Shell-like inclusions with high rigidity: an asymptotic approach

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SUMMARY. We study the problem of an elastic shell-like inclusion with high rigidity in a three-dimensional domain by means of the asymptotic expansion method. The analysis is carried out in a general framework of curvilinear coordinates. After defining a small real parameter ε, we characterize the limit problems when the rigidity of the inclusion has order of magnitude $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^3}$ with respect to the rigidities of the surrounding bodies. Moreover, we prove strong convergence of the solution of the initial three-dimensional problem towards the solution of the simplified limit problem.

1 INTRODUCTION

After the pioneering works of Pham Huy-Sanchez [1], Brezis et al. [2] and Caillerie [3], the thin inclusion of a third material between two other ones when the rigidity properties of the inclusion are highly contrasted with respect to those of the surrounding material has been deeply investigated. More recently, Chapelle-Ferent [4], in order to justify some methods used in the FEM approximation, have studied the asymptotic behavior of a shell-like inclusion of $\frac{1}{\varepsilon^p}$-rigidity ($p = 1$ or $p = 3$) in a three-dimensional domain. In a slightly different geometrical and mechanical context, Bessoud et al. [5] have studied the behavior of a ε-thin three-dimensional layer of $\frac{1}{\varepsilon}$-rigidity. More precisely, they assume that the thin layer can be written as $\omega \times]-\varepsilon, \varepsilon[$ where ω is a projectable two-dimensional surface, and that all the materials are linearly elastic anisotropic. Then the limit problem is a Ventcel-type transmission problem between two three-dimensional linearly elastic anisotropic bodies. When ω is planar and in the isotropic case, the associated surface energy term can be interpreted as the membranal energy of a Kirchhoff-Love plate.

Here we study the situation where the shell-like thin layer is obtained by the translation along the normal direction of a general two-dimensional surface. Using a system of curvilinear coordinates we deduce the formal limit problem for the two cases $p = 1$ and $p = 3$. In this way we obtain the same limit problems as in [4], also if the kinematical assumptions for the physical problem are not the same. Indeed in [4] the authors a priori assume a shell-like energy in the thin layer. As in [4] one must stress that the well-posedness of the limit problems is essentially linked to the well-posedness of the shell models [6, 7]. Afterwards we prove some results of strong convergence in a specific functional framework.

2 GEOMETRICAL PRELIMINARIES

2.1 *Three-dimensional curvilinear coordinates*

This section is aimed at laying down an appropriate ground for the rest of the article. In the sequel, Greek indices range in the set $\{1, 2\}$, Latin indices range in the set $\{1, 2, 3\}$, and the summation convention with respect to the repeated indices is adopted.

Let us consider a three-dimensional Euclidian space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. Let Ω be a non-empty open subset of $\mathbb{R}^3$. A mapping $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ is an immersion if the three vectors $\partial_i \mathbf{\Theta}(x)$ are linearly independent for all $x = (x_i) \in \Omega$. The image $\widehat{\Omega} := \mathbf{\Theta}(\Omega)$ is always an open set immersed in $\mathbb{R}^3$. The three coordinates x_i of a point $x \in \Omega$ represent the curvilinear coordinates of the point $\widehat{x} = \mathbf{\Theta}(x) \in \widehat{\Omega}$, while the three coordinates $\widehat{x}_i$ of the point $\widehat{x} \in \widehat{\Omega}$ are the Cartesian coordinates.

The three vectors $\mathbf{g}_i(x) := \partial_i \mathbf{\Theta}(x)$ form the covariant basis at $\widehat{x} = \mathbf{\Theta}(x)$ and the three vectors $\mathbf{g}^j(x)$, defined by the nine independent relations $\mathbf{g}_i(x) \cdot \mathbf{g}^j(x) = \delta_i^j$ for all $x \in \Omega$, form the contravariant basis at $\widehat{x}$.

The immersion $\mathbf{\Theta}$ induces a Riemannian metric on Ω, defined respectively by its covariant components $g_{ij}(x) := \mathbf{g}_i(x) \cdot \mathbf{g}_j(x)$, and contravariant components $g^{k\ell}(x) := \mathbf{g}^k(x) \cdot \mathbf{g}^\ell(x)$. The contravariant components of this metric can be analogously defined by $(g^{k\ell}(x)) = (g_{ij}(x))^{-1}$ for all $x \in \Omega$.

This metric induces a Levi-Civita connection in the manifold Ω defined by the Christoffel symbols of the second kind $\Gamma^p_{ij} := \mathbf{g}^p \cdot \partial_i \mathbf{g}_j = \Gamma^p_{ji}$.

Let there be given a vector field defined over $\mathbf{\Theta}(\Omega)$. We can rewrite this vector field as a linear combination $\mathbf{v} = v_i \mathbf{g}^i$ of the vector fields $\mathbf{g}^i : \Omega \to \mathbb{R}^3$, where $v_i = \mathbf{v} \cdot \mathbf{g}_i$ are the covariant components of the vector field $\mathbf{v}$. The covariant derivatives $v_{i\|j}$ of the covariant components v_i are defined by $v_{i\|j} := \partial_j v_i - \Gamma^p_{ij} v_p$. The covariant derivatives $T^{ij}\|_k$ of the second-order tensor field with contravariant components T^{ij} are defined by $T^{ij}\|_k := \partial_k T^{ij} + \Gamma^i_{\ell j} T^{\ell k} + \Gamma^j_{\ell k} T^{\ell i}$.

With every displacement field $\mathbf{v}$, we associate the linearized change of metric tensor defined as follows:

$$e_{ij}(\mathbf{v}) := \frac{1}{2}(v_{i\|j} + v_{j\|i}).$$

2.2 *Curvilinear coordinates on a surface*

Let ω be a non-empty open subset in $\mathbb{R}^2$. The coordinates of $\widetilde{x} \in \omega$ are denoted by x_α. A mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ is an immersion if the two vectors $\partial_\alpha \boldsymbol{\theta}(\widetilde{x})$ are linearly independent at each point $\widetilde{x} = (x_\alpha) \in \omega$. The image $S := \boldsymbol{\theta}(\omega)$ is a surface immersed in $\mathbb{R}^3$, equipped with x_α curvilinear coordinates.

The two vectors $\mathbf{a}_\alpha(\widetilde{x}) := \partial_\alpha \boldsymbol{\theta}(\widetilde{x})$ form the covariant basis of the tangent plane to the surface S at $\boldsymbol{\theta}(\widetilde{x})$, and the two vectors $\mathbf{a}^\beta(\widetilde{x})$ defined by the relations $\mathbf{a}_\alpha(\widetilde{x}) \cdot \mathbf{a}^\beta(\widetilde{x}) = \delta^\beta_\alpha$, form the contravariant basis of the tangent plane to the surface S at $\boldsymbol{\theta}(\widetilde{x})$. The unit normal vector to S at $\boldsymbol{\theta}(\widetilde{x})$ is defined by $\mathbf{a}_3(\widetilde{x}) = \mathbf{a}^3(\widetilde{x}) := \frac{\mathbf{a}_1(\widetilde{x}) \wedge \mathbf{a}_2(\widetilde{x})}{|\mathbf{a}_1(\widetilde{x}) \wedge \mathbf{a}_2(\widetilde{x})|}$.

The covariant components of the first fundamental form of the surface are defined by $a_{\alpha\beta}(\widetilde{x}) := \mathbf{a}_\alpha(\widetilde{x}) \cdot \mathbf{a}_\beta(\widetilde{x})$, and its contravariant components are defined by $a^{\alpha\beta}(\widetilde{x}) := \mathbf{a}^\alpha(\widetilde{x}) \cdot \mathbf{a}^\beta(\widetilde{x})$.

The covariant components of the second fundamental form of the surface are defined by $b_{\alpha\beta}(\widetilde{x}) := \partial_\alpha \mathbf{a}_\beta(\widetilde{x}) \cdot \mathbf{a}_3(\widetilde{x})$, and its mixed components are defined by $b^\tau_\alpha(\widetilde{x}) := a^{\tau\beta}(\widetilde{x}) b_{\alpha\beta}(\widetilde{x})$.

The Christoffel symbols on the surface S of the second kind are given by $\Gamma^\tau_{\alpha\beta} := \mathbf{a}^\tau \cdot \partial_\alpha \mathbf{a}_\beta$.

Any vector field on a surface can be written as a linear combination $\boldsymbol{\eta} = \eta_i \mathbf{a}^i$ of the vector field $\mathbf{a}^i : \omega \to \mathbb{R}^3$, where the functions $\eta_i = \boldsymbol{\eta} \cdot \mathbf{a}_i$ are the covariant components of the vector field $\boldsymbol{\eta}$. The covariant derivatives $\eta_{\alpha|\beta}$ of the covariant components η_α are defined by $\eta_{\alpha|\beta} := \partial_\beta \eta_\alpha - \Gamma^\tau_{\alpha\beta}\eta_\tau$. The covariant derivatives $T^{\alpha\beta}|_\tau$ of the second-order tensor field with contravariant components $T^{\alpha\beta}$ are defined by $T^{\alpha\beta}|_\tau := \partial_\tau T^{\alpha\beta} + \Gamma^\alpha_{\beta\sigma}T^{\tau\sigma} + \Gamma^\beta_{\tau\sigma}n^{\alpha\sigma}$. For more details about differential geometry of surfaces, see [6].

With every displacement field $\boldsymbol{\eta}$, we associate the linearized change of metric tensor field defined by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3,$$

and the linearized change of curvature tensor field, defined by

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = \eta_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha \eta_{\sigma|\beta} + b^\tau_\beta \eta_{\tau|\alpha} + (\partial_\alpha b^\tau_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta)\eta_\tau.$$

The symmetric tensor fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ play a key role in the theory of linearly elastic shells (see, e.g., P.G. Ciarlet [6]).

3 POSITION OF THE PROBLEM

Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure and let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an immersion.

Let $0 < \varepsilon < 1$ be an adimensional *small* real parameter. Let us consider $\Omega^{m,\varepsilon} := \omega \times]-\varepsilon, \varepsilon[$ and $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon\}$. Let x^ε denote the generic point in the set $\overline{\Omega}^{m,\varepsilon}$ with $x^\varepsilon_\alpha = x_\alpha$. We consider a *shell-like* domain with middle surface $\boldsymbol{\theta}(\overline{\omega})$ and thickness 2ε, whose reference configuration is the image $\boldsymbol{\Theta}^{m,\varepsilon}(\overline{\Omega}^{m,\varepsilon}) \subset \mathbb{R}^3$ of the set $\overline{\Omega}^{m,\varepsilon}$ through the mapping given by

$$\boldsymbol{\Theta}^{m,\varepsilon}(x^\varepsilon) := \boldsymbol{\theta}(\widetilde{x}) + x^\varepsilon_3 \mathbf{a}_3(\widetilde{x}), \text{ for all } x^\varepsilon = (\widetilde{x}, x^\varepsilon_3) \in \overline{\Omega}^{m,\varepsilon}.$$

We denote by $\Omega^{+,\varepsilon}$ (resp. $\Omega^{-,\varepsilon}$) the translation of Ω^+ (resp Ω^-) in the direction $\mathbf{e}_3$ (resp.-$\mathbf{e}_3$) of the quantity ε.

Moreover, we suppose that there exists an immersion $\boldsymbol{\Theta}^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ defined as follows:

$$\boldsymbol{\Theta}^\varepsilon := \begin{cases} \boldsymbol{\Theta}^{\pm,\varepsilon} & \text{on } \overline{\Omega}^{\pm,\varepsilon} \\ \boldsymbol{\Theta}^{m,\varepsilon} & \text{on } \overline{\Omega}^{m,\varepsilon} \end{cases}, \ \boldsymbol{\Theta}^{\pm,\varepsilon}(S^{\pm,\varepsilon}) = \boldsymbol{\Theta}^{m,\varepsilon}(S^{\pm,\varepsilon}),$$

with $\boldsymbol{\Theta}^{\pm,\varepsilon} : \overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}^3$ immersions over $\overline{\Omega}^{\pm,\varepsilon}$ defining the curvilinear coordinates on $\overline{\Omega}^{\pm,\varepsilon}$. Let us stress that the physical domain of the assembly is obtained by inserting in the direction $\mathbf{a}_3$ the shell within the two bodies, see Figure 1. The structure is clamped on Γ^ε_0 and the complementary part of the boundary is free. Obviously one can there consider other type of boundary conditions. The structure is also submitted to applied body forces f^ε so that the work of the external loading is given by the linear form

$$L^\varepsilon(\mathbf{v}^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} f^\varepsilon_i v^\varepsilon_i \, dx^\varepsilon.$$

We suppose that the materials are linearly elastic and isotropic with Lamé's constants $\lambda^{\pm,\varepsilon}$ and $\mu^{\pm,\varepsilon}$ for $\Omega^{\pm,\varepsilon}$, $\lambda^{m,\varepsilon}$ and $\mu^{m,\varepsilon}$ for $\Omega^{m,\varepsilon}$.

The physical variational problem in curvilinear coordinates defined over the variable domain Ω^ε can be written as

$$\begin{cases} \text{Find } \mathbf{u}^\varepsilon \in V^\varepsilon := \{\mathbf{v}^\varepsilon \in H^1(\Omega^\varepsilon; \mathbb{R}^3); \ \mathbf{v}^\varepsilon_{|\Gamma^\varepsilon_0} = \mathbf{0}\} \text{ such that} \\ A^{-,\varepsilon}(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) + A^{+,\varepsilon}(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) + A^{m,\varepsilon}(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = L^\varepsilon(\mathbf{v}^\varepsilon) \text{ for all } \mathbf{v}^\varepsilon \in V^\varepsilon. \end{cases} \tag{1}$$

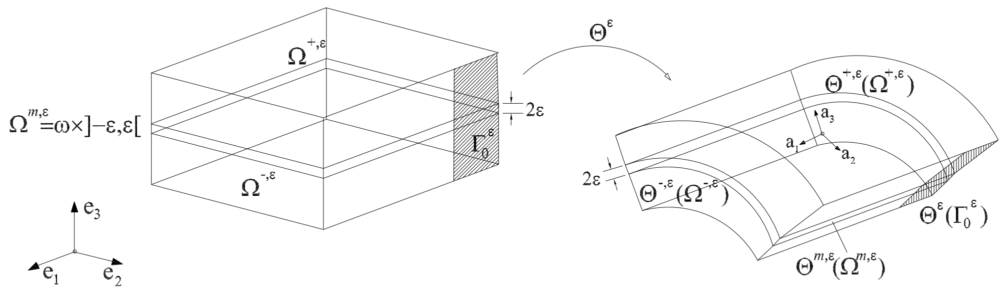

Figure 1: Initial and reference configuration of the assembly

The bilinear forms $A^{\pm,\varepsilon}(\cdot,\cdot)$ and $A^{m,\varepsilon}(\cdot,\cdot)$ are defined by

$$A^{\pm,\varepsilon}(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} A_{\pm}^{ijk\ell,\varepsilon} e^\varepsilon_{k\ell}(\mathbf{u}^\varepsilon) e^\varepsilon_{ij}(\mathbf{v}^\varepsilon)\sqrt{g^{\pm,\varepsilon}}\, dx^\varepsilon,$$
$$A^{m,\varepsilon}(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon) := \int_{\Omega^{m,\varepsilon}} A_{m}^{ijk\ell,\varepsilon} e^\varepsilon_{k\ell}(\mathbf{u}^\varepsilon) e^\varepsilon_{ij}(\mathbf{v}^\varepsilon)\sqrt{g^{m,\varepsilon}}\, dx^\varepsilon.$$

Here $A^{ijk\ell,\varepsilon} := \lambda^\varepsilon g^{ij,\varepsilon} g^{k\ell,\varepsilon} + \mu^\varepsilon(g^{ik,\varepsilon}g^{j\ell,\varepsilon} + g^{i\ell,\varepsilon}g^{jk,\varepsilon})$ are the contravariant components of the elasticity tensor and $g^\varepsilon := \det(g^\varepsilon_{ij})$.

If we suppose that $f_i^\varepsilon \in L^2(\Omega^{\pm,\varepsilon})$, then problem (1) has one and only one solution thanks to Lax-Milgram lemma.

In order to study the asymptotic behavior of the solution of problem (1) when ε tends to zero, we rewrite the problem on a fixed domain Ω independent of ε. By using the approach of [6], we consider the bijection $\pi^\varepsilon : x \in \overline{\Omega} \mapsto x^\varepsilon \in \overline{\Omega}^\varepsilon$ given by

$$\begin{cases} \pi^\varepsilon(x_1,x_2,x_3) = (x_1,x_2,x_3-(1-\varepsilon)), & \text{for all } x\in\overline{\Omega}^+_{tr},\\ \pi^\varepsilon(x_1,x_2,x_3) = (x_1,x_2,\varepsilon x_3), & \text{for all } x\in\overline{\Omega}^m,\\ \pi^\varepsilon(x_1,x_2,x_3) = (x_1,x_2,x_3+(1-\varepsilon)), & \text{for all } x\in\overline{\Omega}^-_{tr},\end{cases}$$

where $\Omega^\pm_{tr} := \{x \pm \mathbf{e}_3,\ x\in\Omega^\pm\}$, $\Omega^m := \omega\times]-1,1[$ and $S^\pm := \omega\times\{\pm1\}$. In order to simplify the notation, we identify $\Omega^\pm_{tr}$ with $\Omega^\pm$, and $\overline{\Omega}$ with $\overline{\Omega}^\pm\cup\overline{\Omega}^m$. Consequently, one has $\partial^\varepsilon_\alpha = \partial_\alpha$ and $\partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3$ in Ω^m.

For ε sufficiently small, we associate with functions $A_\pm^{ijk\ell,\varepsilon},\ g^{\pm,\varepsilon},\ \Gamma^{p,\varepsilon}_{ij} : \overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}$ the functions $A_\pm^{ijk\ell},\ g^\pm,\ \Gamma^p_{ij} : \overline{\Omega}^\pm \to \mathbb{R}$ defined by

$$\begin{aligned} &A_\pm^{ijk\ell}(x) := A_\pm^{ijk\ell,\varepsilon}(x^\varepsilon), && \text{for all } x^\varepsilon = \pi^\varepsilon(x)\in\overline{\Omega}^{\pm,\varepsilon},\\ &g^\pm(x) := g^{\pm,\varepsilon}(x^\varepsilon), && \text{for all } x^\varepsilon = \pi^\varepsilon(x)\in\overline{\Omega}^{\pm,\varepsilon},\\ &\Gamma^p_{ij}(x) := \Gamma^{p,\varepsilon}_{ij}(x^\varepsilon), && \text{for all } x^\varepsilon = \pi^\varepsilon(x)\in\overline{\Omega}^{\pm,\varepsilon},\end{aligned}$$

and we associate with functions $A_m^{ijk\ell,\varepsilon},\ g^{m,\varepsilon},\ \Gamma^{p,\varepsilon}_{ij} : \overline{\Omega}^{m,\varepsilon}\to\mathbb{R}$ the functions $A_m^{ijk\ell}(\varepsilon),\ g^m(\varepsilon),\ \Gamma^p_{ij}(\varepsilon) : \overline{\Omega}^m \to \mathbb{R}$ defined by

$$\begin{aligned} &A_m^{ijk\ell}(\varepsilon)(x) := A_m^{ijk\ell,\varepsilon}(x^\varepsilon), && \text{for all } x^\varepsilon = \pi^\varepsilon(x)\in\overline{\Omega}^{m,\varepsilon},\\ &g^m(\varepsilon)(x) := g^{m,\varepsilon}(x^\varepsilon), && \text{for all } x^\varepsilon = \pi^\varepsilon(x)\in\overline{\Omega}^{m,\varepsilon},\\ &\Gamma^p_{ij}(\varepsilon)(x) := \Gamma^{p,\varepsilon}_{ij}(x^\varepsilon), && \text{for all } x^\varepsilon = \pi^\varepsilon(x)\in\overline{\Omega}^{m,\varepsilon}.\end{aligned}$$

More precisely, one has

$$A_m^{ijk\ell}(\varepsilon) = \frac{1}{\varepsilon^p} A_m^{ijk\ell}(0) + O(\varepsilon^{1-p}) \text{ and } A_m^{\alpha\beta\sigma 3}(\varepsilon) = A_m^{\alpha 333}(\varepsilon) = 0 \text{ for } p \in \{1,3\},$$

where

$$A_m^{\alpha\beta\sigma\tau}(0) := \lambda^m a^{\alpha\beta} a^{\sigma\tau} + \mu^m (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$
$$A_m^{\alpha\beta 33}(0) := \lambda^m a^{\alpha\beta}, \;\; A_m^{\alpha 3\sigma 3}(0) := \mu^m a^{\alpha\sigma}, \;\; A_m^{3333}(0) := \lambda^m + 2\mu^m,$$
$$A_m^{\alpha\beta\sigma 3}(0) = A_m^{\alpha 333}(0) = 0.$$

We suppose that $L^\varepsilon(\mathbf{v}^\varepsilon) = L(\mathbf{v})$. Finally the covariant components of the linearized change of metric tensor $e_{ij}(\varepsilon;\mathbf{v}) \in L^2(\Omega^m)$, transformed by π^ε and associated with the displacement field $\mathbf{v} \in H^1(\Omega^m;\mathbb{R}^3)$, are defined as follows:

$$e_{\alpha\beta}(\varepsilon;\mathbf{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p,$$
$$e_{\alpha 3}(\varepsilon;\mathbf{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma,$$
$$e_{33}(\varepsilon;\mathbf{v}) := \frac{1}{\varepsilon}\partial_3 v_3.$$

According to the previous assumptions, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following re-scaled problem:

$$\begin{cases} \text{Find } \mathbf{u}(\varepsilon) \in V := \{\mathbf{v} \in H^1(\Omega;\mathbb{R}^3);\;\; \mathbf{v}_{|\Gamma_0} = \mathbf{0}\} \;\; \text{such that} \\ A^-(\mathbf{u}(\varepsilon),\mathbf{v}) + A^+(\mathbf{u}(\varepsilon),\mathbf{v}) + \varepsilon A^m(\mathbf{u}(\varepsilon),\mathbf{v}) = L(\mathbf{v}) \;\; \text{for all } \; \mathbf{v} \in V, \end{cases} \tag{2}$$

where

$$A^\pm(\mathbf{u}(\varepsilon),\mathbf{v}) := \int_{\Omega^\pm} A_\pm^{ijk\ell} e_{k\ell}(\mathbf{u}(\varepsilon)) e_{ij}(\mathbf{v}) \sqrt{g^\pm}\, dx,$$
$$A^m(\mathbf{u}(\varepsilon),\mathbf{v}) := \int_{\Omega^m} A_m^{ijk\ell}(\varepsilon) e_{k\ell}(\varepsilon;\mathbf{u}(\varepsilon)) e_{ij}(\varepsilon;\mathbf{v}) \sqrt{g^m(\varepsilon)}\, dx.$$

4 THE LIMIT PROBLEMS

We can now perform an asymptotic analysis of the re-scaled problem (2) and distinguish the two cases when the rigidity of the shell-like layer has its order of magnitude equal to $\frac{1}{\varepsilon}$ or $\frac{1}{\varepsilon^3}$ with respect to the rigidities of the surrounding three-dimensional bodies.

Since the re-scaled problem (2) has a polynomial structure with respect to the small parameter ε, we can look for the solution of the problem as a series of powers of ε:

$$\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + \dots. \tag{3}$$

Hence, by substituting (3) in (2) and by identifying the terms with identical power, we can finally characterize the limit problems for $p = 1$ and $p = 3$.

4.1 *The limit problem for $p = 1$: the membrane transmission condition*

The limit problem when the rigidity of the shell is $\frac{1}{\varepsilon}$ can be written as follows:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in \widehat{V}_M \text{ such that} \\ A^-(\mathbf{u}^0, \mathbf{v}) + A^+(\mathbf{u}^0, \mathbf{v}) + A^m_M(\mathbf{u}^0, \mathbf{v}) = L(\mathbf{v}) \text{ for all } \mathbf{v} \in \widehat{V}_M, \end{cases} \tag{4}$$

with

$$\widehat{V}_M := \{\mathbf{v} \in L^2(\Omega; \mathbb{R}^3);\, \mathbf{v}^\pm \in H^1(\Omega^\pm; \mathbb{R}^3),\ \gamma_{\alpha\beta}(\mathbf{v}^m) \in L^2(\Omega^m),$$
$$\mathbf{v}^\pm_{|S^\pm} = \mathbf{v}^m_{|S^\pm}, L^2(\Omega^m; \mathbb{R}^3) \ni \partial_3 \mathbf{v}^m = 0,\ \mathbf{v}_{|\Gamma_0} = \mathbf{0}\},$$

where $\mathbf{v}^\pm$ (resp. $\mathbf{v}^m$) denotes the restriction of $\mathbf{v}$ to $\Omega^\pm$ (resp. Ω^m) and

$$A^m_M(\mathbf{u}^0, \mathbf{v}) := \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\mathbf{u}^0) \gamma_{\alpha\beta}(\mathbf{v}) \sqrt{a}\, d\widetilde{x},$$
$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda^m \mu^m}{\lambda^m + 2\mu^m} a^{\alpha\beta} a^{\sigma\tau} + 2\mu^m (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

are respectively the bilinear form associated with the membrane behavior of the shell and the contravariant components of the elasticity tensor of the shell, and $a := \det(a_{\alpha\beta})$. We can notice that the space $\widehat{V}_M$ is isomorphic to $V_M := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3); v_{\alpha|\omega} \in H^1(\omega),\ \mathbf{v}_{|\Gamma_0} = \mathbf{0}\}$ which corresponds to the well-known inhibited pure bending space in the classical theory of shells.

Remark 1. In the limit problem for $p = 1$ the behavior of the shell is membrane dominated, thus the pure bending is inhibited. In the simplified model we obtain a membrane transmission condition at the interface between the two three-dimensional bodies. This condition can be interpreted as a curvilinear generalization of the Ventcel-type transmission condition obtained in [5]. Indeed, one has

Elasticity problems in $\Omega^\pm$	**Transmission conditions in ω**
$\begin{cases} -\sigma^{ij}_\pm\|_j = f^i & \text{in } \Omega^\pm, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_0, \end{cases}$	$\begin{cases} [\![\sigma^{\alpha 3}]\!] = n^{\alpha\beta}\|_\beta & \text{in } \omega, \\ [\![\sigma^{33}]\!] = n^{\alpha\beta} b_{\alpha\beta} & \text{in } \omega, \\ \mathbf{u}_{\|\omega} = \boldsymbol{\eta} & \text{in } \omega, \end{cases}$

where $\sigma^{ij}_\pm := A^{ijk\ell}_\pm e_{k\ell}(\mathbf{u})$ and $n^{\alpha\beta} := a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})$ are respectively the contravariant components of the first Piola-Kirchhoff stress tensor and of the membrane stress tensor of the shell, $[\![\sigma^{i3}]\!] := \sigma^{i3}_+ - \sigma^{i3}_-$ represents the stress jump at the interface ω between Ω^+ and Ω^-.

4.2 *The limit problem for $p = 3$: the flexural transmission condition*

The limit problem when the rigidity of the shell is $\frac{1}{\varepsilon^3}$ can be written as follows:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in \widehat{V}_F \text{ such that} \\ A^-(\mathbf{u}^0, \mathbf{v}) + A^+(\mathbf{u}^0, \mathbf{v}) + A^m_F(\mathbf{u}^0, \mathbf{v}) = L(\mathbf{v}) \text{ for all } \mathbf{v} \in \widehat{V}_F, \end{cases} \tag{5}$$

where

$$\widehat{V}_F := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3);\ v^m_3 \in H^2(\Omega^m),\ \gamma_{\alpha\beta}(\mathbf{v}^m) = 0,\ \partial_3 \mathbf{v}^m = 0,\ \mathbf{v}_{|\Gamma_0} = \mathbf{0}\},$$

$$A^m_F(\mathbf{u}^0, \mathbf{v}) := \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\mathbf{u}^0) \rho_{\alpha\beta}(\mathbf{v}) \sqrt{a}\, d\widetilde{x},$$

is the bilinear form associated with the flexural behavior of the shell. We can notice that the space $\widehat{V}_F$ is isomorphic to $V_F := \{\mathbf{v} \in H^1(\Omega;\mathbb{R}^3);\ \boldsymbol{\eta} := \mathbf{v}_{|\omega} \in H^1(\omega;\mathbb{R}^2) \times H^2(\omega),\ \mathbf{v}_{|\Gamma_0} = \mathbf{0},\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$, which is equivalent to the non-inhibited pure bending space in theory of shells.

Remark 2. When $p = 3$, the pure bending of the shell is not inhibited. We can derive from the limit problem a flexural transmission condition between the two three-dimensional bodies as follows:

Elasticity problems in $\Omega^\pm$ **Transmission conditions in ω**

$$\left\{\begin{array}{ll} -\sigma_\pm^{ij}\|_j = f^i & \text{in } \Omega^\pm, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_0, \end{array}\right. \qquad \left\{\begin{array}{ll} [\![\sigma^{\alpha 3}]\!] = (b_\sigma^\alpha m^{\sigma\beta})|_\beta + b_\sigma^\alpha (m^{\sigma\beta}|_\beta) & \text{in } \omega, \\ [\![\sigma^{33}]\!] = b_\alpha^\sigma b_{\sigma\beta} m^{\alpha\beta} - m^{\alpha\beta}|_{\alpha\beta} & \text{in } \omega, \\ \mathbf{u}_{|\omega} = \boldsymbol{\eta} & \text{in } \omega, \end{array}\right.$$

where $m^{\alpha\beta} := \frac{1}{3} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta})$ are the contravariant components of the moment tensor of the shell.

5 CONVERGENCE RESULTS

Let us define the space $\overline{V}$:

$$\overline{V} := \{\mathbf{v} \in L^2(\Omega;\mathbb{R}^3);\ \mathbf{v}^\pm \in H^1(\Omega^\pm;\mathbb{R}^3), \gamma_{\alpha\beta}(\mathbf{v}^m) \in L^2(\Omega^m), \\ \partial_3 v_i \in L^2(\omega),\ \mathbf{v}^\pm_{|S^\pm} = \mathbf{v}^m_{|S^\pm},\ ,\ \mathbf{v}_{|\Gamma_0} = \mathbf{0}\},$$

equipped with $\|\mathbf{v}\|_{\overline{V}} := \{\|e_{ij}(\mathbf{v})\|^2_{0,\Omega^+} + \|e_{ij}(\mathbf{v})\|^2_{0,\Omega^-} + \|\mathbf{v}\|^2_{0,\Omega^m} + \|\gamma_{\alpha\beta}(\mathbf{v})\|^2_{0,\Omega^m} + \|\partial_3 \mathbf{v}\|^2_{0,\Omega^m}\}^{1/2}$. We can prove easily that $\overline{V}$ is complete, due to the presence of Ω^+ and Ω^-. In general we work in a space which is not complete, see [4]: hence, we need to consider the abstract completion of this space with respect to a certain norm, whose limit is not always characterizable. It is necessary to make some regularity assumptions on the completion space on which the admissible displacements for the shell are defined. The space $\widehat{V}_M$, equipped with this norm, is complete, hence the uniqueness of the solution for problem (4) is guaranteed.

The following inequality of Korn's type holds for all $\mathbf{v} \in \overline{V}$:

$$\|\mathbf{v}\|_{\overline{V}} \le C \left(\|e_{ij}(\mathbf{v})\|^2_{0,\Omega^+} + \|e_{ij}(\mathbf{v})\|^2_{0,\Omega^-} + \|e_{ij}(\varepsilon;\mathbf{v})\|^2_{0,\Omega^m} \right)^{1/2}$$

5.1 *Weak and strong convergence for $p = 1$*

For all functions $\mathbf{v}$ defined almost everywhere over $\Omega^m = \omega\times]-1,1[$, we define the average $\overline{\mathbf{v}}(\widetilde{x}) := \frac{1}{2}\int_{-1}^{1} \mathbf{v}(\widetilde{x}, x_3)dx_3$ for all $\widetilde{x} \in \omega$.

Let $\mathbf{u}(\varepsilon)$ be the solution of (2) for $p = 1$. After the assumptions on the loading and the coercivity of the bilinear forms $A^\pm$ and A^m, the following *a priori* estimates hold:

$$\begin{array}{l} \|\mathbf{u}(\varepsilon)\|_{\overline{V}} \le C\ , \\ \|e_{ij}(\varepsilon;\mathbf{u}(\varepsilon))\|_{0,\Omega^m} \le C\ , \\ \|\varepsilon\mathbf{u}(\varepsilon)\|_{1,\Omega^m} \le C. \end{array} \tag{6}$$

Theorem 1 *The sequence $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ converges strongly in $\overline{V}$ to $\mathbf{u}^0 \in \widehat{V}_M$, the unique solution of problem (4).*

Proof. For convenience, the proof is divided into five parts, numbered from (i) to (v).

(i) From (6) we deduce that there exists a subsequence (not relabeled) and $\mathbf{u} \in V$, $\mathbf{u}^{-1} \in \overline{V}$, $e_{ij} \in L^2(\Omega^m)$ such that

$$\begin{array}{ll} \mathbf{u}(\varepsilon) \rightharpoonup \mathbf{u} & \text{in } \overline{V}, \\ \varepsilon\mathbf{u}(\varepsilon) := \mathbf{u}^{-1}(\varepsilon) \rightharpoonup \mathbf{u}^{-1} & \text{in } H^1(\Omega;\mathbb{R}^3), \\ e_{ij}(\varepsilon;\mathbf{u}(\varepsilon)) \rightharpoonup e_{ij} & \text{in } L^2(\Omega^m), \\ \partial_3 u_3(\varepsilon) = \varepsilon e_{33}(\varepsilon;\mathbf{u}(\varepsilon)) \to 0 & \text{in } L^2(\Omega^m), \end{array} \tag{7}$$

We can easily prove that $\mathbf{u}^{-1} = \mathbf{0}$.

(ii) We show that $\partial_3\mathbf{u} = \mathbf{0}$ in Ω^m. From (6) and (7), one has

$$\begin{array}{ll} \partial_3 u_3(\varepsilon) = \varepsilon e_{33}(\varepsilon;\mathbf{u}(\varepsilon)) \to 0 & \text{in } \ L^2(\Omega^m), \\ \partial_3 u_\alpha(\varepsilon) = 2\varepsilon e_{\alpha 3}(\varepsilon;\mathbf{u}(\varepsilon)) - \partial_\alpha(\varepsilon u_3(\varepsilon)) + 2\Gamma^\sigma_{\alpha 3}(\varepsilon)\varepsilon u_\sigma(\varepsilon) \rightharpoonup 0 & \text{in } L^2(\Omega^m). \end{array}$$

Thus, thanks to the definition of convergence in $\overline{V}$, $\partial_3\mathbf{u}(\varepsilon) \rightharpoonup \partial_3\mathbf{u} = \mathbf{0}$ in $L^2(\Omega^m)$ and $\mathbf{u} \in \widehat{V}_M$.

(iii) The limits $e_{\alpha\beta}$ satisfy the relation $e_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u})$. Using the definition of the average and after some technicalities, we deduce that

$$\|\overline{e_{\alpha\beta}(\varepsilon;\mathbf{u}(\varepsilon))} - \gamma_{\alpha\beta}(\overline{\mathbf{u}(\varepsilon)})\|_{0,\omega} \leq C\varepsilon\Big(\|\varepsilon u_\alpha(\varepsilon)\|_{0,\Omega^m} + \|\partial_3 u_i(\varepsilon)\|_{0,\Omega^m}\Big),$$

which tends to zero as $\varepsilon \to 0$. By definition of the norm $\|\cdot\|_{\overline{V}}$, $\gamma_{\alpha\beta}(\mathbf{u}(\varepsilon)) \rightharpoonup \gamma_{\alpha\beta}(\mathbf{u})$ in $L^2(\Omega^m)$, which implies that $\gamma_{\alpha\beta}(\overline{\mathbf{u}(\varepsilon)}) \rightharpoonup \gamma_{\alpha\beta}(\overline{\mathbf{u}}) = \gamma_{\alpha\beta}(\mathbf{u})$ in $L^2(\omega)$ and, hence, $\overline{e_{\alpha\beta}} = \gamma_{\alpha\beta}(\mathbf{u})$ in $L^2(\omega)$.

According to theorem 5.2.1 in [6],

$$\|\partial_3 e_{\alpha\beta}(\varepsilon;\mathbf{u}(\varepsilon)) + \varepsilon\rho_{\alpha\beta}(\mathbf{u}(\varepsilon))\|_{-1,\Omega^m} \leq C\varepsilon\Big(\|e_{i3}(\varepsilon;\mathbf{u}(\varepsilon))\|_{0,\Omega^m} + \|\varepsilon u_\alpha(\varepsilon)\|_{0,\Omega^m} + \|\varepsilon u_3(\varepsilon)\|_{1,\Omega^m}\Big).$$

It is evident that the second member tends to zero as $\varepsilon \to 0$. Thanks to the continuity of the operator $\partial_3 : L^2(\Omega^m) \to H^{-1}\Omega)$, and since $\varepsilon\rho_{\alpha\beta}(\mathbf{u}(\varepsilon)) \rightharpoonup 0$ in $H^{-1}(\Omega^m)$, we derive that $\partial_3 e_{\alpha\beta} = 0$ and finally

$$e_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}) \ \text{ in } \ L^2(\omega). \tag{8}$$

(iv) Multiplying problem (2) by ε and letting $\varepsilon \to 0$ yield to the relations

$$e_{\alpha 3} = 0 \ \text{ and } \ e_{33} = -\frac{\lambda^m}{\lambda^m + 2\mu^m} a^{\alpha\beta} e_{\alpha\beta}. \tag{9}$$

By choosing in (2) test functions $\mathbf{v}$ independent of x_3 in Ω^m and by applying the limit as $\varepsilon \to 0$, we obtain:

$$A^+(\mathbf{u},\mathbf{v}) + A^-(\mathbf{u},\mathbf{v}) + \int_{\Omega^m} \left(A^{\alpha\beta\sigma\tau}(0)e_{\alpha\beta}\gamma_{\sigma\tau}(\mathbf{v}) + A^{\sigma\tau 33}(0)e_{33}\gamma_{\sigma\tau}(\mathbf{v})\right)\,dx = L(\mathbf{v}).$$

From (8) and (9), we infer that $A^+(\mathbf{u},\mathbf{v}) + A^-(\mathbf{u},\mathbf{v}) + 2\displaystyle\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\mathbf{u})\gamma_{\sigma\tau}(\mathbf{v})\sqrt{a}\,d\widetilde{x} = L(\mathbf{v})$. Hence, by virtue of the uniqueness of the solution, we deduce that $\mathbf{u} = \mathbf{u}^0$.

(v) It remains to show the strong convergence. Let $(\mathbf{w}^\eta)_{\eta>0}$ be a sequence in $\mathcal{D}(\omega,\mathbb{R}^3)$ such that

$$\left\{\begin{array}{ll} w^\eta_\alpha \to -\partial_\alpha u^0_3 - 2b^\tau_\alpha u^0_\tau & \text{in } \ L^2(\omega), \\ w^\eta_3 \to e_{33} & \text{in } \ L^2(\omega), \end{array}\right.$$

and let $(\boldsymbol{\phi}^\eta)_{\eta>0}$ be a sequence in $\mathcal{D}(\Omega,\mathbb{R}^3)$ such that $\boldsymbol{\phi}^\eta(x) := x_3\mathbf{w}^\eta(\widetilde{x})$ pour tout $x \in \Omega^m$. Then $\mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\phi}^\eta \in V$. Setting $A(\cdot,\cdot) := A^+(\cdot,\cdot) + A^-(\cdot,\cdot) + \varepsilon A^m(\cdot,\cdot)$, by virtue of the coercivity, we obtain: $A(\mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\phi}^\eta, \mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\phi}^\eta) \geq \|\mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\phi}^\eta\|^2_{\overline{V}}$. By letting $\varepsilon \to 0$ and by using a standard diagonalization argument, one has

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0} A(\mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\phi}^\eta, \mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\phi}^\eta) =$$
$$= L(\mathbf{u}^0) - A^+(\mathbf{u}^0,\mathbf{u}^0) - A^-(\mathbf{u}^0,\mathbf{u}^0) - \int_{\Omega^m} a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\mathbf{u}^0)\gamma_{\sigma\tau}(\mathbf{u}^0)\,dx = 0.$$

Hence the announced strong convergence holds.

5.2 *Weak and strong convergence for $p = 3$*

Let $\mathbf{u}(\varepsilon)$ be the solution of (2) for $p = 3$, then we can write the following *a priori* estimates:

$$\begin{aligned} &\|e_{ij}(\mathbf{u}(\varepsilon))\|^2_{0,\Omega^+} + \|e_{ij}(\mathbf{u}(\varepsilon))\|^2_{0,\Omega^-} + \frac{1}{\varepsilon^2}\|e_{ij}(\varepsilon;\mathbf{u}(\varepsilon)\|^2_{0,\Omega^m} \leq C,\\ &\frac{1}{\varepsilon^2}\|e_{ij}(\varepsilon;\mathbf{u}(\varepsilon))\|^2_{0,\Omega^m} \leq C. \end{aligned} \tag{10}$$

Theorem 2 *The sequence $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ converges strongly in $H^1(\Omega;\mathbb{R}^3)$ to $\mathbf{u}^0 \in \widehat{V}_F$, the unique solution of problem (5).*

Proof. For the sake of clarity, the proof is divided into four parts numbered from (i) to (iv).

(i) The *a priori* bound (10_1) and the Korn's inequality establish that there exist $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ and a subsequence not relabeled such that $\mathbf{u}(\varepsilon) \rightharpoonup \mathbf{u}$ in $H^1(\Omega;\mathbb{R}^3)$. Estimate (10_2) implies that $\partial_3 u_3(\varepsilon) \to 0$ in $L^2(\Omega^m)$, thus $\partial_3 u_3 = 0$, and that $e_{\alpha 3}(\varepsilon;\mathbf{u}(\varepsilon)) \to 0$ in $L^2(\Omega^m)$, hence $\partial_3 u_\alpha(\varepsilon) = \varepsilon\big(2e_{\alpha 3}(\varepsilon;\mathbf{u}(\varepsilon)) - \partial_\alpha u_3(\varepsilon) + 2\Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)\big) \to 0$ in $L^2(\Omega^m)$, so that $\partial_3 u_\alpha = 0$. Finally, one has $\gamma_{\alpha\beta}(\mathbf{u}) = 0$ so that $\mathbf{u} \in \{\mathbf{v} \in H^1(\Omega,\mathbb{R}^3) : \partial_3 v_i = \gamma_{\alpha\beta}(\mathbf{v}) = 0 \text{ in } \Omega^m,\ \mathbf{v}_{|\Gamma_0} = \mathbf{0}\}$. Besides, estimate ($10_2$) also implies the existence of $z_{ij} \in L^2(\Omega^m)$ such that $\frac{1}{\varepsilon}e_{ij}(\varepsilon;\mathbf{u}(\varepsilon)) \rightharpoonup z_{ij}$ in $L^2(\Omega^m)$.

(ii) By multiplying (2) by ε^2 and by letting $\varepsilon \to 0$, we deduce that

$$z_{33} = -\frac{\lambda^m}{\lambda^m + 2\mu^m}a^{\alpha\beta}z_{\alpha\beta} \ \text{ and } \ z_{\alpha 3} = 0.$$

Afterwards, by multiplying (2) by ε, by letting $\varepsilon \to 0$ and by choosing test functions such that $\partial_3 v_3 = 0$, we obtain:

$$\lim_{\varepsilon\to 0}\int_{\Omega^m}\frac{1}{\varepsilon}A^{\alpha 3\sigma 3}(0)e_{\alpha 3}(\varepsilon;\mathbf{u}(\varepsilon))e_{\sigma 3}(\varepsilon;\mathbf{v})\,dx = -\int_{\Omega^m} a^{\alpha\beta\sigma\tau}z_{\alpha\beta}\gamma_{\sigma\tau}(\mathbf{v})\,dx. \tag{11}$$

According to theorem 5.2.1 in [6], we can prove that $\rho_{\alpha\beta}(\mathbf{u}) = -\partial_3 z_{\alpha\beta}$ in $L^2(\Omega^m)$. By choosing test functions such that $v_\alpha = \eta_\alpha - x_3\theta_\alpha$ and $v_3 = \eta_3$, where $\theta_\alpha := \partial_\alpha\eta_3 + 2b^\sigma_\alpha\eta_\sigma$, with $\boldsymbol{\eta} \in H^1(\omega,\mathbb{R}^2) \times H^2(\omega)$ such that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \partial_3\eta_i = 0$, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega^m} A^{\alpha 3\sigma 3}(0)e_{\sigma 3}(\varepsilon,\mathbf{u}(\varepsilon))\frac{1}{2}\theta_\alpha\,dx = \frac{2}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\mathbf{u})\widetilde{\gamma}_{\sigma\tau}\,d\widetilde{x}. \tag{12}$$

where $\widetilde{\gamma}_{\sigma\tau} := \frac{1}{2}(\partial_\sigma\theta_\tau + \partial_\tau\theta_\sigma) - \Gamma^\mu_{\sigma\tau}\theta_\mu$.

(iii) If we choose in (2) $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$ such that $v_3^m \in H^2(\Omega^m)$ and $\partial_3 v_i = \gamma_{\alpha\beta}(\mathbf{v}) = 0$ in Ω^m, passing to limit we get:

$$A^+(\mathbf{u},\mathbf{v}) + A^-(\mathbf{u},\mathbf{v}) + \int_{\Omega^m} a^{\alpha\beta\sigma\tau} z_{\alpha\beta} x_3 \big(b^\mu_\tau|_\sigma v_\mu + b^\mu_\sigma b_{\mu\tau} v_3\big)\, dx \\ + \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{\Omega^m} A^{\alpha 3\sigma 3}(0) e_{\sigma 3}(\varepsilon; \mathbf{u}(\varepsilon)) \big(\frac{1}{2}\partial_\alpha v_3 + b^\tau_\alpha v_\tau\big)\, dx = L(\mathbf{v}).$$

Let $\frac{1}{2}\theta_\alpha := \frac{1}{2}\partial_\alpha v_3 + b^\tau_\alpha v_\tau$. Then, using (12) and the relation $\widetilde{\gamma}_{\sigma\tau} - b^\mu_\tau|_\sigma v_\mu - b^\mu_\sigma b_{\mu\tau} v_3 = \rho_{\sigma\tau}(\mathbf{v})$, we obtain that

$$A^+(\mathbf{u},\mathbf{v}) + A^-(\mathbf{u},\mathbf{v}) + \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\mathbf{u}) \rho_{\sigma\tau}(\mathbf{v})\, d\widetilde{x} = L(\mathbf{v}),$$

thus, $\mathbf{u} = \mathbf{u}^0$, the one and only one solution of problem (5).

(iv) It remains to prove the strong convergence. Let $(\boldsymbol{\phi}^\eta)_{\eta>0} \subset V$ such that $\phi^\eta_\alpha(x) = x_3^2(b^\sigma_\alpha \partial_\sigma u^0_3(\widetilde{x}) + b^\sigma_\alpha b^\tau_\sigma u^0_\tau(\widetilde{x}))$ and $\phi^\eta_3(x) = \frac{x_3^2}{2} w^\eta(\widetilde{x})$ for all $x \in \Omega^m$, where $(w^\eta)_{\eta>0}$ is a sequence in $\mathcal{D}(\omega)$ which satisfies $w^\eta \to \partial_3 z_{33}$ in $L^2(\omega)$. Let $\boldsymbol{\psi} \in V$ such that $\psi_\alpha(x) = -x_3 \partial_\alpha u^0_3(\widetilde{x})$ and $\psi_3(x) = 0$ for all $x \in \Omega^m$. Then $\mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\psi} - \varepsilon^2\boldsymbol{\phi}^\eta \in V$ and by applying the previous reasoning, we show that

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0} A(\mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\psi} - \varepsilon^2\boldsymbol{\phi}^\eta, \mathbf{u}(\varepsilon) - \mathbf{u}^0 - \varepsilon\boldsymbol{\psi} - \varepsilon^2\boldsymbol{\phi}^\eta) = \\ = L(\mathbf{u}^0) - A^+(\mathbf{u}^0,\mathbf{u}^0) - A^-(\mathbf{u}^0,\mathbf{u}^0) - \int_{\Omega^m} x_3^2 a^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\mathbf{u}^0) \rho_{\sigma\tau}(\mathbf{u}^0)\, dx = 0,$$

which completes the proof.

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