A linearized biphasic poroelastic model and its calibration by experimental measures

Roberto Serpieri[1], Luciano Rosati[2]
[1]*Dipartimento di Ingegneria, Università degli Studi del Sannio, Benevento, Italy*
E-mail: roberto.serpieri@unisannio.it

[2]*Dipartimento di Ingegneria Strutturale, Università degli Studi di Napoli, Italy*
E-mail: rosati@unina.it

Keywords: biphasic media, finite deformations, poroelasticity.

ABSTRACT. We specialize to the case of infinitesimal perturbations a finite deformation model for the dynamic behaviour of fluid saturated porous biphasic media. The original model exploits eight kinematic scalar fields: the displacement field of the solid phase, the velocity and the density of the fluid, and an additional scalar field associated with the microscopic volumetric deformation of the solid phase. We also discuss a procedure for the determination of the elastic parameters that characterize the present linearized model; it is based on the set of experimental measures originally considered by Biot and Willis [1] for calibrating Biot's seminal model. In particular, we select from this set the shear test, the unjacketed compressibility test and two jacketed tests in which drainage is either completely allowed or prevented.

1 INTRODUCTION

Gas filled open cell flexible polymeric foams are widely employed for absorption of impact energy due to their excellent dissipation properties which originate from several mechanisms, involving, among others, the deformation driven viscous flow of gas through interconnected pores which may be expelled or drawn into the solid [2]. These materials can undergo large deformations and find applications in cushioning systems to minimize injuries of passengers during impact; actually, by properly selecting the polymer, density and cell morphology, the microstructure can be tailored so as to achieve desired stiffness and damping properties [3].

The development of mechanical models for the prediction of the mechanical response of this class of materials requires a biphasic formulation able to describe pore-gas interaction at finite deformations [4, 5]. This family of mechanical problems is also of primary interest in several other fields of engineering and physics science; for instance, in biomechanics, multiphase formulations have been used for simulating the deformation of soft biological tissues, such as heart muscle and cartilage [6] and to account for mass transport through tissues associated with cell nutrition. Biphasic formulations in the finite deformation regime enter also in the analysis of the loss of strength of saturated soils subjected to seismic excitations, referred to in geotechnics as liquefaction [7].

A biphasic formulation in the finite deformations regime has been developed by the authors [8], [9], with the main objective of predicting the response of polyurethanic open cell foams by means of finite element techniques. Such formulation exploits eight kinematic scalar fields: the displacement field of the solid phase, the velocity and the density of the fluid, and an additional scalar field associated with the microscopic volumetric deformation of the solid phase. The proposed formulation is also suitable for describing the response of biphasic media for which the ratio between volumetric unjacketed and jacketed compressibility is different from zero.

Besides being totally formulated in terms of work-conjugate variables, the model is sufficiently general to account for the volumetric compressibility of the bulk material constituting the solid phase, so that the elastic response of the medium to isotropic infinitesimal loading conditions depends upon four elastic coefficients as in the original work by Biot. Moreover, the formulation is completely biphasic, i.e. the independent constitutive behavior of each phase is separately introduced thus allowing one to separately identify the response of the isolated phases and to combine them to obtain the total response of the medium. This property is made possible since the equations of motions are written separately for the solid and the fluid phases and no use is made of the Cauchy total stress tensor [10] . Furthermore, no restriction is imposed on the compressibility of the fluid and of the solid skeleton.

In the present contribution, we focus on the specialization of the above mentioned formulation to infinitesimal perturbations and report some results concerning the linearized model. We first illustrate the set of linear PDEs that arises from the original system of governing equations when this is subjected to infinitesimal perturbations of the state fields starting from a base configuration which, in general, may also differ from the reference one. The interest in the linearized system is twofold since, on one hand, a linearized system turns out to be a convenient simplification of the most general problem when the objective of the analysis is a medium actually experimenting infinitesimal strains due to the small entity of the perturbations compared with the stiffness of the medium. This class of problems has been the subject of fundamental researchs in the past [11] and is encountered, for instance, when studying the propagation of elastic waves in a shock tube filled with sand particles [12]. On the other hand, the linearized system is also required in step-wise methods for the incremental solution of geometrically nonlinear media in the context of finite element methods, e.g. following the strategy illustrated in [7] based on a hypoelastic formulations.

Owing to space reasons we report here only the set of equations resulting fom the linearization around the reference configuration under the additional hypotheses that typically hold in static laboratory tests of initially motionless, stress-free, isotropic and homogeneous medium. The reader is referred to [9] for the most general linearized set of equations and further details on their derivation.

A procedure is also presented for the determination of the elastic parameters that characterize the present linearized model; it is based on the set of experimental measures originally considered by Biot and Willis [1] for calibrating Biot's seminal model. In particular, we select from this set the shear test, the unjacketed compressibility test and two jacketed tests in which drainage is either completely allowed or prevented.

2 SUMMARY OF THE FINITE DEFORMATION MODEL

The model presented in [8] is based on a macroscopic continuum formulation which accounts for the presence of a heterogeneous structure at the microscale. The state of the solid phase is described by the relevant average macroscopic configuration change $\bar{\boldsymbol{\phi}}^{(s)}$ and by an additional macroscopic field $\hat{J}^{(s)}$ defined as the Jacobian of the transformation of the solid phase averaged over the part of the RVE occupied by the solid phase [9]. A local hyperelastic isotropic behaviour, with uncoupled spheric and deviatoric constitutive responses, is assumed for the solid phase according to the following strain energy density function:

$$\bar{\psi}^{(s)} = \bar{\psi}^{(s)}_{dev}\left(\bar{\mathbf{F}}_{dev}\right) + \bar{\psi}^{(s)}_{sph}\left(\bar{J},\, \hat{J}^{(s)}\right) \tag{1}$$

where $\bar{\mathbf{F}}_{dev} = \bar{J}^{-\frac{1}{3}}\bar{\mathbf{F}}$ is the deviatoric part of the macroscopic deformation gradient $\bar{\mathbf{F}}$ and $\bar{J} = \det \bar{\mathbf{F}}$ is the macroscopic Jacobian.

It is further assumed the existence of a kinetic energy density field $\bar{\chi}^{(s)}$ which, apart from the macroscopic velocity $\bar{\mathbf{v}}^{(s)}$ of the RVE, also depends on the time rate $\dot{\hat{J}}^{(s)}$

$$\bar{\chi}^{(s)} = \frac{1}{2}\bar{\rho}^{(s)}||\bar{\mathbf{v}}^{(s)}||^2 + \bar{\chi}_{add}^{(s)}(\dot{\hat{J}}^{(s)}) \tag{2}$$

where $\bar{\rho}^{(s)}$ is the macroscopic mass density of the solid in the current configuration and $\bar{\chi}_{add}^{(s)}$ is an additional kinetic energy density, associated with $\hat{J}^{(s)}$, which accounts for the microscopic fluctuations of the velocity field inside the RVE.

The equations governing the motion of the biphasic media have been presented in [8] and the reader may refer to [9] for a thorough discussion on these results and on the relevant derivation.

$$n^{(f)} = 1 - \hat{J}^{(s)} n_0^{(s)} \frac{1}{\bar{J}} \tag{3}$$

$$\left(\bar{J}\bar{\rho}^{(f)}\right)^{\cdot(s)} + \frac{\partial}{\partial X_j}\left[\bar{J}\frac{\partial X_j}{\partial x_i}\bar{\rho}^{(f)}(\hat{v}_i^{(f)} - \bar{v}_i^{(s)})\right] = 0 \tag{4}$$

$$\begin{aligned}\left[\bar{J}\bar{\rho}^{(f)}\left(\hat{v}_i^{(f)} - \bar{v}_i^{(s)}\right)\right]^{\cdot(s)} + \frac{\partial}{\partial X_K}\left[\bar{\Pi}_{iK} + \bar{J}\frac{\partial X_K}{\partial x_j}\bar{\rho}^{(f)}\left(\hat{v}_i^{(f)} - \bar{v}_i^{(s)}\right)\left(\hat{v}_j^{(f)} - \bar{v}_j^{(s)}\right)\right] = \\ = \bar{J}\bar{b}_i^{(fs\mathbf{n})} + \bar{J}\bar{b}_i'^{(fs)} + \bar{J}\bar{b}_i^{(f)(ext)} - \left(\bar{J}\bar{\rho}^{(f)}\right)\dot{\bar{v}}_i^{(s)} - \left[\bar{J}\bar{\rho}^{(f)}\left(\hat{v}_j^{(f)} - \bar{v}_j^{(s)}\right)\right]\frac{\partial \bar{v}_i^{(s)}}{\partial x_j}\end{aligned} \tag{5}$$

$$\bar{\rho}_0^{(s)}\ddot{\bar{u}}_i^{(s)} = \frac{\partial \hat{P}_{iJ}^{(s)}}{\partial X_J} + \bar{J}\bar{b}_i^{(sf\mathbf{n})} + \bar{J}\bar{b}_i'^{(sf)} + \bar{J}\bar{b}_i^{(s)(ext)} \tag{6}$$

$$\left(\bar{J}\bar{q}_{add.}^{(s)}\right)^{\cdot} + S_{\hat{J}^{(s)}} = 0 \tag{7}$$

Equation (3) expresses the condition of complete saturation while equations (4)-(5) express, respectively, the local form of the macroscopic mass balance and momentum balance for the fluid phase with respect to the reference configuration of the solid phase. The last two equations (6) and (7) express the local form of the Euler-Lagrange momentum balance for the solid phase stemming from the satisfaction of Hamilton's principle respectively with respect to infinitesimal variations of of the macroscopic displacement field and to infinitesimal variations of the field $\hat{J}^{(s)}$. The symbols in this set of equations have following meaning: $\mathbf{x} = \bar{\boldsymbol{\phi}}^{(s)}(\mathbf{X})$ indicates the vector defining the current position of point $\mathbf{X}$ while $\bar{\mathbf{v}}^{(s)} = \dot{\bar{\phi}}^{(s)}$ denotes the macroscopic velocity. $n^{(f)}$ and $n_0^{(s)}$ respectively denote the current void volume fraction and the volume fraction of the solid phase in the reference configuration. The field $\bar{\rho}^{(f)}$ is the macroscopic density of the fluid phase obtained by averaging over the entire volume of the RVE while $\hat{\mathbf{v}}^{(f)}$ indicated the macroscopic velocity of the fluid obtained by averaging over the void parts of the RVE. The notation $\overset{\cdot(s)}{(\bullet)}$ denotes the *solid time derivative* [9] and represents, from the physical point of view, the rate measured by an observer attached to the solid phase. The term $\bar{\mathbf{\Pi}}$ in (5) denotes the pull-back of the tensor $n^{(f)}\hat{p}\delta_{ij}$,

$$\bar{\Pi}_{iJ} = \bar{J}\frac{\partial X_J}{\partial x_i} n^{(f)}\hat{p} \tag{8}$$

where $\hat{p}$ is the interstitial pressure of the fluid.

The body force terms appearing on the right hand side of (5)$_3$ have different physical origins. Actually, the vectors $\bar{\mathbf{b}}^{(f)(ext)}$ and $\bar{\mathbf{b}}^{(s)(ext)}$ respectively collect all body forces per unit total RVE current volume which are exerted by the external environment on the fluid and on the solid, while $\bar{\mathbf{b}}^{(fs\mathbf{n})}$ and $\bar{\mathbf{b}}'^{(fs)}$ are internal interaction body forces between fluid and solid, so that by virtue of the action-reaction principle one has $\bar{\mathbf{b}}^{(fs\mathbf{n})} = -\bar{\mathbf{b}}^{(sf\mathbf{n})}$ and $\bar{\mathbf{b}}'^{(fs)} = -\bar{\mathbf{b}}'^{(sf)}$. In particular, $\bar{\mathbf{b}}'^{(fs)}$ collects all drag body force terms per unit total current volume originated by the fluid-solid relative motion, while $\bar{\mathbf{b}}^{(fs\mathbf{n})} = \hat{p}\frac{\partial n^{(f)}}{\partial \mathbf{x}}$ is an interaction body force which fluid and solid reciprocally exchange also in absence of a relative motion and originates from the porosity gradient. The remaining symbols are $\hat{\mathbf{P}}^{(s)}$, which is the two-point stress tensor work conjugated to $\bar{\mathbf{F}}$, $S_{\hat{J}^{(s)}}$ and $\bar{q}^{(s)}_{add.}$ which are respectively the stress and the linear momentum term associated with $\hat{J}^{(s)}$

$$\hat{P}^{(s)}_{iJ} = \frac{\partial\left(\bar{J}\bar{\psi}^{(s)}\right)}{\partial \bar{F}_{iJ}}, \qquad S_{\hat{J}^{(s)}} = \frac{\partial \bar{J}\bar{\psi}^{(s)}}{\partial \hat{J}^{(s)}}, \qquad \bar{q}^{(s)}_{add.} = \frac{\partial \bar{\chi}^{(s)}_{add.}}{\partial \dot{\hat{J}}^{(s)}} \tag{9}$$

3 THE LINEARIZED MODEL

Let us now proceed to the linearization of the set of equations $(3)-(7)$ starting from a generic base configuration, denoted in the sequel by a suffix (B), in which a perturbation is applied.

3.1 *Kinematic description of the linearized model*

The following representation is considered for the perturbed state fields:

$$\bar{\mathbf{u}}^{(s)} = \bar{\mathbf{u}}^{(s)}_B + d\bar{\mathbf{u}}^{(s)}, \qquad \bar{\rho}^{(f)} = \bar{\rho}^{(f)}_B + d\bar{\rho}^{(f)}, \qquad \hat{\mathbf{v}}^{(f)} = \hat{\mathbf{v}}^{(f)}_B + d\hat{\mathbf{v}}^{(f)}, \qquad \hat{J}^{(s)} = \hat{J}^{(s)}_B + d\hat{J}^{(s)} \tag{10}$$

where $\bar{\mathbf{u}}^{(s)}$ is the macroscopic displacements field and the prefix d indicates the infinitesimal variation fields. In place of $d\hat{J}^{(s)}$, the infinitesimal increment $d\hat{\varepsilon}^{(s)}_v$

$$d\hat{\varepsilon}^{(s)}_v = \frac{d\hat{J}^{(s)}}{\hat{J}^{(s)}_B} = \frac{\displaystyle\int_{\Omega^{(s)}_{0_{RVE}}(\mathbf{X})} J_B d\varepsilon^{(s)}_v \, dV}{\displaystyle\int_{\Omega^{(s)}_{0_{RVE}}(\mathbf{X})} J_B \, dV} \tag{11}$$

where $\varepsilon^{(s)}_v$ is the infinitesimal local increment of volumetric strain, may be alternatively employed and J_B is the local Jacobian of the deformation of the solid phase associated with the base configuration.

The first order approximation of the macroscopic Green-Lagrange tensor of the solid phase is $\bar{\mathbf{E}}^{(s)} = \bar{\mathbf{E}}^{(s)}_B + \bar{\mathbf{F}}^t_B d\bar{\boldsymbol{\varepsilon}}^{(s)}\bar{\mathbf{F}}_B$ where $d\bar{\boldsymbol{\varepsilon}}^{(s)}$ is the infinitesimal increment of the macroscopic strain tensor defined as the symmetric part of the macroscopic displacements gradient. The first order approximation of the macroscopic Jacobian is

$$\bar{J} = \bar{J}_B + \bar{J}_B d\bar{\varepsilon}^{(s)}_v \tag{12}$$

where $d\bar{\varepsilon}^{(s)}_v = \mathrm{tr}\, d\bar{\boldsymbol{\varepsilon}}^{(s)} = \frac{\partial d\bar{u}^{(s)}_l}{\partial x_l}$ is the infinitesimal increment of the apparent macroscopic volumetric strain.

3.2 *Linearized constitutive equations*

In the finite formulation the following spatial stress terms are employed

$$\hat{\sigma}^{(s)}_{ij} = \frac{1}{\bar{J}} \hat{S}^{(s)}_{PQ} \frac{\partial x_i}{\partial X_P} \frac{\partial x_j}{\partial X_Q}, \qquad \sigma_{\hat{J}^{(s)}} = \frac{\partial \bar{\psi}^{(s)}}{\partial \hat{J}^{(s)}} \tag{13}$$

where $\hat{\mathbf{S}}^{(s)}$ is derivated from the potential (1). As pointed out in [9], the tensor $\hat{\boldsymbol{\sigma}}^{(s)}$ is a spatial tensor which, in general, is different from the Cauchy stress $\boldsymbol{\sigma}^{(s)}$. The former is related to $\hat{\mathbf{P}}^{(s)}$ and to its corresponding material reference tensor $\hat{\mathbf{S}}^{(s)}$ by the usual transformations holding between the ordinary Cauchy stress tensor and the relevant first- and second-Piola transforms. Analogous transformations allow one to convert $\sigma_{\hat{J}^{(s)}}$ in alternative stress measures associated with the material and the spatial settings

$$S_{\hat{\varepsilon}^{(s)}_v} = \frac{\partial \bar{J} \bar{\psi}^{(s)}}{\partial d\hat{\varepsilon}^{(s)}_v}, \qquad \sigma_{\hat{\varepsilon}^{(s)}_v} = \frac{\partial \bar{\psi}^{(s)}}{\partial d\hat{\varepsilon}^{(s)}_v} \tag{14}$$

In the linearized theory the stress increments $d\hat{\boldsymbol{\sigma}}^{(s)}$ and $d\sigma_{\hat{J}^{(s)}}$, respectively work-conjugated to $\bar{\varepsilon}^{(s)}$ and $\hat{J}^{(s)}$, have to be computed

$$\hat{\boldsymbol{\sigma}}^{(s)} = \hat{\boldsymbol{\sigma}}^{(Bs)} + d\hat{\boldsymbol{\sigma}}^{(s)}, \qquad \sigma_{\hat{J}^{(s)}} = \sigma^{(B)}_{\hat{J}^{(s)}} + d\sigma_{\hat{J}^{(s)}} \tag{15}$$

The quantity $d\hat{\boldsymbol{\sigma}}^{(s)}$ is computed by linearizing equation $(13)_1$ around the base configuration

$$d\hat{\sigma}^{(s)}_{ij} = \partial_{\left(\bar{u}^{(s)}_l,\, \hat{J}^{(s)}\right)} \left(\frac{1}{\bar{J}} \hat{S}^{(s)}_{PQ} \frac{\partial x_i}{\partial X_P} \frac{\partial x_j}{\partial X_Q} \right) [d\bar{u}^{(s)}_l,\, d\hat{J}^{(s)}] \tag{16}$$

what provides, for a generic hyperelastic law

$$d\hat{\sigma}^{(s)}_{ij} = \bar{\mathcal{D}}^{(B)}_{ijlm} d\bar{\varepsilon}^{(s)}_{lm} + \bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}ij} d\hat{\varepsilon}^{(s)}_v + \hat{\sigma}^{(Bs)}_{mj} \frac{\partial d\bar{u}^{(s)}_i}{\partial x_m} + \hat{\sigma}^{(Bs)}_{im} \frac{\partial d\bar{u}^{(s)}_j}{\partial x_m} - \hat{\sigma}^{(Bs)}_{ij} \frac{\partial d\bar{u}^{(s)}_l}{\partial x_l} \tag{17}$$

It can be recognized that, after moving to the LHS the last three terms appearing on the RHS, a term reminiscent of the Truesdell rate of the Cauchy-like stress tensor $\hat{\boldsymbol{\sigma}}^{(s)}$ is obtained. The stiffness operators appearing in (17) are given by

$$\bar{\mathcal{D}}^{(B)}_{ijlm} = \frac{1}{\bar{J}} \frac{\partial x_i}{\partial X_P} \frac{\partial x_j}{\partial X_Q} \bar{\mathcal{C}}^{(B)}_{PQHK} \frac{\partial x_l}{\partial X_H} \frac{\partial x_m}{\partial X_K}, \qquad \bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}ij} = \frac{1}{\bar{J}} \frac{\partial x_i}{\partial X_P} \frac{\partial x_j}{\partial X_Q} \bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}PQ} \tag{18}$$

and depend in turn on the following fourth-order and third-order material stiffness operators

$$\bar{\mathcal{C}}^{(B)}_{PQHK} = \left. \frac{\partial^2 \bar{J} \bar{\psi}^{(s)}}{\partial \bar{E}^{(s)}_{PQ} \partial \bar{E}^{(s)}_{HK}} \right|_{\bar{\mathbf{F}}_B, \hat{J}^{(s)}_B}, \qquad \bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}PQ} = \left. \frac{\partial^2 \bar{J} \bar{\psi}^{(s)}}{\partial \bar{E}^{(s)}_{PQ} \partial \hat{J}^{(s)}} \right|_{\bar{\mathbf{F}}_B, \hat{J}^{(s)}_B} \tag{19}$$

Analogously, the linearization of $\sigma_{\hat{J}^{(s)}}$ around a generic base configuration provides

$$d\sigma_{\hat{J}^{(s)}} = \bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}lm} d\bar{\varepsilon}^{(s)}_{lm} + \bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}\hat{J}^{(s)}} d\hat{J}^{(s)} - \sigma^{(B)}_{\hat{J}^{(s)}} \frac{\partial d\bar{u}^{(s)}_l}{\partial x_l} \tag{20}$$

where

$$\bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}lm} = \frac{1}{\bar{J}} \bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}HK} \frac{\partial x_l}{\partial X_H} \frac{\partial x_m}{\partial X_K}, \qquad \bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}\hat{J}^{(s)}} = \frac{1}{\bar{J}} \bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}\hat{J}^{(s)}} \tag{21}$$

and the material stiffness coefficients in (21) are defined by

$$\bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}HK} = \left.\frac{\partial^2 \bar{J}\bar{\psi}^{(s)}}{\partial \hat{J}^{(s)} \partial \bar{E}^{(s)}_{HK}}\right|_{\bar{\mathbf{F}}_B, \hat{J}^{(s)}_B}, \qquad \bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}\hat{J}^{(s)}} = \left.\frac{\partial^2 \bar{J}\bar{\psi}^{(s)}}{\partial \hat{J}^{(s)} \partial \hat{J}^{(s)}}\right|_{\bar{\mathbf{F}}_B, \hat{J}^{(s)}_B} \tag{22}$$

Finally, the following linearized expression around the base configuration holds for $\bar{q}^{(s)}_{add.}$

$$\bar{q}^{(s)}_{add.} = \bar{q}^{(s)}_{add.B} + d\bar{q}^{(s)}_{add.} \tag{23}$$

where $d\bar{q}^{(s)}_{add.} = \bar{\rho}^{(s)}_{add.} d\dot{\hat{J}}^{(s)}$ being, on account of (2)

$$\bar{\rho}^{(s)}_{add.} = \frac{\partial^2 \bar{\chi}^{(s)}_{add.}}{\partial \dot{\hat{J}}^{(s)} \partial \dot{\hat{J}}^{(s)}}. \tag{24}$$

We remark that, although the notation are similar, there is no relation between $\bar{\rho}^{(s)}_{add.}$ and the added mass term introduced by Biot in [11].

3.3 *Linearization around the reference configuration. The special case of an isotropic, homogeneous, initially motionless and stress-free base configuration*

The present subsection illustrates the expressions obtained by specializing the most general linearized equations, reported in [9], to the case in which base and reference configuration of the solid phase do coincide; hence $\bar{\mathbf{u}}^{(s)}_B = \mathbf{0}$, $\hat{J}^{(s)}_B = 1$ and indexes B and 0 are equivalent. We also introduce the further hypotheses that such configuration is initially motionless, being all time rates of the base configuration zero, and stress-free, i.e. $\hat{\boldsymbol{\sigma}}^{(Bs)} = \mathbf{0}$.

Owing to the coincidence of reference, base and current configurations, all stress tensors collapse to the same tensor $\hat{\boldsymbol{\sigma}}^{(s)} \equiv \hat{\mathbf{P}}^{(s)} \equiv \hat{\mathbf{S}}^{(s)}$and the same occurrs for the $\hat{J}^{(s)}$-conjugated stress scalars $S_{\hat{J}^{(s)}} \equiv S_{\hat{\varepsilon}^{(s)}_v} \equiv \sigma_{\hat{J}^{(s)}} \equiv \sigma_{\hat{\varepsilon}^{(s)}_v}$ as well as for the stiffness operators $\bar{\mathcal{C}}^{(B)} \equiv \bar{\mathcal{D}}^{(B)}, \quad \bar{\mathcal{D}}_{\hat{J}^{(s)}} \equiv \bar{\mathcal{C}}_{\hat{J}^{(s)}}, \quad \bar{\mathcal{D}}^{(B)}_{\hat{J}^{(s)}\hat{J}^{(s)}} \equiv \bar{\mathcal{C}}^{(B)}_{\hat{J}^{(s)}\hat{J}^{(s)}}$. Employing the usual engineering notation for stress and strain quantities, suitably augmented to account for the additional state variable $\hat{\varepsilon}^{(s)}_v$, the linearized constitutive relation is accordingly represented by a 7×7 symmetrix matrix. The potential (1), after invoking the isotropy hypothesis, entails a stress-strain relation that admits the following matrix representation

$$\begin{bmatrix} \hat{\sigma}^{(s)}_x \\ \hat{\sigma}^{(s)}_y \\ \hat{\sigma}^{(s)}_z \\ \hat{\tau}^{(s)}_{yz} \\ \hat{\tau}^{(s)}_{zx} \\ \hat{\tau}^{(s)}_{xy} \\ \sigma_{\hat{\varepsilon}^{(s)}_v} \end{bmatrix} = \begin{bmatrix} 2\bar{G}_0 + \bar{Z}_0 & \bar{Z}_0 & \bar{Z}_0 & 0 & 0 & 0 & \bar{K}^{\bar{\varepsilon}_v\hat{\varepsilon}_v}_0 \\ \bar{Z}_0 & 2\bar{G}_0 + \bar{Z}_0 & \bar{Z}_0 & 0 & 0 & 0 & \bar{K}^{\bar{\varepsilon}_v\hat{\varepsilon}_v}_0 \\ \bar{Z}_0 & \bar{Z}_0 & 2\bar{G}_0 + \bar{Z}_0 & 0 & 0 & 0 & \bar{K}^{\bar{\varepsilon}_v\hat{\varepsilon}_v}_0 \\ 0 & 0 & 0 & 2\bar{G}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\bar{G}_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\bar{G}_0 & 0 \\ \bar{K}^{\hat{\varepsilon}_v\bar{\varepsilon}_v}_0 & \bar{K}^{\hat{\varepsilon}_v\bar{\varepsilon}_v}_0 & \bar{K}^{\hat{\varepsilon}_v\bar{\varepsilon}_v}_0 & 0 & 0 & 0 & \bar{K}^{\hat{\varepsilon}_v\hat{\varepsilon}_v}_0 \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}^{(s)}_x \\ \bar{\varepsilon}^{(s)}_y \\ \bar{\varepsilon}^{(s)}_z \\ \frac{\bar{\gamma}^{(s)}_{yz}}{2} \\ \frac{\bar{\gamma}^{(s)}_{zx}}{2} \\ \frac{\bar{\gamma}^{(s)}_{xy}}{2} \\ \hat{\varepsilon}^{(s)}_v \end{bmatrix} \tag{25}$$

We remark that, although this matrix representation is formally similar to the one considered by Biot and Willis in [1] for the description of the constitutive response of the model devised by Biot [11], relation (25) describes the elastic response to infinitesimal strains of the solid phase alone, indipendently from the fluid phase.

Regarding the physical meaning of $\sigma_{\hat{\varepsilon}_v^{(s)}}$, it is shown in [9] that the stress term $\sigma_{\hat{\varepsilon}_v^{(s)}}$ turns out to be related by the following relation

$$\sigma_{\hat{\varepsilon}_v^{(s)}} = -\hat{p}n^{(s)} = -\hat{p}\left(1 - n^{(f)}\right) \tag{26}$$

to the intestitial pressure $\hat{p}$ of the fluid in the interconnected cavities.

Finally, having neglected viscous contribution with respect to the drag interaction forces originated from relative motions between solid and fluid phase, the constitutive relation of the fluid phase is a simple linear relation connecting the interstitial pressure $\hat{p}$ and the infinitesimal volumetric strain of the fluid $\hat{\varepsilon}_v^{(f)}$

$$\hat{p} = -K_{v0}^{(f)}\hat{\varepsilon}_v^{(f)} \tag{27}$$

3.4 *Linearized governing equations*

The linearized governing equations reported in the present work are obtained by specializing the more general linearized equations, reported in [9], according to the additional above mentioned hypotheses that characterize experimental static laboratory tests, i.e. both fluid and solid are at rest and stress-free in the base configuration, homogeneity and isotropy. The linear equations resulting from the linearization of the set of equations (7) under the above mentioned assumptions are reported hereafter in the same order:

$$dn^{(f)} = \left(\bar{\varepsilon}_v^{(s)} - \hat{\varepsilon}_v^{(s)}\right) n_0^{(s)} \tag{28}$$

$$d\bar{\rho}^{(f)} = dn^{(f)}\hat{\rho}_0^{(f)} + n_0^{(f)} d\hat{\rho}^{(f)} \tag{29}$$

$$\overset{\cdot(s)}{d\bar{\rho}^{(f)}} + \bar{\rho}_0^{(f)}\dot{\bar{\varepsilon}}_v^{(s)} + \frac{\partial}{\partial x_i}\left[\bar{\rho}_0^{(f)}\left(d\hat{v}_i^{(f)} - d\bar{v}_i^{(s)}\right)\right] = 0 \tag{30}$$

$$-n_0^{(f)}\frac{\partial d\hat{p}}{\partial x_i} + d\bar{b}_i^{\prime(fs)} + d\bar{b}_i^{(f)(ext)} = \bar{\rho}_0^{(f)} \overset{\cdot(s)}{d\hat{v}_i^{(f)}} \tag{31}$$

$$\bar{\rho}_0^{(s)}\ddot{\bar{u}}^{(s)}{}_i = \bar{G}_0\frac{\partial^2\bar{u}_i^{(s)}}{\partial x_l \partial x_l} + \left(\bar{G}_0 + \bar{Z}_0\right)\frac{\partial^2\bar{u}_l^{(s)}}{\partial x_i \partial x_l} + \bar{K}_0^{\bar{\varepsilon}_v\hat{\varepsilon}_v}\frac{\partial\hat{\varepsilon}_v^{(s)}}{\partial x_i} + d\bar{b}_i^{(sf\mathbf{n})} + d\bar{b}_i^{\prime(sf)} + d\bar{b}_i^{(s)(ext)} \tag{32}$$

$$\bar{\rho}_{add.0}^{(s)}\ddot{\hat{\varepsilon}}_v^{(s)} + \bar{K}_0^{\hat{\varepsilon}_v\bar{\varepsilon}_v}\bar{\varepsilon}_v^{(s)} + \bar{K}_0^{\hat{\varepsilon}_v\hat{\varepsilon}_v}\hat{\varepsilon}_v^{(s)} = 0 \tag{33}$$

3.5 *Relation between the constitutive laws of the solid phase alone*

Relation (26) allows one to analyze the connection between the classic Cauchy stress $\boldsymbol{\sigma}^{(s)}$ and the tensor $\hat{\boldsymbol{\sigma}}^{(s)}$. As a matter of fact, the real Cauchy stress is the macroscopic stress when a deformation is applied on the porous medium in absence of fluid. The absence of fluid is characterized by the condition $\hat{p} = 0$ and, consequently, by virtue of (26), by the condition $\sigma_{\hat{\varepsilon}_v^{(s)}} = 0$. It is thus recognized that the elastic matrix relating the Cauchy stress to the strain $\bar{\varepsilon}^{(s)}$ of the porous solid phase alone is simpy obtained by static condensation of the elastic matrix (25) in which the term $\sigma_{\hat{\varepsilon}_v^{(s)}}$ is set equal to zero. The seventh elastic relation in (25) accordingly provides

$$\hat{\varepsilon}_v^{(s)} = -\frac{\bar{K}_0^{\hat{\varepsilon}_v\bar{\varepsilon}_v}}{\bar{K}_0^{\hat{\varepsilon}_v\hat{\varepsilon}_v}}\left(\bar{\varepsilon}_x^{(s)} + \bar{\varepsilon}_y^{(s)} + \bar{\varepsilon}_z^{(s)}\right) \tag{34}$$

and, after substitution, an elastic isotropic stress-strain law is obtained for the solid phase alone which is characterized by the following Lamé coefficients:

$$\bar{G}^{(s)} = \bar{G}_0, \qquad \bar{\lambda}^{(s)} = \bar{Z}_0 - \frac{(\bar{K}_0^{\hat{\varepsilon}_v \bar{\varepsilon}_v})^2}{\bar{K}_0^{\hat{\varepsilon}_v \hat{\varepsilon}_v}} \tag{35}$$

3.6 *Experimental calibration of the elastic coefficients of the linearized constitutive equations*

A procedure for the calibration of the constitutive coefficients is illustrated in this section and the evaluation of the macroscopic response of the medium is derived without resorting to homogenization techniques. This task is carried out in relation to the condition that typically characterize the specimens in experimental static laboratory tests: small deformations, macroscopically homogeneous, initially motionless configuration of the medium, isotropy of both the bulk solid material and of the macroscopic behaviour; finally, it is further assumed that no initial stress is applied before the test starts. Under these conditions both $\sigma^{(0)}_{\hat{J}^{(s)}} = 0$ and $\hat{\boldsymbol{\sigma}}^{(0s)} = \mathbf{0}$ and the response to small perturbations is governed by equations (28)-(33).

The constitutive parameters, entering such linearized formulation, which have to be measured are

$$\bar{G}_0,\ \bar{Z}_0,\ \bar{K}_0^{\bar{\varepsilon}_v \hat{\varepsilon}_v},\ \bar{K}_0^{\hat{\varepsilon}_v \hat{\varepsilon}_v},\ K_{v0}^{(f)},\ \bar{\rho}_0^{(s)},\ \bar{\rho}_{add.}^{(s)},\ n_0^{(f)} \tag{36}$$

In particular, we focus on the determination of the parameters entering the static behaviour, namely $\bar{G}_0,\ \bar{Z}_0,\ \bar{K}_0^{\bar{\varepsilon}_v \hat{\varepsilon}_v},\ \bar{K}_0^{\hat{\varepsilon}_v \hat{\varepsilon}_v},\ K_{v0}^{(f)},\ n_0^{(f)}$, leaving to a following specific work a study of the coefficient $\bar{\rho}_{add.}^{(s)}$ which essentially affects the dynamic response of the medium.

The volumetric stiffness of the fluid $K_{v0}^{(f)}$ and the initial density $\bar{\rho}_0^{(s)}$ can be determined by customary tests directly carried out on the fluid phase alone. The porosity $n_0^{(f)}$ can be measured by exploiting any of the well codified procedures available based on weight or volumetric comparisons. Uncoupling between volumetric and deviatoric response allow one to measure the shear modulus $\bar{G}_0$ by means of a simple shear test.

Observing that the remaining three coefficients $\bar{Z}_0,\ \bar{K}_0^{\bar{\varepsilon}_v \hat{\varepsilon}_v},\ \bar{K}_0^{\hat{\varepsilon}_v \hat{\varepsilon}_v}$ in (36) are all associated with the constitutive response under isotropic loading conditions, it is convenient to refer to the following isotropic reduced representation of the elastic response of the solid medium associated with the isotropic strain measures of the solid phase $\bar{\varepsilon}_v^{(s)}$ and $\hat{\varepsilon}_v^{(s)}$

$$\begin{bmatrix} -p^{(s)} \\ -\hat{p}n^{(f)} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\bar{G}_0 + \bar{Z}_0 & \bar{K}_0^{\hat{\varepsilon}_v \bar{\varepsilon}_v} \\ \bar{K}_0^{\hat{\varepsilon}_v \bar{\varepsilon}_v} & \bar{K}_0^{\hat{\varepsilon}_v \hat{\varepsilon}_v} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_v^{(s)} \\ \hat{\varepsilon}_v^{(s)} \end{bmatrix} \tag{37}$$

which is obtained from (25). In (37) $p^{(s)}$ is simply defined as $p^{(s)} = -1/3\mathrm{tr}\,\hat{\boldsymbol{\sigma}}^{(s)}$.

Although these parameters in the formulation are specific property of the solid medium, since they only depend on the elastic properties of the bulk solid matrix and the particular realization of the RVE geometry, the tests described hereafter are based on experiments in which the voids are saturated by a fluid. This is motivated by the need of indirectly controlling the kinematic parameter $\hat{\varepsilon}_v^{(s)}$, which is related to $\hat{p}$ by (26).

In particular we consider the unjacketed compressibility test devised in [1] and two jacketed compressibility tests characterized by drainage which is either completely allowed or completely prevented.

We use $\Omega_{b0} = \Omega_{b0}^{(s)} \cup \Omega_{b0}^{(f)}$ to refer to the reference domain of the biphasic specimen, and $\partial\Omega_{b0}^{(sf)}$ to refer to the part of the boundary of $\Omega_{b0}^{(s)}$ which is in contact with fluid phase, using $\partial\Omega_{b0}^{(s\,ext.)} \subseteq \partial\Omega_{b0}$ to refer to the remaining external boundary.

3.7 *Unjacketed compressibility test*

In this test a specimen of the solid material is immersed in a fluid subject to a pressure $\hat{p}$ and full drainage is allowed. Since no costraint exists on the motion of the fluid phase a relative fluid-solid motion is activated by the application of the pressure.

Since a uniform pressure field $\hat{p}$ exists in the fluid domain $\Omega_{b0}^{(f)}$, a uniform boundary pressure is applied to the boundary of the solid phase $\partial\Omega_{b0}^{(s)}$, i.e. on both $\partial\Omega_{b0}^{(sf)}$ and $\partial\Omega_{b0}^{(s\,ext.)}$. It can be recognized that, on account of the isotropy of the bulk material, the solution to this elastic problem for the solid domain $\Omega_{b0}^{(s)}$ in terms of stress field turns out to be the uniform isotropic stress field $-\hat{p}\delta_{ij}$. In other words, as a matter of fact, the pressure field simply extends from the fluid to the solid phase. As a direct consequence, since the bulk solid is isotropic, the solution in terms of strain field in $\Omega_{b0}^{(s)}$ is the uniform field $\varepsilon_{ij}^{(s)} = \dfrac{\varepsilon_v^{(s)}}{3}\delta_{ij}$ while the solution in terms of displacements is

$$\bar{\mathbf{u}}^{(s)}(\mathbf{X}) = \frac{\varepsilon_v^{(s)}}{3}\mathbf{X} \tag{38}$$

where $\boldsymbol{\varepsilon}^{(s)}$ is the local microscopic value of the infinitesimal strain and $\varepsilon_v^{(s)}$ the associated volumetric strain. It is also recognized that the extension of the displacement field to Ω_{b0} is still provided by (38), and consequently still characterized by a uniform volumetric strain. This particular condition of uniformity for $\varepsilon_v^{(s)}$ over $\Omega_{b0}^{(s)}$ and, after extension over the entire Ω_{b0}, entails the kinematic constraint $\hat{\varepsilon}_v^{(s)} = \bar{\varepsilon}_v^{(s)}$. Under this constraint the second equation of (37) provides the equation

$$\frac{\bar{K}_0^{\hat{\varepsilon}_v\bar{\varepsilon}_v} + \bar{K}_0^{\hat{\varepsilon}_v\hat{\varepsilon}_v}}{n^{(f)}} = -\frac{\hat{p}}{\bar{\varepsilon}_v^{(s)}} \tag{39}$$

where the ratio on the RHS represents the experimentally measured quantity.

3.8 *Drained jacketed isotropic compressibility test*

In the jacketed compressibility tests an external hydrostatic stress is applied to the biphasic specimen by means of a film or an equivalent sealing device that covers the external boundary $\partial\Omega_{b0}$ and completely prevents drainage through this surface. However, a communicating duct exists between the internal fluid and an external chamber. This condition implies that $\hat{p} = 0$ before and throughout the test, so that the presence of the fluid phase can be neglected in all stress balances as if the test were carried on the solid phase alone. On account of (35) and (37) an equation relating the unknown coefficients to the ratio between the spherical component of the externally applied loads and the measured volumetric strain is obtained

$$\frac{2}{3}\bar{G}_0 + \bar{Z}_0 - \frac{(\bar{K}_0^{\hat{\varepsilon}_v\bar{\varepsilon}_v})^2}{\bar{K}_0^{\hat{\varepsilon}_v\hat{\varepsilon}_v}} = -\frac{p^{(s)}}{\bar{\varepsilon}_v^{(s)}} \tag{40}$$

3.9 *Undrained jacketed isotropic compressibility test*

In the undrained jacketed compressibility, as in the drained one, a film or an equivalent device that completely prevents drainage through the boundary is applied. However, differently from the previous test, no communicating duct exists for the fluid. Solid and fluid external boundaries undergo the same macroscopic deformation and, as shown in [9], solid and fluid undergo the same macroscopic volumetric strain, a condition that specializes to $\bar{\varepsilon}_v^{(s)} = \bar{\varepsilon}_v^{(f)}$ under infinitesimal deformations. Subsequently, an external hydrostatic stress $\hat{p}^{ext}$ is applied. The equilibrium of this mechanical system has been analyzed in [9] and the following equilibrium condition has been inferred

$$\frac{2}{3}\bar{G}_0 + \bar{Z}_0 + 4n^{(f)}K_{v0}^{(f)} - \frac{\left(\bar{K}_0^{\hat{\varepsilon}_v\bar{\varepsilon}_v} - 2n^{(f)}K_{v0}^{(f)}\right)^2}{\bar{K}_0^{\hat{\varepsilon}_v\hat{\varepsilon}_v} + n^{(f)}K_{v0}^{(f)}} = -\frac{\hat{p}^{ext}}{\bar{\varepsilon}_v^{(s)}} \tag{41}$$

in which the ratio on the RHS is experimentally determined.

Formula (41) together with (39) and (40) provide an algebraic system that can be solved to obtain the constituve coefficients $\bar{Z}_0$, $\bar{K}_0^{\hat{\varepsilon}_v\hat{\varepsilon}_v}$ and $\bar{K}_0^{\hat{\varepsilon}_v\bar{\varepsilon}_v}$, as shown in [9].

References

[1] Biot M.A., Willis D.G., "The elastic coefficients of the theory of consolidation", *Journal of Applied Mechanics*, **24**, 594-601 (1957).

[2] Gibson L., Ashby M., *Cellular solids - Structure and properties*, Cambridge University Press, (1999).

[3] Sorrentino L., Aurilia M., Iannace S., "A simple method to predict high strain rates mechanical behavior of low interconnected cell foams", *Polymer Testing*, **26**, 878-885 (2007).

[4] Landervik M., Larsson R., "Pore-gas interaction in foams at finite deformation using staggered solution techniques", *Comput. Methods Appl. Mech. Engrg.*, **197**, 148-159 (2007).

[5] Li C., Borja R., Regueiro R., "Dynamics of porous media at finite strain", *Comput. Methods Appl. Mech. Engrg.*, **193**, 3837-3870 (2004).

[6] Almeida E.S., Spilker R.L., "Finite element formulations for hyperelastic transversely isotropic biphasic soft tissues", *Comput. Methods Appl. Mech. Engrg.*, **151**, 513-538 (1998).

[7] Zienkiewicz O.C., Chan A.H.C., Pastor M., Paul D.K., Shiomi T., "Static and dynamic behaviour of soils, a rational approach to quantitative solutions", *Proc. Royal Society of London*, A 429, 285-309 (1990).

[8] Serpieri R., Rosati L., "A finite deformation model for the dynamic behaviour of fluid saturated porous biphasic media" in *Proc. VII EUROMECH Solid Mechanics Conference*, Lisbon, Portugal, Sept. 7-11, (2009).

[9] Serpieri, R., Rosati, L., "A finite deformation model for open cell biphasic media", *in preparation*.

[10] Gray, W.G., Schrefler, B.A., Pesavento, F., "The solid phase stress tensor in porous media mechanics and the Hill-Mandel condition" *Jour. Mech. Phys. Sol.*, **57**, 539-554 (2009).

[11] Biot M.A., "Theory of propagation of elastic waves in a fluid saturated porous solid - Part I", *Jour. Acoust. Soc. America*, **28**, 168-191 (1956).

[12] van der Grinten, J.G.M., van Dongen, M.E.H., van der Kogel, H., "A shock-tube technique for studying pore-pressure propagation in a dry and water-saturated porous medium", *J. Appl. Phys.*, **58**, 2937-2942, (1985).