An analytic solution method for a biharmonic elastic problem

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SUMMARY. The problem of equilibrium on an elastic cylinder of finite length subjected to a surface load is one of the oldest problems in the theory of elasticity. If the cylinder is subject to axisymmetric conditions, the use of the Love representation function reduces the three-dimensional elastic problem to the search for a biharmonic function on a cylindrical domain. In this paper, we discuss a method to find the solution of the axisymmetric boundary value problem for a finite elastic cylinder with assigned stress and/or displacements acting on the ends and the mantle.

1 INTRODUCTION

The boundary value problems describing the deformation of a symmetrically loaded elastic circular cylinder are very complex, and a solution that rigorously satisfies all the boundary conditions on the side surface and on the ends of a cylinder is known only for certain particular cases [1,2]. Various approaches are used to find solutions of practical interest; the usual boundary conditions are in terms of tractions, and some cases are obtained by assuming surface forces over the plane ends of the cylinder and zero stress on the side surface (Saint-Venant problem) or by assuming zero stress at the ends and assigned surface tractions on the side surface (Almansi-Michell problem; Filon problem). If the cylinder is subject to axisymmetric conditions, the use of the Love representation function reduces the three-dimensional axisymmetric elastic problem to the search for a biharmonic function on a cylindrical domain and several approaches exist for constructing explicit solutions [3,4,5,6].

In this contribution we discuss a technique recently presented in [7]. Let us consider the elastic problem of an homogenous isotropic cylinder, of radius *b* and length *h*, subjected to axisymmetric boundary conditions. By using a polar coordinate system, we use the biharmonic Love function which we choose to write in the form of a zero-order Bessel expansion.

The usual process of construction of the solution assumes to write the Love function as

$$\Phi(r,z)=\sum_{k=1}^{\infty}\Phi_k\left(z\right)J_0\left(\phi_k r\right), \tag{1.1}$$

where $\phi_k = z_k^0/b$ and z_k^0 denote the positive zeros of the Bessel functions $J_0\left(x\right)$; successively one computes the derivatives by differentiating equation (1.1) term-by-term.

This procedure is restrictive since it implies

$$\Phi(b,z)=0, \qquad \nabla_r^2\Phi(b,z)=0. \tag{1.2}$$

Since these equations are not true in general, they clearly represent additive assumptions which restrict the class of biharmonic equation solutions and will allow satisfaction of only a limited set of boundary value conditions [8,9,10].

To overcome this difficulty and so avoid the restrictions (1.2), we propose a solution method which takes into account the fact that, in general, the requirements allowing the term-by-term differentiation of the expansion are not fulfilled [11]. To this end, we explicitly construct a radial Laplace formula obtained without these assumptions [12]. By introducing two functions, $\alpha(z)$ and $\beta(z)$, we are able to write the Bessel expansion associated with the Love function in a form which is convergent for all r and allows to match all smooth boundary conditions on the side and on the ends of the cylinder.

We employ the solution method to obtain an exact explicit solution of practical interest concerning a cylinder with mixed boundary conditions [7].

2 PROBLEM FORMULATION

Let us consider a circular, finite, homogeneous, and isotropic elastic cylinder of length h and radius b and further consider only axisymmetric elastic states. We introduce a cylindrical coordinate system (0; r, ϑ, z) with the origin at the centre of the top plane end and we denote with $u(r, z)$ and $w(r, z)$ the radial and longitudinal components of the displacement. The isotropic material is described by the Poisson's ratio ν and the shear modulus μ.

By using the Love function $\Phi(r, z)$, we express the displacement and the stress components, without body forces, in the form

$$u(r,z) = -\frac{1}{2\mu}\frac{\partial^2\Phi}{\partial r\partial z}, \qquad w(r,z) = \frac{1}{2\mu}\left(2(1-\nu)\nabla_r^2\Phi + (1-2\nu)\frac{\partial^2\Phi}{\partial z^2}\right), \tag{2.1}$$

and

$$\begin{aligned} \sigma_r &= \frac{\partial}{\partial z}\left(\nu\left(\nabla_r^2\Phi + \frac{\partial^2\Phi}{\partial z^2}\right) - \frac{\partial^2\Phi}{\partial r^2}\right), & \sigma_z &= \frac{\partial}{\partial z}\left((2-\nu)\left(\nabla_r^2\Phi + \frac{\partial^2\Phi}{\partial z^2}\right) - \frac{\partial^2\Phi}{\partial z^2}\right), \\ \sigma_\vartheta &= \frac{\partial}{\partial z}\left(\nu\left(\nabla_r^2\Phi + \frac{\partial^2\Phi}{\partial z^2}\right) - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right), & \tau_{rz} &= \frac{\partial}{\partial r}\left((1-\nu)\left(\nabla_r^2\Phi + \frac{\partial^2\Phi}{\partial z^2}\right) - \frac{\partial^2\Phi}{\partial z^2}\right), \end{aligned} \tag{2.2}$$

where we introduced the radial Laplace operator

$$\nabla_r^2(\cdot) = \frac{\partial^2}{\partial r^2}(\cdot) + \frac{1}{r}\frac{\partial}{\partial r}(\cdot). \tag{2.3}$$

The function $\Phi(r, z)$ is biharmonic [1]:

$$\nabla^2\nabla^2\Phi(r,z) = 0. \tag{2.4}$$

3 METHOD OF SOLUTION

We consider the Love function $\Phi(r,z)$ and we denote $\beta(z) = \Phi(b,z)$. Since the function $\Phi(r,z) - \beta(z)$ vanishes for r=b, we write the Bessel expansion

$$\Phi(r,z)-\beta(z)=\sum_{k=1}^{\infty}\Phi_k(z)J_0(\phi_k r), \tag{3.1}$$

which is convergent for $0\le r\le b$ [11]. The coefficients assume the following form

$$\Phi_k(z)=\frac{2}{b^2\left[J_1(\phi_k b)\right]^2}\int_0^b(\Phi(\rho,z)-\beta(z))J_0(\phi_k\rho)\rho d\rho, \tag{3.2}$$

with $\phi_k=z_k^0/b$ where z_k^0 denote the positive zeros of the Bessel functions $J_0(x)$.

In order to construct the solution of the biharmonic equation (2.4), we need a formula for $\nabla_r^2\Phi(r,z)$; since the convergence of the expansion (3.1) does not guarantee verification that the conditions allowing term-by-term differentiation of the expansion with respect to r are satisfied (see, for example [11] p.143), we introduce the quantity $\alpha(z)=\nabla_r^2\Phi(b,z)$ and again expand the function $\nabla_r^2\Phi(r,z)-\alpha(z)$, for $0\le r\le b$, as

$$\nabla_r^2\Phi(r,z)-\alpha(z)=\sum_{k=1}^{\infty}d_k(z)J_0(\phi_k r), \tag{3.3}$$

where

$$d_k(z)=\frac{2}{b^2\left[J_1(\phi_k b)\right]^2}\int_0^b\left(\nabla_r^2\Phi(\rho,z)-\alpha(z)\right)J_0(\phi_k\rho)\rho d\rho. \tag{3.4}$$

By writing the explicit formula of the radial Laplace operator and performing two part-by-part integrations, we get:

$$d_k(z)=-\left(\phi_k{}^2\Phi_k(z)+\frac{2\alpha(z)}{b\phi_k J_1(\phi_k b)}\right), \tag{3.5}$$

and so, the formula for the radial Laplace operator becomes:

$$\nabla_r^2\Phi(r,z)=\alpha(z)-\sum_{k=1}^{\infty}\left(\phi_k{}^2\Phi_k(z)+\frac{2\alpha(z)}{b\phi_k J_1(\phi_k b)}\right)J_0(\phi_k r). \tag{3.6}$$

Similarly, by repeating this procedure two times, after setting $\gamma(z)=\nabla_r^2\nabla_r^2\Phi(b,z)$, we get:

$$\nabla_r^2\nabla_r^2\Phi(r,z)=\gamma(z)+\sum_{k=1}^{\infty}\left(\phi_k{}^4\Phi_k(z)+\frac{2\alpha(z)\phi_k}{bJ_1(\phi_k b)}-\frac{2\gamma(z)}{\phi_k bJ_1(\phi_k b)}\right)J_0(\phi_k r). \tag{3.7}$$

The biharmonic equation (2.4) can be written in the form

$$\nabla^2\nabla^2\Phi(r,z) = \nabla_r^2\nabla_r^2\Phi(r,z) + \frac{\partial^4}{\partial z^4}\Phi(r,z) + 2\frac{\partial^2}{\partial z^2}\nabla_r^2\Phi(r,z) = 0. \tag{3.8}$$

By substituting equations (3.6) and (3.7) in (3.8) we get

$$\begin{aligned}\gamma(z) &+ \sum_{k=1}^{\infty}\left(\phi_k{}^4\Phi_k(z) + \frac{2\alpha(z)\phi_k}{bJ_1(\phi_k b)} - \frac{2\gamma(z)}{\phi_k bJ_1(\phi_k b)}\right)J_0(\phi_k r) + \frac{d^4}{dz^4}\beta(z) + \sum_{k=1}^{\infty}\frac{d^4}{dz^4}\Phi_k(z)J_0(\phi_k r) + \\ &+ 2\frac{d^2\alpha(z)}{dz^2} - 2\sum_{k=1}^{\infty}\left(\phi_k{}^2\frac{d^2}{dz^2}\Phi_k(z) + \frac{2}{b\phi_k J_1(\phi_k b)}\frac{d^2\alpha(z)}{dz^2}\right)J_0(\phi_k r) = 0.\end{aligned} \tag{3.9}$$

For $r=b$, this equation gives

$$\gamma(z) = -\frac{d^4}{dz^4}\beta(z) - 2\frac{d^2}{dz^2}\alpha(z), \tag{3.10}$$

which can be substituted in equation (3.9) to get

$$\sum_{k=1}^{\infty}\left(\frac{d^4}{dz^4}\Phi_k(z) - 2\phi_k{}^2\frac{d^2}{dz^2}\Phi_k(z) + \phi_k{}^4\Phi_k(z) + \frac{2\phi_k}{b\phi_k{}^4J_1(\phi_k b)}\left(\frac{d^4}{dz^4}\beta(z) + \phi_k{}^2\alpha(z)\right)\right)J_0(\phi_k r) = 0. \tag{3.11}$$

and so

$$\frac{d^4}{dz^4}\Phi_k(z) - 2\phi_k{}^2\frac{d^2}{dz^2}\Phi_k(z) + \phi_k{}^4\Phi_k(z) + \frac{2\phi_k}{b\phi_k{}^4J_1(\phi_k b)}\left(\frac{d^4}{dz^4}\beta(z) + \phi_k{}^2\alpha(z)\right) = 0, \qquad (k = 1,2,..), \tag{3.12}$$

whose solutions are

$$\Phi_k(z) = (A_k + zC_k)\cosh(\phi_k z) + (B_k + zD_k)\sinh(\phi_k z) + \frac{R_k(z)}{b\phi_k^4 J_1(\phi_k b)}, \qquad (k = 1,2,..), \tag{3.13}$$

where we have set

$$R_k(z) = \int_0^z \left(\frac{d^4}{dZ^4}\beta(Z) + \phi_k{}^2\alpha(Z)\right)\left(\phi_k(Z-z)\cosh(\phi_k(Z-z)) - \sinh(\phi_k(Z-z))\right)dZ. \tag{3.14}$$

By substituting equation (3.13) in (3.1), we get an expression for the Love function:

$$\Phi(r,z) = \beta(z) + \sum_{k=1}^{\infty}\left((A_k + zC_k)\cosh(\phi_k z) + (B_k + zD_k)\sinh(\phi_k z)\right)J_0(\phi_k r) + \sum_{k=1}^{\infty}\frac{R_k(z)}{b\phi_k^4 J_1(\phi_k b)}J_0(\phi_k r). \tag{3.15}$$

The four series of coefficients A_k, B_k, C_k, D_k and the two functions $\alpha(z)$ and $\beta(z)$ must be determined by the boundary conditions of the specific elastic problem.

We explicitly write the displacement components:

$$u(r,z)=\frac{1}{2\mu}\sum_{k=1}^{\infty}\Big(\big(\phi_k A_k+z\phi_k C_k+D_k\big)\sinh(\phi_k z)+\big(\phi_k B_k+C_k+z\phi_k D_k\big)\cosh(\phi_k z)\Big)\phi_k J_1(\phi_k r)+ \\ +\frac{1}{2\mu}\sum_{k=1}^{\infty}\frac{dR_k(z)}{dz}\frac{J_1(\phi_k r)}{\phi_k^3 b J_1(\phi_k b)}, \tag{3.16}$$

$$w(r,z)=-\frac{1}{2\mu}\sum_{k=1}^{\infty}\Big(\big(\phi_k B_k-2(1-2\nu)C_k+z\phi_k D_k\big)\sinh(\phi_k z)+\big(\phi_k A_k+z\phi_k C_k-2(1-2\nu)D_k\big)\cosh(\phi_k z)\Big)\phi_k J_0(\phi_k r)+ \\ +\frac{1}{2\mu}\sum_{k=1}^{\infty}\left((1-2\nu)\frac{d^2}{dz^2}R_k(z)-2\phi_k^2(1-\nu)R_k(z)\right)\frac{J_0(\phi_k r)}{\phi_k^4 b J_1(\phi_k b)}+ \\ -\frac{(1-\nu)}{\mu}\left(\sum_{k=1}^{\infty}\frac{2J_0(\phi_k r)}{\phi_k b J_1(\phi_k b)}-1\right)\alpha(z)+\frac{(1-2\nu)}{2\mu}\frac{d^2}{dz^2}\beta(z). \tag{3.17}$$

4 EXPLICIT SOLUTION

We use this method of solution to find the explicit form for the class of solutions concerning a cylinder with assigned radial displacement on the side ($r=b$):

$$w(b,z)=0, \\ u(b,z)=\xi(z), \tag{4.1}$$

and free ends:

$$\sigma_z(r,0)=0, \qquad \sigma_z(r,h)=0, \\ \tau_{rz}(r,0)=0, \qquad \tau_{rz}(r,h)=0. \tag{4.2}$$

The condition on the transversal displacement (4.1) implies

$$\frac{d^2\beta(z)}{dz^2}=\frac{2(1-\nu)}{(2\nu-1)}\alpha(z). \tag{4.3}$$

On the other hand, the boundary conditions (4.2) give

$$\alpha(0)=0, \qquad \frac{d}{dz}\alpha(0)=0, \qquad \alpha(h)=0, \qquad \frac{d}{dz}\alpha(h)=0, \tag{4.4}$$

and the following conditions on the coefficients:

$$A_k=-\frac{2\nu}{\phi_k}D_k, \qquad C_k=-\frac{\big(F_k(h)\phi_k h-G_k(h)\big)\sinh(\phi_k h)+\phi_k h\,G_k(h)\cosh(\phi_k h)}{(1-2\nu)\Big(\sinh(\phi_k h)^2-\phi_k{}^2h^2\Big)\phi_k b\,J_1(\phi_k b)}, \\ B_k=\frac{(1-2\nu)}{\phi_k}C_k, \qquad D_k=\frac{\big(F_k(h)+G_k(h)\phi_k h\big)\sinh(\phi_k h)+F_k(h)\phi_k h\cosh(\phi_k h)}{(1-2\nu)\Big(\sinh(\phi_k h)^2-\phi_k{}^2h^2\Big)\phi_k b\,J_1(\phi_k b)}, \tag{4.5}$$

where

$$F_k(h)=\int_0^h \alpha(Z)\left(\phi_k(Z-h)\sinh\left(\phi_k(Z-h)\right)+2(1-\nu)\cosh\left(\phi_k(Z-h)\right)\right)dZ,$$
$$G_k(h)=\int_0^h \alpha(Z)\left(\phi_k(Z-h)\cosh\left(\phi_k(Z-h)\right)+(3-2\nu)\sinh\left(\phi_k(Z-h)\right)\right)dZ. \qquad (4.6)$$

The expression (3.14), by using (4.4), furnishes the Love function:

$$\Phi(r,z)=\beta(z)+\sum_{k=1}^{\infty}\left(\left(z\cosh(\phi_k z)+\frac{(1-2\nu)}{\phi_k}\sinh(\phi_k z)\right)C_k+\left(z\sinh(\phi_k z)-\frac{2\nu}{\phi_k}\cosh(\phi_k z)\right)D_k\right)J_0(\phi_k r)+$$
$$+\frac{1}{(2\nu-1)}\sum_{k=1}^{\infty}\left(\phi_k\int_0^z \alpha(Z)\cosh\left(\phi_k(Z-z)\right)(Z-z)\,dZ+(3-4\nu)\int_0^z \alpha(Z)\sinh\left(\phi_k(Z-z)\right)dZ\right)\frac{J_0(\phi_k r)}{b\phi_k^2 J_1(\phi_k b)}.$$

We remark that the Love function will be completely determined only by the specific choice of the function $\alpha(z)$ that gives the function $\beta(z)$ by using the condition (4.3).

Now we study the remaining boundary condition (4.1), namely $u(b,z)=\xi(z)$, and determine $\alpha(z)$ from $\xi(z)$. This procedure is, in general, very hard to perform and so we adopt an *inverse method*: accordingly, by considering an explicit form of the function $\alpha(z)$, verifying conditions (4.4) and eventually, depending on some parameters, we restrict our attention to a subclass of solution for the problem (4.1-2). In this way, we get the explicit form of the Love function and the radial displacement expression $\xi(z)$ for the surface side of the cylinder. In other words, for any choice of the function $\alpha(z)$, verifying conditions (4.4), we get $u(b,z)=\xi(z)$.

We now present an explicit solution, which is the simplest polynomial choice for $\alpha(z)$ satisfying conditions (4.4). We choice:

$$\alpha(z)=\frac{q}{h^5}z^2(h-z)^2, \qquad (4.8)$$

where q is a positive parameter. By using expression (4.3) we get:

$$\beta(z)=\frac{q(\nu-1)z^4}{(1-2\nu)h^5}\left(\frac{1}{15}z^2-\frac{1}{5}hz+\frac{1}{6}h^2\right). \qquad (4.9)$$

The Love function corresponding to this choice is

$$\Phi(r,z)=q\sum_{k=1}^{\infty}\left(z\tilde{D}_k+\frac{(1-2\nu)}{\phi_k}\tilde{C}_k-\frac{2(z(12+h^2\phi_k^2)+6h(1+4\nu))}{(1-2\nu)\phi_k^6 h^5 bJ_1(\phi_k b)}\right)\sinh(\phi_k z)J_0(\phi_k r)+$$
$$+q\sum_{k=1}^{\infty}\left(z\tilde{C}_k-\frac{2\nu}{\phi_k}\tilde{D}_k-\frac{4(h\phi_k^2(3z+2h\nu)+12(1+2\nu))}{(1-2\nu)\phi_k^7 h^5 bJ_1(\phi_k b)}\right)\cosh(\phi_k z)J_0(\phi_k r)-\frac{qz^2(b^2-r^2)(h-z)^2}{4h^5}+$$
$$-\frac{qz^4(1-\nu)(2z^2-6hz+5h^2)}{30h^5(1-2\nu)}+\frac{q\nu(3b^2-r^2)(b^2-r^2)(h^2-6hz+6z^2)}{16h^5(1-2\nu)}+\frac{q(1+2\nu)(b^2-r^2)(r^4-8b^2r^2+19b^4)}{96h^5(1-2\nu)},$$

where the coefficients $\tilde{C}_k$ and $\tilde{D}_k$, are:

$$\tilde{C}_k = \frac{4\left(3(1+2\nu)\phi_k h \sinh(\phi_k h) - \left(12(1+\nu)+\nu\phi_k^2 h^2\right)\cosh(\phi_k h) + \left(12(1+\nu)+(3+\nu)\phi_k^2 h^2\right)\right)}{(1-2\nu)\left(\phi_k h - \sinh(\phi_k h)\right) h^5 b \phi_k^6 J_1(\phi_k b)},$$
$$\tilde{D}_k = \frac{2\left(\left(12(3+2\nu)+(1+2\nu)\phi_k^2 h^2\right)\sinh(\phi_k h) - 12(1+\nu)\phi_k h \cosh(\phi_k h) - \left(12(2+\nu)\phi_k h + \phi_k^3 h^3\right)\right)}{(1-2\nu)\left(\phi_k h - \sinh(\phi_k h)\right) h^5 b \phi_k^6 J_1(\phi_k b)}. \tag{4.10}$$

The radial displacement component in $r=b$ is linear in q (positive) and assumes the form

$$u(b,z) = \frac{q}{2\mu}\sum_{k=1}^{\infty}\left(\left(z\phi_k \sinh(\phi_k z) + 2(1-\nu)\cosh(\phi_k z)\right)\tilde{C}_k + \left(z\phi_k \cosh(\phi_k z) + (1-2\nu)\sinh(\phi_k z)\right)\tilde{D}_k\right)\phi_k J_1(\phi_k b) +$$
$$+\frac{qb}{\mu h^2}\left(-\frac{3b^2\nu}{4(1-2\nu)h^2} + \frac{h^2(1-2\nu)-3b^2\nu}{2(1-2\nu)h^2}\frac{z}{h} + \frac{3}{2}\frac{z^2}{h^2} - \frac{z^3}{h^3}\right),$$

where $\tilde{C}_k$ and $\tilde{D}_k$ are given by (4.10). Detailed expressions for the stresses are presented in [7].

5 NUMERICAL RESULTS

We analyze the radial displacement at $r=b$ for different ratio b/h and we observe a “linear type” behavior, in the thickness z of the mantle, for the case of plate-like body (see Figure 1).

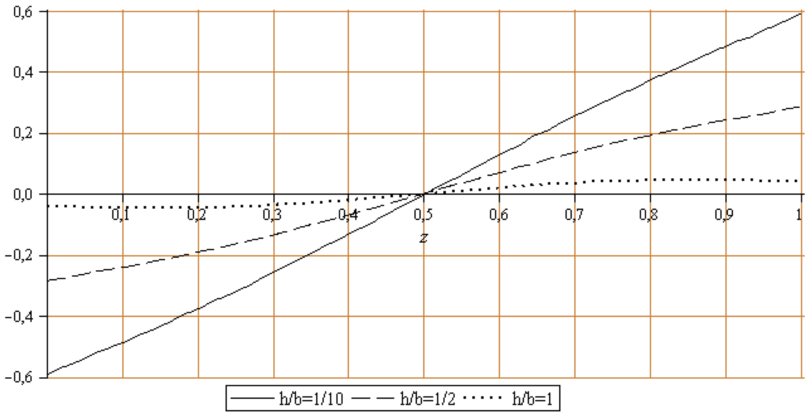

Figure 1: Radial displacement on the side $u(b,z)$ for different h/b ratios.

The expression of $u(b, z)$ can be indeed written as

$$u(b,z) = \frac{qb(1-\nu)^2}{10h^3\mu(2\nu-1)}\left(z-\frac{h}{2}\right) + O^4\left(\frac{h}{b}\right). \tag{5.1}$$

In Figure 2 and 3, we show the behavior of the radial and circumferential stresses in the thickness of the plate near the mantle (for h/b=0.1, q=1, ν=0.3).

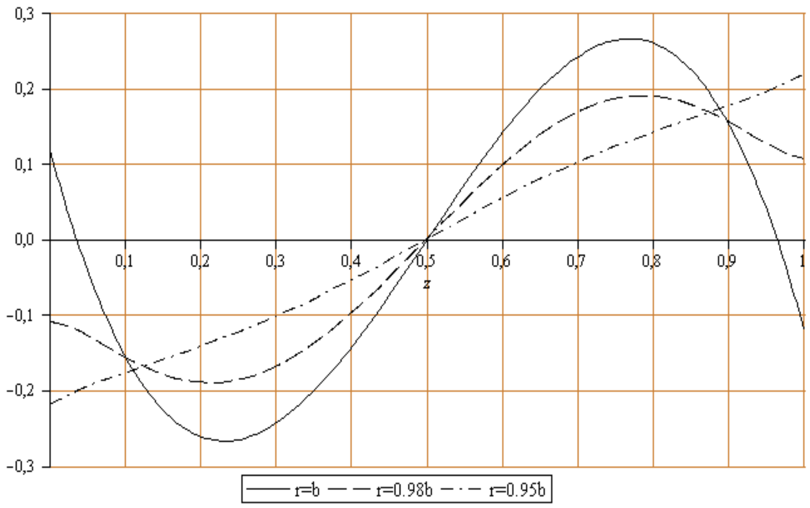

Figure 2: Radial stress $\sigma_r(r,z)$ near the mantle of the plate (for h=0.1b).

We can so observe the edge localized effects on the mantle of the plate (z=0, z=h) and that, travelling away from the side, the radial and circumferential stresses assume quickly the classical behavior, linear in z and constant in r, predicted by the structural mechanics.

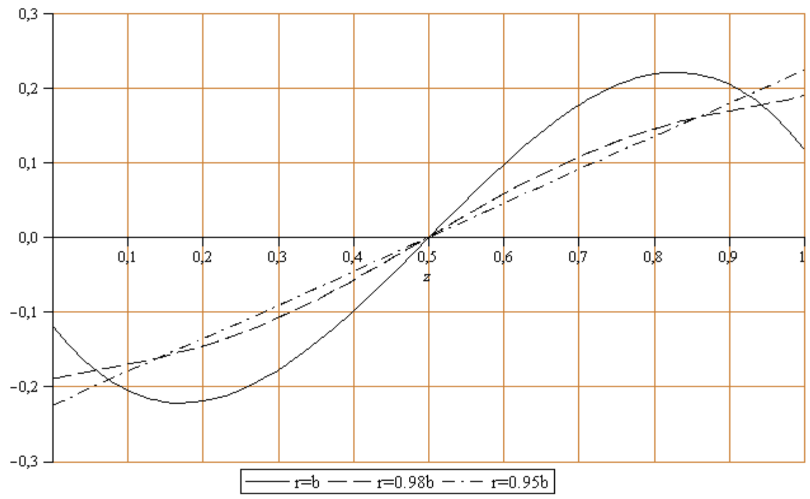

Figure 3: Circumferential stress $\sigma_\vartheta(r,z)$ near the mantle of the plate (for h=0.1b).

In Figure 4 we show that the shear stress in the thickness of the plate near the mantle presents a self equilibrated distribution in the thickness and has a maximum value in the point in which the radial stress vanishes; next, travelling away from the side, after a first region in which an increasing is present, $\sigma_{rz}(r,z)$ quickly vanishes.

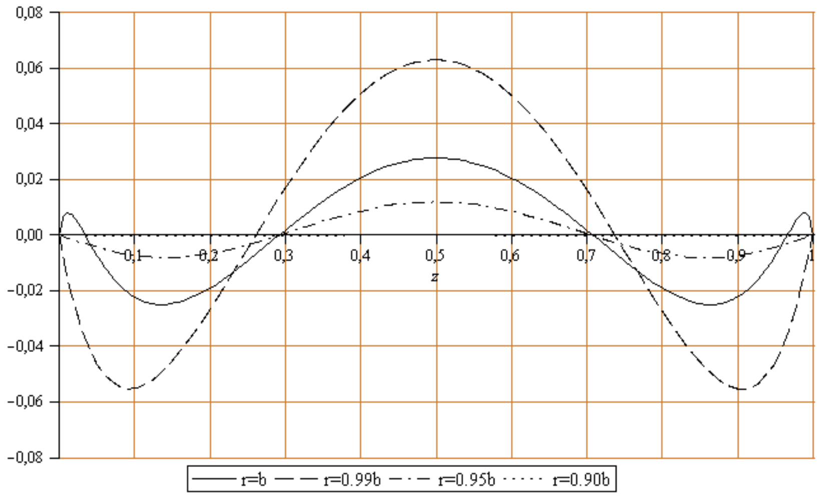

Figure 4: Shear stress $\tau_{rz}(r,z)$ near the mantle of the plate (for *h*=0.1*b*).

In Figure 5 we compare the shear stress, for *z*=*h*/2, and the radial stress, for *z*=0 and *z*=*h*, in the cases *h*/*b*=1/2 and *h*/*b*=1/10, in the region near the mantle (0.9*b*<*r*<*b*). We remark that the edge effects near the mantle for a thin plate are greater than the edge effects of a thick plate; however, they decrease more quickly as we travel away from the side.

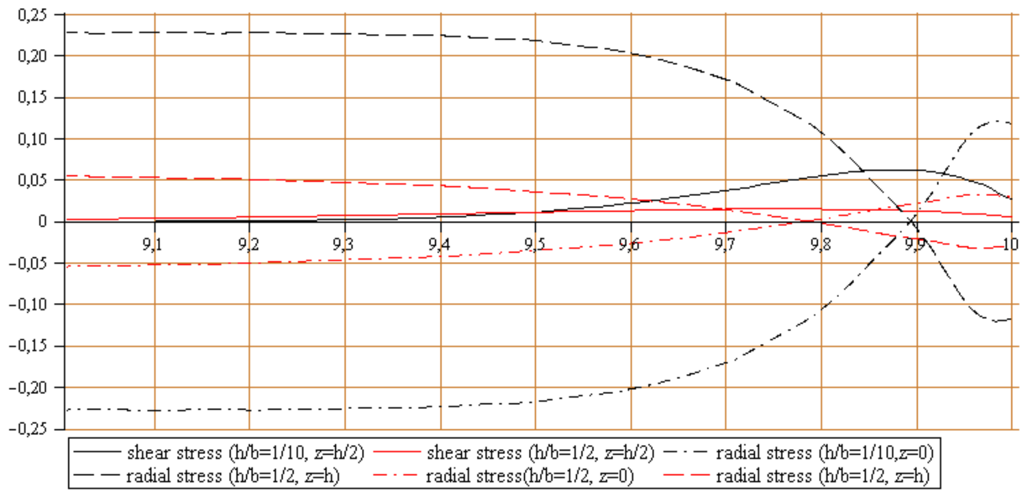

Figure 5: Plots of radial and shear stresses as a function of *r* near the mantle.

6 CONCLUSIONS

In this paper we present an analytical method used to solve the axisymmetric biharmonic problem. In order to employ the solution method, we explicitly study a thin cylinder with free loading applied on the ends and a radial displacement applied on the side. The analytical solution provides a full description of the local behavior of the stresses at the sides of type-plate cylinders.

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