Waves propagation in a fractional viscoelastic continuum

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SUMMARY. In this paper the analysis of waves scattering in a fractional-type viscoelastic material is analyzed. Such a material involves, in the constitutive equation, the presence of non-integer order derivatives of the strain filed yielding a memory-type behavior of the material model. The presence of such a term has been also justified experimentally reporting the relaxation modulus of polymeric materials, obtained from experimental test, that are well-fitted by a power-law of fractional order. Some numerical applications reporting the standing-waves condition of an 1D solid varying the fractional differentiation order has also been reported in the paper.

1 INTRODUCTION

Viscoelastic materials have been more and more used nowadays for their low-cost productions as well as for their dissipative capabilities that may be coupled with others, more performing materials, to form complex-type engineering elements. The main feature of viscoelastic behavior is the relaxation of the stress state and the creep of the strain field that may be experienced, respectively, in hard or soft test devices. Such phenomenological consideration has been extensively analyzed yet at the beginning of the twentieth century and simple rheological models representing linear, constitutive, stress-velocity relations of the studied material have been proposed. Moreover the rheological relations have been also represented by a linear mechanical model, represented by a linear, viscous, dashpot relating the relative speed of its components to the applied load by a viscous coefficient. Combination of the dashpot, respectively in parallel or in series with a linear elastic spring corresponds to the Kelvin-Voigt or to the Maxwell model, respectively.

Large use of linear viscoelastic materials have been reported in scientific literature and engineering applications yet at the end of the fifties of the last century [1, 2]. These applications used the high damping characters of viscoelastic materials to provide passive controls of engineering systems in the form of bearing support or artificial dampers. The main feature of the viscoelastic material model provided with the linear models, either Kelvin-Voigt and Maxwell models, is related to the presence of a relaxation time that is characteristic parameter of the model.

Despite their wide diffusion, linear-type viscoelastic models do not match experimental evidences and suitable modifications of the rheological relations have been proposed in scientific literature as reported by several authors. Such improvements of the rheological relations have been obtained generalizing the Kelvin-Voigt model with different combinations of other mechanical elements, either dashpots or linear springs. In this way the mechanical response is obtained as combination of the responses of the various elements and the presence of multiple relaxation times is involved.

The generalization of such an approach, with an infinite number of Kelvin-Voigt or Maxwell elements yields a response function obtained as a convolution integral of an exponential-type function, as representing the kernel, and the external load applied to the element.

A different choice of the kernel may, in the form of a power-law decay, corresponds to a Riemann-Liouville fractional derivative of the external load [3]. Such a model of fractional viscoelasticity has been introduced, recently, to represent the rheological behavior of more complexes viscoelastic media.

The fractional model of viscoelasticity, introduced on mathematical basis, possesses an equivalent self-similar mathematical structure so that a fractal-type mechanical model has been introduced to represent such a kind of rheological model [4].

In this study axial waves propagation analysis will be faced in a fractional viscoelastic continuum in the simple monodimensional case. Such a problem is ruled by a fractional differential equation of second-order in the axial displacement function whose solution will be performed by Fourier transform for stationary, standing-waves analysis in unbounded domain, whereas in bounded domains, standing waves analysis will be conducted resorting to a suitable, spatial projection of the displacement function, and subsequent analysis of the time-dependent fractional differential equation.

Some numerical analysis will be reported also provided to include the effect of fractional viscoelasticity in the context of waves propagation.

2 REMARKS ON FRACTIONAL CALCULUS

In this section we give some details on fractional calculus and we consider functions defined in a finite interval. Given a Lebesgue measurable function $w(x)$ on the closed interval $[a,b]$, briefly $w(x) \in Leb_1([a,b])$ it is possible to define the left–handed RL fractional derivative, $\left(\mathcal{D}_{a+}^{\gamma} w\right)(x)$ with $\gamma \in \Re$, given by:

$$\left(\mathcal{D}_{a+}^{\gamma} w\right)(x) \overset{def}{=} \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \frac{w(\xi)\, d\xi}{(x-\xi)^{\gamma}} = \frac{d}{dx}\left(I_{a^+}^{1-\gamma} w\right)(x) \tag{1}$$

and the right–handed RL fractional derivative, $\left(\mathcal{D}_{b-}^{\gamma} w\right)(x)$ in the form:

$$\left(\mathcal{D}_{b-}^{\gamma} w\right)(x) = \frac{(-1)}{\Gamma(1-\gamma)} \frac{d}{dx} \int_x^b \frac{w(\xi)\, dt}{(\xi-x)^{\gamma}} = \frac{d}{dx}\left(I_{b^-}^{1-\gamma} w\right)(x) \tag{2}$$

with $0<\gamma<1$ and the latter terms in eqs. (1, 2) are the so-called RL fractional integrals that reads:

$$\left(I_{a^+}^{\gamma} w\right)(x) \overset{def}{=} \frac{1}{\Gamma(\gamma)} \int_a^x \frac{w(\xi)}{(x-\xi)^{\gamma}} d\xi \;\; ; \;\; \left(I_{b^-}^{\gamma} w\right)(x) \overset{def}{=} \frac{1}{\Gamma(\gamma)} \int_x^b \frac{w(\xi)}{(\xi-x)^{\gamma}} d\xi \tag{3 a, b}$$

Useful representations of Eqs.(1, 2) are:

$$\left(\mathcal{D}_{a+}^{\gamma} w\right)(x) = \frac{1}{\Gamma(1-\gamma)}\left[\frac{w(a)}{(x-a)^{\gamma}} + \int_a^x \frac{w'(\xi)\,d\xi}{(x-\xi)^{\gamma}}\right] \tag{4 a}$$

$$\left(\mathcal{D}_{b-}^{\gamma} w\right)(x) = \frac{1}{\Gamma(1-\gamma)}\left[\frac{w(b)}{(b-x)^{\gamma}} - \int_x^b \frac{w'(\xi)\,d\xi}{(\xi-x)^{\gamma}}\right] \tag{4 b}$$

as reported in [3]. In order to extend the definition of fractional derivative of order greater than 1, first, we recall a standard notation, indicating with $[\gamma]$ the integer part of a real number and with $\{\gamma\}$ the fractional part, that is $\gamma = [\gamma] + \{\gamma\}$. Then, for every positive real number γ the Riemann–Liouville fractional derivatives are defined as:

$$\left(\mathcal{D}_{a+}^{\gamma} w\right)(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_a^x \frac{w(\xi)\,d\xi}{(x-\xi)^{\gamma-n+1}} \tag{5 a}$$

$$\left(\mathcal{D}_{b-}^{\gamma} w\right)(x) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_x^b \frac{w(\xi)\,d\xi}{(\xi-x)^{\gamma-n+1}} \tag{5 b}$$

where $n = [\gamma]+1$. Comparing the definitions, it follows that the fractional derivatives and fractional integrals are related by the simple relations:

$$\left(\mathcal{D}_{a+}^{\gamma} w\right)(x) = \frac{d^n}{dx^n}\left(I_{a+}^{n-\gamma} w\right)(x) \tag{6 a}$$

$$\left(\mathcal{D}_{b-}^{\gamma} w\right)(x) = (-1)^n \frac{d^n}{dx^n}\left(I_{b-}^{n-\gamma} w\right)(x) \tag{6 b}$$

The presence of the derivatives of order *n* in the fractional derivatives definitions involves more strict conditions to the existence of the fractional derivative. A sufficient condition is the function having continuous derivatives up to the order $[\alpha]-1$.

$$\left(\mathcal{D}_{\pm}^{\gamma} w\right)(x) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^{\infty} \frac{w(x \mp \xi)\,d\xi}{\xi^{\gamma}} \tag{7}$$

for $0 < \gamma < 1$ or, for $\gamma > 0$, it is expressed as:

$$\left(\mathcal{D}_{\pm}^{\gamma}w\right)(x)=\frac{(\pm1)^{n}}{\Gamma(n-\gamma)}\frac{d^{n}}{dx^{n}}\int_{0}^{\infty}\xi^{n-\gamma-1}w(x\mp\xi)\,d\xi \qquad (8)$$

On the real axis the Eq.(8) can be written in more convenient form, working out a little on definition and those considerations, together with other useful properties and definitions may be found in basic books on fractional calculus.
Another useful definition of fractional derivatives, often used in a context of viscoelasticity and hereditary materials is represented by the Caputo's version of fractional derivatives, that reads:

$$\left({}_{C}\mathcal{D}_{a}^{\gamma}\right)f(x)=\frac{1}{\Gamma(\alpha-n)}\int_{a}^{x}\frac{f^{(n)}(\xi)}{(x-\xi)^{\alpha+1-n}}d\xi \quad ; \quad n-1\le\alpha\le n \qquad (9)$$

That coincides with the Riemann-Liouville and Marchaud fractional derivative in presence of unbounded domains as $a\to-\infty$. An useful property of the fractional differential operators defined in eqs.(7-9) are represented by the Fourier and Laplace transforms, denoted respectively, $\Im[\bullet]$ and $\wp[\bullet]$ as:

$$\Im\left[\left(\mathcal{D}_{+}^{\gamma}f\right)(t)\right]=\Im\left[\left({}_{C}\mathcal{D}_{+}^{\gamma}f\right)(t)\right]=(i\omega)^{\gamma}\,\Im\left[f(t)\right]=(i\omega)^{\gamma}F(\omega) \qquad (10\text{ a})$$

$$\wp\left[\left({}_{C}\mathcal{D}_{0^{+}}^{\gamma}f\right)(x)\right]=(p)^{\gamma}\,\wp\left[f(x)\right]=(p)^{\gamma}\hat{f}(p) \qquad (10\text{ b})$$

The basic equations of the 1D linear elastodynamics in presence of fractional damping will be outlined in sec.3 whereas the origin of the fractional model of linear viscoelasticity based upon experimental set ups will be reported in the next section.

3 THE FRACTIONAL MODEL OF LINEAR VISCOELASTICITY

As afore mentioned in the introduction, all type of material behaviour are usually schematized starting from spring and dashpot and then generalizing a several of both in parallel or in series. In any case from the simplest to the generalized one, the analytical formulation or briefly the constitutive law is governed by derivatives of displacement of order zero, one or of integer n. This differential equation also finds an integral representation that is the Duhamell integral response to a mathematical impulse having an exponential function as kernel. This schematization is also wide used dealing with viscoelastic materials as rubber, polymers, concrete, bitumen and etc., however the experimental data, pertaining these viscoelastic materials, lead to different considerations as above. For instance in fig.(1 a) it is depicted a relaxation test of a rubber with density 180 (Kg/m3). From a sharp observation of this fig.(1a) it is apparent that, experimental results are best fitted by a power law and not by an exponential function. These results show just one test done in the lab of university of Palermo, pertaining a wide campaign of tests to capture the viscoelastic behaviour of rubber with different value of density. Here it has been chosen just one because all tests show the same behaviour: the best fit is obtained by a decay power law. Of course the order parameter of this function was changing test by test, depending on the density of the rubber. But it has to be stressed that all of these parameters had a common feature: they were not integer numbers but a number between zero and one, for instance the power law of the relaxation modulus depicted in fig..(1 b) has been obtained with values:

$$E(t) = a\, t^{-\beta}, \quad \text{where } a = 10,64010 \;\; Ns^{\beta}/cm^{2} \;\; \text{and } \beta = 0,02883; A = 1\, cm^{2} \tag{11}$$

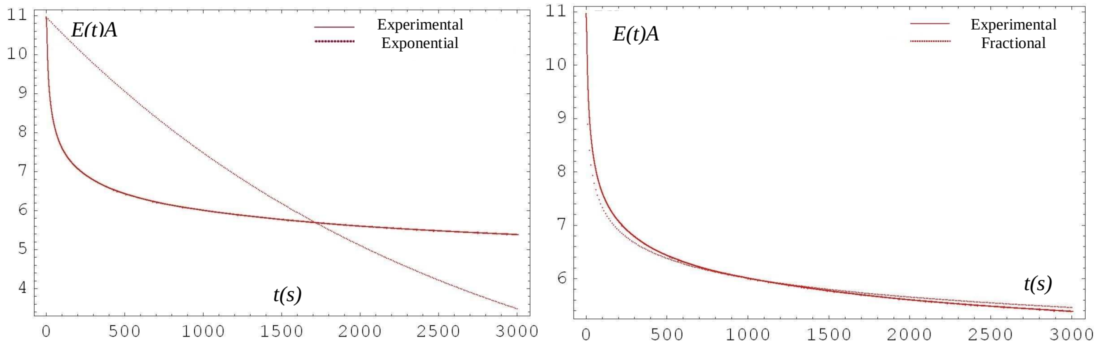

Figure 1: Static set-up on viscoleastic rubber: a) Continuous line experimental, dashed-line exponential best fit, b) Continuous line experimental, dashed fractional-power law best fit

as already observed in other pioneeristic studies [5]. In order to find the Duhamell integral we have to get the creep compliance function $J(t)$ starting from $E(t)$. This may be easily done by considering that creep compliance and relaxation modulus must obey to the relationship $E(s)J(s) = s^{-2}$ where $E(s)$ and $J(s)$ are the Laplace transform of $E(t)$ and $J(t)$ respectively. The Laplace transform of $E(t)$ is readily found $E(s) = a\, s^{\beta-1}\,\Gamma(1-\beta)$. It follows that after some straightforward manipulations, the creep compliance function reads:

$$J(t) = \frac{t^{2-\beta}}{a\,\Gamma(1-\beta)\,\Gamma(3-\beta)} \tag{12}$$

and the impulse response function $\dot{J}(t)$ is then provided in the form:

$$\dot{J}(t) = \frac{(2-\beta)\,t^{1-\beta}}{a\,\Gamma(1-\beta)\,\Gamma(3-\beta)} \tag{13}$$

By denoting as $c_{\beta} = \Gamma(\beta)(2-\beta)/a\,\Gamma(1-\beta)\Gamma(3-\beta)$ eq.(13) is then written as:

$$\varepsilon(t) = \frac{c_{\beta}}{\Gamma(\beta)}\int_{0}^{t} \frac{\sigma(\tau)}{(t-\tau)^{\beta-1}}\, d\tau = c_{\beta}\left(I_{0^{+}}^{\beta}\sigma\right)(t) \tag{14}$$

involving an RL fractional integral of the time-varying stress. Such a relation will be used in the next section to study the wave propagation in a viscolelastic material.

4 1D WAVES PROPAGATION IN FRACTIONAL VISCOELASTIC MATERIAL

In this section the 1D formulation of waves analysis in a fractional viscoelastic continuum will be reported for longitudinal (P) waves propagation in an 1D solid. To this aim let us consider the 1D solid reported in fig.(2a) of length L and referred to a coordinate system positive rightward. Let us assume, for shortness, that the solid is homogeneous with uniform cross-section A and uniform mass-density $\rho(x)=\bar{\rho}$.

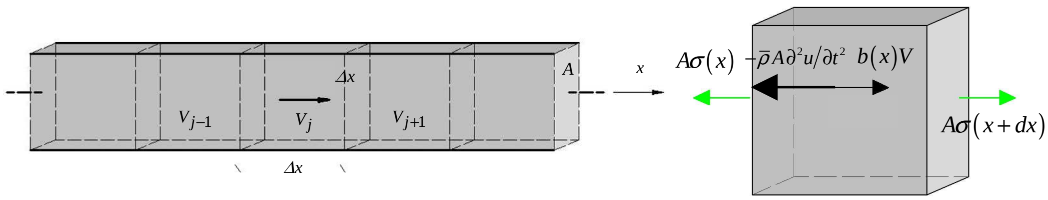

Figure 2: a) 1D model of viscoelastic bar; b) Equilibrium of 1D volume element

The elastodynamics of the solid is ruled, straightforwardly, by the axial equilibrium equation of the discrete volume element of the body, $\Delta V = A\Delta x$ with $\Delta x = L/m$ and m is the number of elements considered to discretize the solid reported in fig.(2b) that reads:

$$A\Delta\sigma(x_j,t) = -q(x_j,t)\Delta x A \tag{15}$$

where $\sigma(x_j,t)$ is the axial stress at abscissa $x_j=(j-1)\Delta x$ $(j=1,2,...,m)$, $q(x_j,t)$ is the volume force applied at element ΔV and $\Delta\sigma(x_j,t)=\sigma(x_j,t)-\sigma(x_{j-1},t)$. The body force field named $q(x_j,t)$ includes both the external force field, named $b(x_j,t)$ as well as the inertial body forces as:

$$q(x_j,t) = b(x_j,t) - \bar{\rho}\frac{\partial^2 u(x_j,t)}{\partial t^2} \tag{16}$$

in which we denoted $u(x_j,t)$ the axial displacement of the volume at abscissa x_j at time t. Introducing Eq.(15) into Eq.(16) and letting $\Delta x \to 0$ the following differential equation is obtained:

$$\frac{\partial\sigma}{\partial x} - \bar{\rho}\frac{\partial^2 u}{\partial t^2} = -b(x,t) \tag{17}$$

The field equation of the linear elastodynamics is obtained as we introduce a relation between the stress field and the axial strain to be included in eq.(16). Thus, the rheological behavior of the material must be included in such stress-strain condition and it will be assumed as in eq.(14) to model the viscous behavior of the material. After some straightforward manipulations, by using the composition rules of fractional calculus [3] the constitutive, rheological conditions between the stress and the strain is assumed in the form:

$$\sigma(x,t) = E_0\varepsilon(x,t) + E_\alpha \left({}_C\mathcal{D}_{0^+}^{\alpha}\varepsilon \right)(x,t) \quad ; \quad 0 \le \alpha \le 1 \tag{18}$$

where $\left({}_C\mathcal{D}_{0^+}^{\alpha}\varepsilon \right)(x,t)$ is the Caputo fractional derivative of order α with respect to time, that coincide with the RL derivative for vanishing initial conditions, of the strain field $\varepsilon(x,t)$ and $[E_\alpha] = [1/c_\alpha] = FT^{\alpha-1}/L^2$ is an anomalous force coefficient. Introducing eq.(14) into eq.(13) and accounting for the kinematic restraint $\varepsilon(x,t) = \partial u(x,t)/\partial x$, the governing equation of the displacement field $u(x,t)$ reads:

$$\overline{\rho}\frac{\partial^2 u(x,t)}{\partial t^2} - E_\alpha \frac{\partial^2}{\partial x^2}\left[\left({}_C\mathcal{D}_{0^+}^{\alpha} u \right)(x,t) \right] - E_0 \frac{\partial^2 u(x,t)}{\partial x^2} = -b(x,t) \tag{19}$$

Representing the governing equation of the linear 1D elastodynamics of a fractional viscoelastic continuum. Eq.(19) must be supplemented by the relevant initial and boundary conditions as::

$$u(x,0) = \overline{u}(x) \quad ; \quad \left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0} = \overline{v}(x) \tag{20 a, b}$$

$$u(0,t) = u_0(t) \quad ; \quad u(L,t) = u_L(t) \tag{21 a, b}$$

with Eqs.(21 a, b) representing the kinematic (essential) boundary conditions to be replaced, eventually, with the mechanical (natural) boundary conditions that reads:

$$A\left\{ E_0 \frac{\partial u}{\partial x} + E_\alpha \frac{\partial}{\partial x}\left({}_C\mathcal{D}_{0^+}^{\alpha} u \right)(x,t) \right\}\Bigg|_{x=0} = -F_0(t) \tag{22 a}$$

$$A\left\{ E_0 \frac{\partial u}{\partial x} + E_\alpha \frac{\partial}{\partial x}\left({}_C\mathcal{D}_{0^+}^{\alpha} u \right)(x,t) \right\}\Bigg|_{x=L} = F_L(t) \tag{22 b}$$

Waves analysis will be limited in this paper to steady-state motion so that the dynamic equilibrium equation reported in eq.(22) may be assumed in the form:

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j(x)\, y_j(t) \tag{23}$$

where $\varphi_j(x)$ are the standing wave shapes of the undamped solid $(E_\alpha = 0)$ and $y_j(t)$ are time-amplitude, unknown functions that are provided introducing Eq.(22) into Eq.(19) and projecting the resulting expression over the complete space spanned by the function set $\varphi_k(x)$ $(k = 1, 2, \ldots)$ yielding:

$$\sum_{j=1}^{\infty}\bar{\rho}\frac{d^2y_j(t)}{dt^2}\int_0^L\varphi_k(x)\varphi_j(x)dx-\sum_{j=1}^{\infty}E_0y_j(t)\int_0^L\varphi_k(x)\frac{d^2\varphi_j(x)}{dx^2}dx$$
$$-\sum_{j=1}^{\infty}E_\alpha\left({}_C\mathcal{D}_{0^+}^{\alpha}y_j\right)(t)\int_0^L\varphi_k(x)\frac{d^2\varphi_j(x)}{dx^2}dx=-\int_0^L\varphi_k(x)b(x,t)dx \tag{24}$$

Accounting for the orthogonality conditions between the eigenfunctions that read:

$$\int_0^L\varphi_k(x)\varphi_j(x)dx=\delta_{jk}\quad;\quad\int_0^L\varphi_k(x)\frac{d^2\varphi_j(x)}{dx^2}dx=\omega_k^2\delta_{jk} \qquad (25\ a\ ,b)$$

where δ_{jk} is the Kronecker delta $\omega_k^2=\kappa_k^2c_0^2=\kappa_k^2\,E_0/\bar{\rho}$ is the circular frequency associated to the eigenfunction $\varphi_k(x)$ and κ_k is the wavenumber associated to the *k*-th standing wave $\varphi_k(x)$, Eq.(20) may be written in the form:

$$\frac{d^2y_k(t)}{dt^2}+\omega_k^2y_k(t)-C_\alpha^2\left({}_C\mathcal{D}_{0^+}^{\alpha}y\right)(t)=g_k(t)/\bar{\rho}\;\;;\;(\;k=1,2,...) \tag{26}$$

with $C_\alpha^2=E_\alpha/\bar{\rho}$ and the load function $g_k(t)$ is provided by the right-hand side of Eq.(24). The associated initial conditions are reported in Eq.(20 a) since the essential boundary conditions are accounted for in the shape functions and the natural boundary conditions in the external load field. In the following the differential equation providing the time-amplitude function $y_k(t)$ (Eq.26) will be solved in Laplace domain yielding, accounting for Eq.(11 b), the following relation between the Laplace transform of the amplitude function $\hat{y}_k(s)$ and the Laplace transform of the load as:

$$\hat{y}_k(p)=-\frac{\hat{g}_k(p)}{p^2+c_0^2\omega_k^2-C_\alpha^2(p)^\alpha}\quad;\quad(\;k=1,2,...) \tag{27}$$

The effect of the fractional viscoelasticity may be highlighted from the observation of the steady-state response of a clamped-free 1D solid with harmonically time-varying axial force at the free end $(x=L)$ modeled as $b(x,t)=\bar{b}\,\delta(x-L)\sin(\Omega t)$ with $\bar{b}$ is the amplitude and $\delta(\bullet)$ is the Dirac's delta. In this context the amplitude functions $\hat{y}_k(s)$ obtained from eq.(23) yields, after straightforward manipulations:

$$\hat{y}_k(p)=\frac{\bar{b}\varphi_k(L)}{\bar{\rho}}\frac{\Omega}{p^2+\Omega^2}\frac{1}{p^2+\omega_k^2-C_\alpha^2(p)^\alpha}\quad;\quad(\;k=1,2,...) \tag{28}$$

so that the vibrating response of the bar, in the standing waves case, is obtained as the inverse Laplace transform of each of the Laplace components described in eq.(28) as:

$$u(x,t)=\frac{\bar{b}\varphi_k(L)\Omega}{\bar{\rho}}\sum_{j=1}^{\infty}\varphi_j(x)\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}\frac{1}{p^2+\Omega^2}\frac{1}{p^2+\omega_k^2-C_\alpha^2(p)^\alpha}dp \tag{29}$$

with the integration carried out in the complex *p*-plane along an Henkel contour of real abscissa γ. The effect of the fractional differentiation order on the steady-state waves condition may be observed from figs.(3 a, b) reporting the vibration response of the column obtained from the standing waves analysis outlined above for $\alpha = 0.3$ (fig. 3a) and for $\alpha = 0.5$ (fig. 3b) . The numerical tests have been conducted for an 1D prismatic bar with geometrical parameters $A = 1\ cm^2$; $L = 100\ cm$ whereas the mechanical characteristic used for the numerical analysis have been selected as $E_0 = 5000\ Kg/cm^2$; $E_\alpha = 10\ Kg\ s^\alpha/cm^2$; $\bar{\rho} = 4.0\ 10^{-3}\ Kg\ /cm^3$.

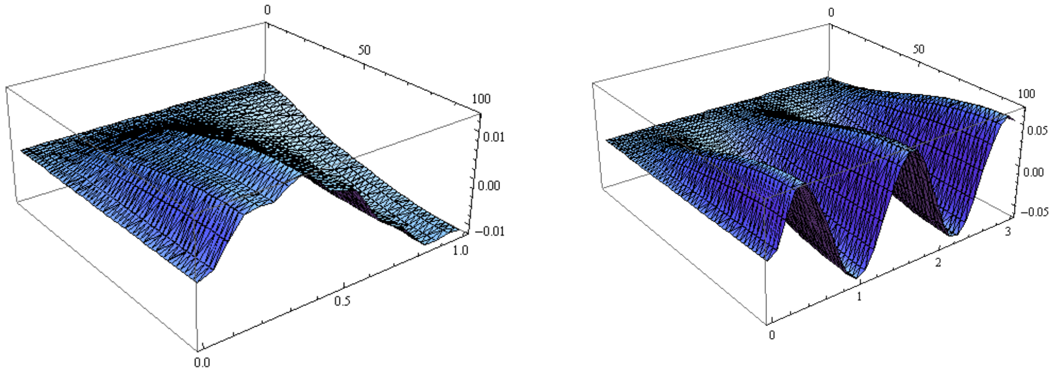

Figure 3 a,b: Standing-waves under harmonic time-varying force

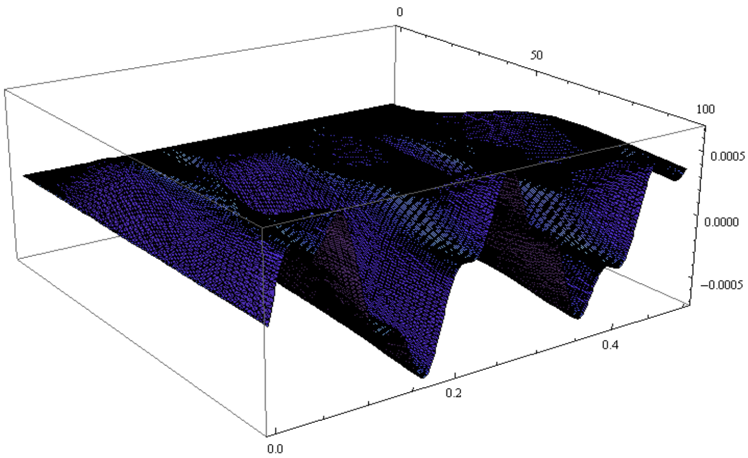

Figure 4 a: Standing-waves under time-window force for $\alpha = 0.3$

The observation of fig.(3 a,b) shows that as the fractional differentiation α decreases toward zero, then the material behavior tends to be more and more elastic with less dissipation characteristic whereas as α increases the dissipation characters are more pronounced. This may be observed by the fact that the bar initial conditions are maintained longer in the system prior the steady-state condition. Such a consideration is more and more evident observing the behavior of the material under a windows force for different values of the fractional exponent as reported in figs.(4 a, b).

5 CONCLUSIONS

In this paper the basic equations of the elastodynamics in a fractional viscoelastic material have been outlined and numerical analysis has been conducted under standing-waves condition. The considered problem is concerned with an 1D domain and the constitutive equations of the viscoelastic material have been assumed in terms of fractional-order Caputo derivatives instead than the well-known Kelvin-Voigt model of material damping Such a choice has been justified from experimental tests conducted on polymeric materials that yields the relaxation modulus well-described by a power-law function of the time with the exponent of the decay that is not an integer power. The fractional power is the equivalent, in the Laplace domain to an impulse response function of the material, involved in the Duhamel integral of the response, that is described by a power-law kernel yielding the governing equations in the form of fractional differential equations. The numerical analysis conducted in the paper has shown that varying the fractional differential exponent the response of the material to applied forces may be very different and it must be calibrated after accurate experimental set ups.

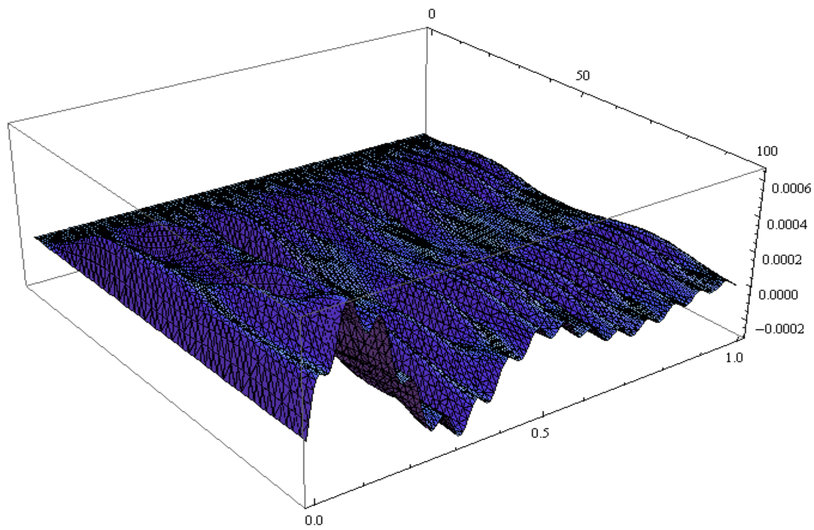

Figure 4 b: Standing-waves under time-window force for $\alpha = 0.5$

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