in general characterized by nonlinear elastic behavior. In the case that the displacements undergone by the arch are sufficiently small in comparison to those of the wall, it can be shown that the distribution of the loads transmitted by the wall to the arch differs considerably from the distribution deduced by assuming each vertical strip of wall to be sustained directly by the underlying arch element. Though rather small, such differences sometimes involve a critical difference in the load bearing capacity of the arch, calculated under the assumption that the arch's constituent material cannot withstand traction and has limited compressive strength. The model enables, among other things, evaluating the effect on bearing capacity of substituting a semicircular arch with an ogival one of equal thickness and span. The results of such comparison clearly reveal the superiority of the latter arch type over the former in terms of the maximum possible height of the overlying wall under equilibrium conditions.

1. INTRODUCTION

The façades and dividing walls of many historical and monumental masonry buildings commonly contain arches, built with the aim of accommodating various types of openings, such as doors, windows and colonnades. In such cases, the arch must bear the weight of the overlying masonry wall, and the problem therefore arises of studying the interactions between these two elements, each exhibiting its own particular characteristics, combining to form an arch-wall 'system'. A similar structural problem is encountered in the study of masonry arch bridges, for which it is anything but simple to evaluate the contribution to the arch's bearing capacity of the so-called fill [1].

Figure 1: an example arch-wall system: the interior colonnade of the church of *San Giovanni e Santa Reparata* in Lucca.

In fact, determining the load transmitted by the wall to the arch is by no means a trivial matter, and for this reason, approximate solutions obtained by means of finite-element computation codes are often considered acceptable (see, for instance, [2]). Alternatively, elementary calculation schemes may be chosen, for instance, by assuming the wall to be divisible into vertical strips, the weight of each being directly sustained by the underlying arch element [3]. Although such approaches undoubtedly have the virtue of simplicity, they completely neglect the real distribution of the loads through the wall and the way they are transferred from the wall to the arch.

2. THE MODEL

The problem of determining the effective stress field established in the structural system made up of a masonry arch and the overlying vertical wall has no simple solution. Although the problem can easily be addressed in terms of the corresponding plane elastic problem, for which an exact solution can be had relatively simply, here we undertake to determine an approximate expression for the distribution of the real actions that the elastic wall effectively transmits to the arch. Our conviction is that only by knowing the loads actually transferred to the arch can an accurate evaluation be made of the degree of safety afforded by such structures.

The arch-wall system is schematized as a composite structure made up of an elastic, one-dimensional element with curvilinear axis, which is inextensible and shear indeformable (the arch), and an elastic one-dimensional element with rectilinear axis (the overlying wall). Because the wall's height is usually comparable (and often superior) to its length, it has accordingly been assumed to be deformable to shear alone. It has moreover been assumed that the wall is connected to the underlying arch through a continuous distribution of vertical and horizontal elastic elements (Figure 2), so as to account for (albeit in a rather simplistic way) the overall deformations affecting the wall and the effects that such deformations have on the actual distribution of the loads transmitted by the wall to the underlying arch.

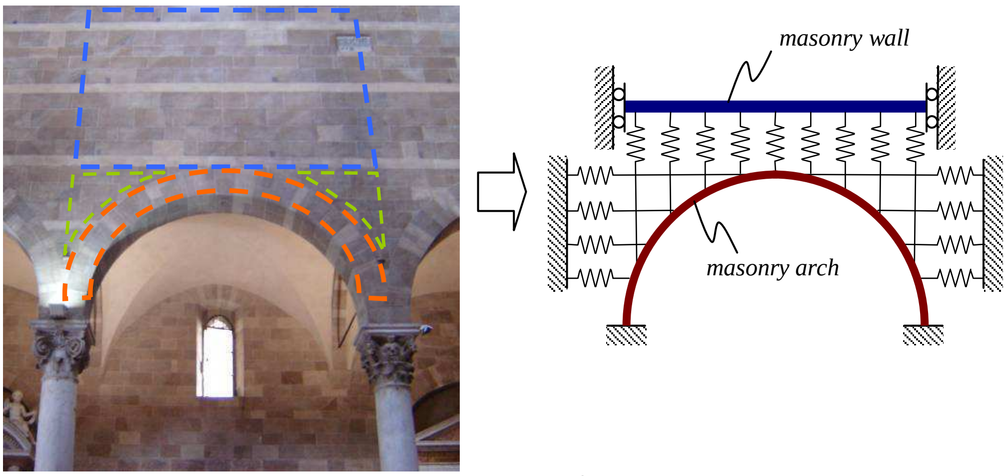


Figure 2: structural scheme of the arch-wall system.

In the following we will denote with θ the anomaly at any given transverse section of the arch ABC (see fig. 3), with $u(\theta)$ and $v(\theta)$ the displacement components of the points along its axis in the tangential and radial directions, respectively. For simplicity's sake, the radius, R, and bending stiffness, EJ, of the arch are both assumed to be constant. For the one-dimensional element, DE, representing the overlying wall, we denote with v_p the transverse displacement of the points along its axis, GA_T its shear stiffness and $p(\theta)$ the vertical load, varying along the beam itself, represented by the weight of each vertical strip of wall. For simplicity's sake, the thickness of the wall and the arch orthogonal to the diagram's plane has been assumed to be one.

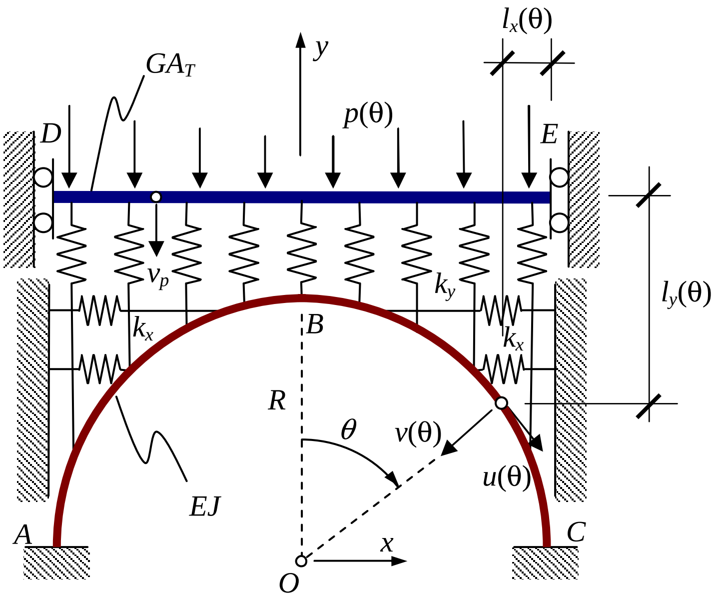


Figure 3: symbols and notation

For the arch ABC the usual equilibrium equations hold:

$$\frac{dN}{d\theta} - T = -p_t, \quad \frac{dT}{d\theta} + N = -p_r, \quad T = \frac{dM}{d\theta}\frac{1}{R}, \tag{1}$$

and, under the assumption of an inextensible, shear indeformable arch, we also have the well-known kinematics relations:

$$v = \frac{du}{d\theta}, \quad \varphi = \frac{u}{R} + \frac{1}{R}\frac{d^2u}{d\theta^2}, \quad \chi = -\frac{1}{R}\frac{d\varphi}{d\theta}, \tag{2}$$

In the preceding relations N, T and M are respectively the normal and shear stresses, and the bending moment; p_r and p_t are the components of any distributed loads in the radial and circumferential directions, φ is the rotation of the transverse section, positive if clockwise, and χ is the change in curvature.

The deformations of the horizontal element DE are instead described by the differential equation

$$GA_T \frac{d^2 v_p}{dx^2} = q_p, \tag{3}$$

where q_p is the distributed load acting in the transverse direction.

Lastly, let $k_x(\theta)$ and $k_y(\theta)$ indicate the stiffnesses per unit length of the elastic elements connecting the two structural elements in the horizontal and vertical directions, respectively. E_p and ν_p indicate the Young's modulus and Poisson's coefficient of the wall's constituent material, assumed for simplicity's sake to be linearly elastic, homogeneous and isotropic. Now, with $l_x(\theta)$ and $l_y(\theta)$ the lengths shown in the figure, chosen so as to account for deformation of the wall as a whole, we have:

$$k_x(\theta) = \frac{E_p}{(1-\nu^2)l_x(\theta)}, \quad k_y(\theta) = \frac{E_p}{(1-\nu^2)l_y(\theta)}.$$

By denoting

$$f_x = k_y \left(v_p - \cos\theta \frac{du}{d\theta} - u \sin\theta \right), \tag{4}$$

and

$$f_y = k_x \left(u \cos\theta - \sin\theta \frac{du}{d\theta} \right), \tag{5}$$

as the actions transferred, respectively in the horizontal and vertical direction, at corresponding points on arch *ABC* and the wall *DE*, simple calculations, omitted here for the sake of brevity, enable arriving at a differential system with unknowns $u(\theta)$ and $v_p(\theta)$:

$$\begin{cases} \dfrac{d^6 u}{d\theta^6} + 2\dfrac{d^4 u}{d\theta^4} + \dfrac{d^2 u}{d\theta^2} = \dfrac{R^4}{EJ}\left(3(f_x - f_y) + \dfrac{df_y}{d\theta} + \dfrac{\sin 2\theta}{2}\dfrac{df_x}{d\theta} \right), \\ \dfrac{GA_T}{R^2 \cos^2\theta}\left(\dfrac{d^2 v_p}{d\theta^2} + \tan\theta \dfrac{dv_p}{d\theta} \right) = f_y - p, \end{cases} \tag{6}$$

which is supplemented by the boundary conditions

$$\begin{array}{ll} u(\pm\pi) = 0, & \\ \dfrac{du}{d\theta}(\pm\pi) = 0, & \dfrac{dv_p}{d\theta}(\pm\pi) = 0. \\ \dfrac{d^2 u}{d\theta^2}(\pm\pi) = 0, & \end{array} \tag{7}$$

Once solved, the differential problem allows for evaluating the true distribution of the actions exchanged between the arch and the wall, which will depend on the shape and geometry of the arch, the height of the wall and the mechanical properties of their constituent masonry.

In the following, we make a first comparison between the bearing capacity of different arch-wall systems, specifically the common semicircular and ogival types. The case study is limited to the situation in which the displacements of the points along the arch axis can be considered negligible in comparison to those undergone by the points of the wall. This added hypothesis allows reducing system (6) to a single differential equation of the second order with the single unknown function v_p,

$$GA_T \frac{d^2 v_p}{dx^2} - \frac{E_p}{(1-\nu^2)\left[\dfrac{H}{2} + R\left(1 - \sqrt{1 - \left(\dfrac{x}{R} \right)^2} \right) \right]} v_p = -p, \tag{8}$$

where H is the height of the wall, as measured from the arch's keystone. Because expression (8) is also difficult to integrate, we have limited the treatment here to the search for a numerical solution based on application of a standard finite difference method.

3. A COMPARISON BETWEEN SEMICIRCULAR AND OGIVAL ARCH-WALL SYSTEMS

If, as stated, we entirely ignore the deformations undergone by the arch, the load transmitted to the arch by the overlying wall is limited to the vertical component f_y alone. By way of example, figure 4 shows the plot of load component $f_y(x)$, evaluated via expression (5) and using the approximate solution from differential equation (8) for the case of a round arch and an ogival arch with a rise-to-span ratio of 7/6, both with a net span of 3 m and both overlain by a 5m-high wall, as measured from the arch keystone. The solution (blue curve) has been calculated using values drawn from the literature, E_p = 1.1 GPa and G = 0.18 GPa, and assuming a specific weight value of 18 kN/m^2 for the wall's constituent material. For the sake of comparison, the same figure also shows the plot of the same load component determined by assuming, as is common in applications, that the weight of each vertical strip of wall is sustained directly by the underlying arch element (red curve).

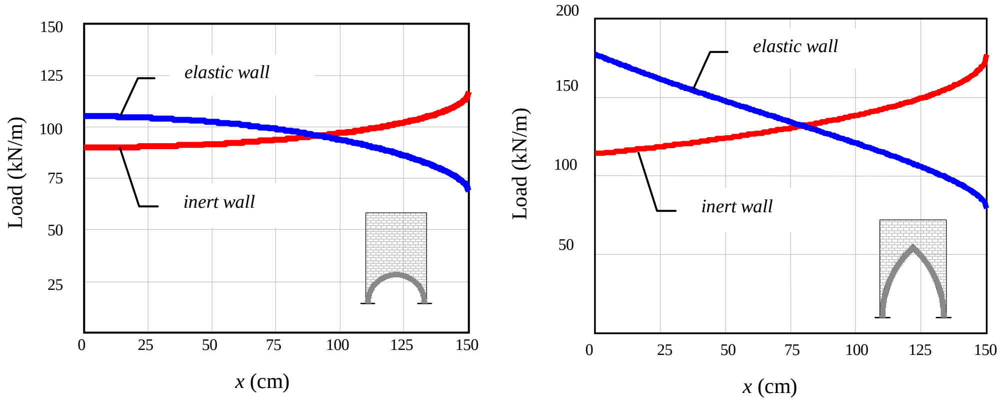


Figure 4: plots of the loads transmitted by the overlying wall
to a semicircular (left) and ogival (right) arch.

The distributions of the loads transmitted by the wall to the arch obtained by integrating differential equation (8) differs clearly, though not dramatically, from those deduced by assuming each vertical strip of wall to be sustained directly by the underlying arch element. The results, also calculated for numerous other choices of geometric and mechanical parameters, have also been compared to those resulting from numerical solution via a finite element model of a two-dimensional wall using an accurate calculation code. The comparisons, omitted here for the sake of brevity, reveal good agreement between the results of the numerical finite element model and those deduced by the simple mechanical model adopted here, in which the wall and arch are represented by interconnected one-dimensional elastic elements.

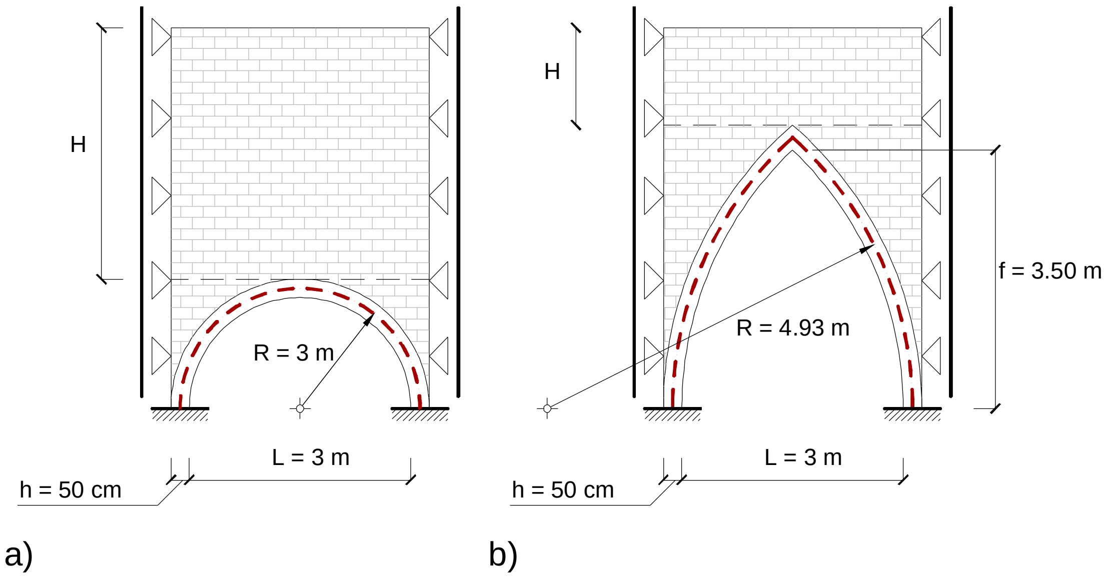
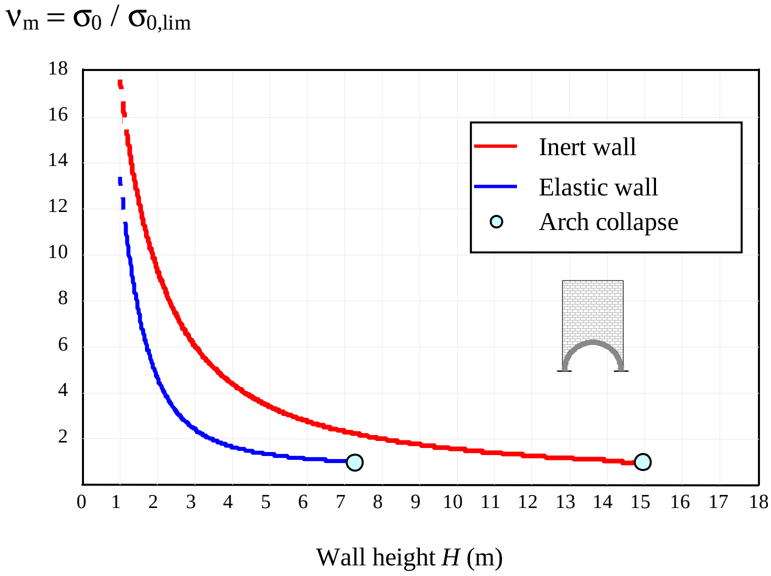


Figure 5: the arch-wall system for a semicircular (left) and ogival (right) arch.

Once the load transmitted by the wall to the arch has been determined, an estimate of the arch's bearing capacity can be made.



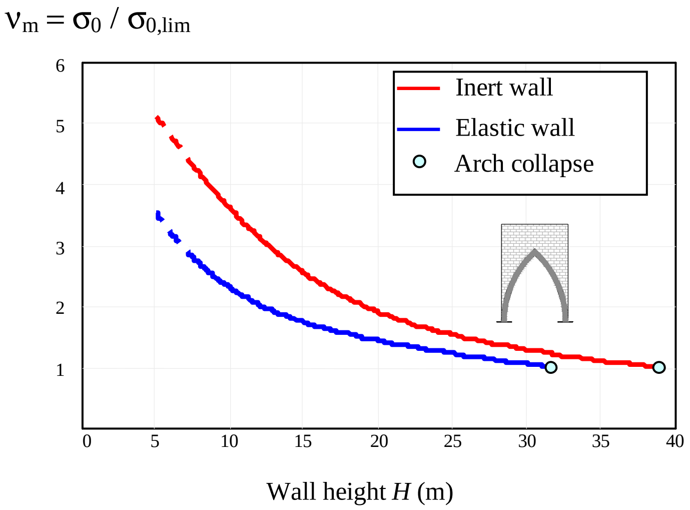


Figure 6: bearing capacity of the semicircular (above) and ogival (below) arch.

For simplicity's sake, a safety coefficient, ν_m, has been adopted as a measure of the arch's safety. It has been defined as the ratio between the compressive strength of the arch masonry, σ_0. and the limit value of the this same strength, $\sigma_{0,lim}$, which would correspond to collapse of the arch. This last value has been evaluated via an analytical variation of the so-called "stability area" method, under the hypothesis that the constituent material of the arch offers no resistance to traction and has limited compressive strength. The stability area method is a derivation of the Durand-Claye method: a graphical procedure to define the so-called area of stability at the crown section of a symmetrical arch, that is to say, the area within which the extremes of the vectors representing the crown thrust must be included in order that both the global equilibrium of the structure and the limited strength of masonry be respected. By scanning each joint of the arch and each eccentricity of the thrust, the set of all admissible thrust values may be found. The locus of the extremes of the vectors representing such forces constitutes the so-called area of stability. When this area shrinks to a point, the limit condition for the arch is attained and a unique admissible thrust line exists (for a more detailed description of the method, see, for example, [3]).

Regarding the geometric and mechanical characteristics of the arch, we have assumed a transverse section height of 50 cm and a compressive strength σ_0 = 10 MPa. Lastly, it is worth emphasizing that, for the sake of simplicity, the arch's collapse has been determined ignoring the contribution to overall strength afforded by the overlying wall.

Figure 6 shows a plot of the safety coefficient ν_m as a function of the height of the wall, measured from its keystone section. The figure clearly shows that the limit height of the wall, which corresponds to arch collapse ($\nu_m = 1$), depends heavily on the load distribution considered. In effect, the maximum height for a wall modeled as an elastic element overlying a semicircular arch is nearly half that obtained by applying the so-called inert wall model: only about 8 meters as opposed to nearly 16 meters.

A second noteworthy aspect is the influence of the arch's line of axis on its ultimate bearing capacity. To this end, we return once again to the two arch-wall systems shown in figure 5.

A comparison between the bearing capacities of the round and ogival arches is shown in figure 7, which plots the safety coefficient values determined first by assuming the weight of each vertical wall strip to be directly sustained by the underlying arch segment (figure 7a), and then by applying the elastic wall model (Figure 7b). The clear superiority, in terms of safety, of the ogival arch that emerges from figure 7 is further emphasized in figure 8, which represents the two arch-wall systems – semicircular and ogival – under conditions of imminent collapse.

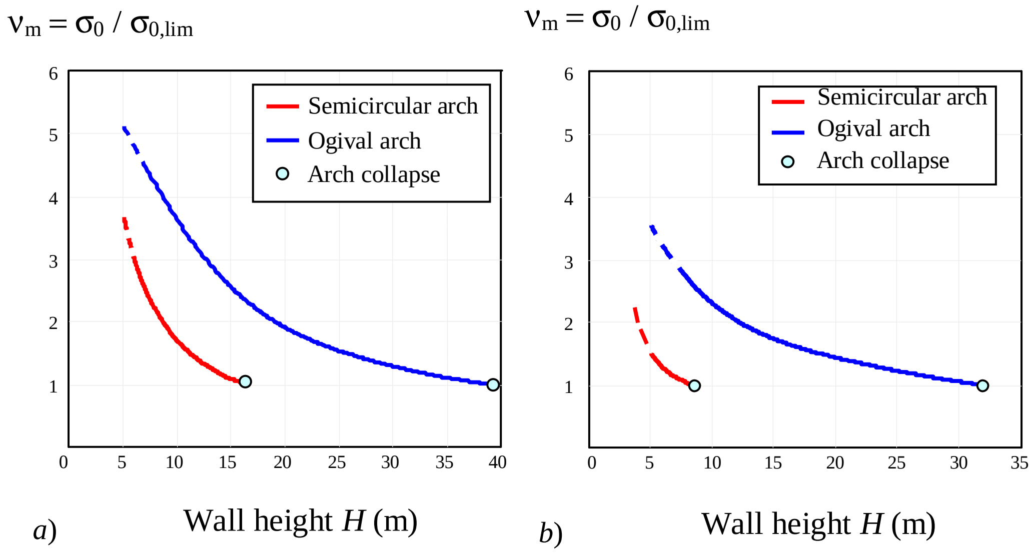


Figure 7: bearing capacity of semicircular and ogival arches when the wall is schematized as a vertical load (left) or as an elastic element (right).

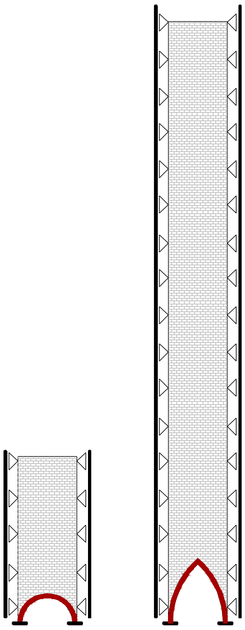


Figure 8: semicircular and ogival arches under conditions of imminent collapse.

4. CONCLUDING REMARKS

Evaluating the bearing capacity of a masonry arch-wall system seems all but banal, especially considering the great influence that relatively modest changes in the load distribution bearing down on the arch have on the (nonlinear) mechanical response of the system as well as on the collapse load value itself. Although the distribution of the loads transmitted to the arch by the wall obtained when the elastic deformations of the overlying wall are accounted for does not differ greatly from that deduced by assuming each vertical wall strip to be directly sustained by the underlying arch element, the resulting alteration of the shape of the load distribution, though relatively modest, generally leads to significant alterations in the arch's load bearing capacity. Surprisingly, such alterations may be critical, as happens in the case of semicircular masonry arches.

A further aspect worth stressing is the direct comparison made of the bearing capacities of ogival and semicircular arches of equal thickness making up otherwise identical arch-wall systems. The results highlight the clear superiority of the former over the latter, thus confirming a widely held conviction.

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