Complex analysis for the solution of torsion problems: a comparison among three methods

Giorgio Barone, Antonina Pirrotta, Roberta Santoro

Dipartimento di Ingegneria Strutturale, Aerospaziale e Geotecnica, Università di Palermo, Viale delle Scienze, I-90128, Palermo.
E-mail:giorgio.barone@unipa.it,antonina.pirrotta@unipa.it, roberta.santoro@unipa.it

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SUMMARY. In this paper Saint-Venant torsion problem has been solved taking the advantage of complex analysis. In particular three methods framed into complex analysis have been compared: the Complex Variable Boundary Element Method (CVBEM), the Complex Polynomial Method (CPM) and the Line Element-less Method (LEM), the latter has been proposed very recently. The CVBEM takes advantage of the Cauchy's integral formula for the solution of Laplace equations when mixed boundary conditions on both real and imaginary parts of the complex potential are known. The CPM involves the expansion of the complex potential in Taylor series, computing the unknown coefficients by means of collocation points on the boundary. LEM expand the complex potential function by a double-ended Laurent series involving harmonic polynomials. Numerical implementation of all methods demonstrate the efficiency and accuracy of them.

1 INTRODUCTION

This paper aims at showing how complex analysis can be used for the solution of torsion problems, defining a complex analytic potential involving the warping and its harmonic conjugate function. Three numerical methods will be discussed and compared as regards: the Complex Variable Boundary Element Method (CVBEM), the Complex Polynomial Method (CPM) and the Line Element-less Method (LEM) .

The CVBEM takes advantage of the Cauchy's integral formula for the solution of Laplace equations when mixed boundary conditions on both real and imaginary parts of the complex potential are known. The CPM involves the expansion of the complex potential in Taylor series, computing the unknown coefficients by means of collocation points on the boundary. The order of the truncation of the series is strictly related to the number on chosen nodes, related to the discretization of the domain. Even though the method is characterized by a remarkable computational efficiency, it can be applied only for convex simply connected domains. Recently it has been developed the Line Element-less Method, a numerical method for finding an approximate distribution of the shear stresses in Saint-Venant cylinder. The method is based on the definition of a novel complex potential function involving directly shear stresses, that has been expanded in Laurent series. The unknown coefficients of the series can be evaluated requiring that the square of the net flux across the boundary of the cross-section is minimum under the static equivalence relation. The minimization of the so defined functional involves a system of linear algebric equations with symmetric and positive definite matrix. In this paper LEM was applied to a classical complex potential function, related to the warping function and its harmonic conjugate function, in order to have a straight comparison of the results obtained by CVBEM and CPM that were applied to this potential function. Also by using this classical potential function, all integrals are conducted to line integrals avoiding any discretization procedure of the domain or the boundary. The LEM gives directly a complete description of the shear stresses vector field, twist angle and torsional stiffness factor, while other boundary method involving complex analysis, the CVBEM and CPM, return results only in terms of Prandtl function (apart from the constant value $G\theta$). Moreover, for these last two methods, domain integrals are required to evaluate the torsional stiffness factor, and then the static equivalence have to be considered to determine the effective distribution of the shear stresses.

2 THEORETICAL BACKGROUND

Let us consider a linearly elastic and isotropic De Saint-Venant cylinder of length L and arbitrary cross section A twisted by a moment $M_z(L)$ applied at its end. Cylinder is referred to a counter-clockwise coordinate system with x and y axes coincident as customary with the principal axes of inertia of the cross section.

Stress field in Saint-Venant cylinder is completely defined by

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \;; \tau_{xz} = G\theta\left(\frac{\partial\omega}{\partial x} - y\right) \;;\; \tau_{yz} = G\theta\left(\frac{\partial\omega}{\partial y} + x\right) \tag{1}$$

where $\omega(x,y)$ represents the *warping function*, G the shear modulus of the material and θ the unitary twist angle. The general three-dimensional equilibrium equations of elasticity, with no body forces and particularized through (1), are given by:

$$\frac{\partial\tau_{zx}}{\partial z} = 0 \quad ; \quad \frac{\partial\tau_{zy}}{\partial z} = 0 \quad ; \quad \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} = 0 \qquad \text{in A} \tag{2}$$

in addition the stress components should satisfy the free-traction boundary conditions on the surface C of the bar, namely:

$$\tau_{zx} n_x + \tau_{zy} n_y = \boldsymbol{\tau}^{\mathrm{T}}\mathbf{n} = 0 \qquad \text{on } C \tag{3}$$

where $\mathbf{n}^{\mathrm{T}} = [n_x, n_y]$ is the outward normal vector to the external surface at any point on C.

In terms of the warping function , $\omega(x,y)$ the latter may be determined by solving the following problem

$$\nabla^2\omega = 0 \quad in\, A; \quad \frac{\partial\omega}{\partial n} = y n_x - x n_y \;\; on\, C \tag{4}$$

known as a Neumann problem, being $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Alternatively we can have a Dirichlet boundary value problem, through the so-called Prandtl stress function $\psi(x,y)$

$$\nabla^2\psi(x,y) = -2G\theta \qquad \text{in } A; \quad \psi(x,y) = const \qquad \text{on } C \tag{5}$$

being

$$\tau_{zx} = \frac{\partial \psi}{\partial y}; \quad \tau_{zy} = -\frac{\partial \psi}{\partial x} \tag{6}$$

Moreover the static equivalence condition on the cross-section is defined as:

$$M_z = \int_A \boldsymbol{\tau}^{\mathrm{T}} \mathbf{g} dA = 2\int_A \psi(x,y)\, dA = G\theta J_t \tag{7}$$

where $\mathbf{g}^T = [-y \quad x]$; J_t is usually referred to as torsional stiffness factor.

3 COMPLEX POTENTIAL FUNCTION FORMULATION

The elastic equilibrium problem for Saint-Venant cylinder may be framed into complex analysis introducing a complex analytic function $U(\hat{z})$ of the complex variable $\hat{z} = x + iy$ defined in the cross section domain[1].

$$U(\hat{z}) = \omega(x,y) + i\varphi(x,y) \tag{8}$$

where $\omega(x,y)$ is the previously defined warping function and $\varphi(x,y)$ its harmonic conjugate, related to the latter by Cauchy-Riemann conditions:

$$\frac{\partial \omega}{\partial x} = \frac{\partial \varphi}{\partial y} \quad ; \quad \frac{\partial \omega}{\partial y} = -\frac{\partial \varphi}{\partial x} \tag{9}$$

The torsion function $U(\hat{z})$ contains information on both displacements and stresses. In fact, $\varphi(x,y)$ is related to Prandtl function by:

$$\psi(x,y) = G\theta\left(\varphi(x,y) - \frac{1}{2}\left(x^2 + y^2\right)\right) \tag{10}$$

According to the previous relation between $\varphi(x,y)$ and $\psi(x,y)$, the Dirichlet problem in eq.(5) assumes the following form:

$$\nabla^2 \varphi(x,y) = 0 \qquad \text{in } A; \qquad \varphi(x,y) = \frac{1}{2}\left(x^2 + y^2\right) \qquad \text{on } C \tag{11}$$

For both Neumann and Dirichlet problems exact solutions are available only for a restricted class of cross-sectional geometries namely elliptical, circular and equilateral triangular cross-sections. When dealing with domains of a complex shape, numerical methods such as the Boundary Difference Method, and the Finite Element Method, are widely used. However, both methods need a discretization of the whole domain very time consuming and with this respect Boundary Element Methods make it possible to restrict oneself to the discretization of only the boundary domain. In this context the Complex Polynomial Method (CPM), the Complex Variable Boundary Element Method (CVBEM) and the Line Element-less Method (LEM) can be considered efficient tools for the numerical analysis of the Laplace equation solution improving computational facility by use of complex analysis.

3.1 *The CVBEM for torsion problem*

The CVBEM has been developed by Hromadka [2] for the solution of general problems involving Laplace or Poisson equations. Let A be a simply connected domain with a piecewise continuously differentiable boundary C, which is a simple closed curve of finite length.

Chosen n nodes $\hat{z}_k$ on the contour C, the following approximate potential function may be defined:

$$\breve{U}(\hat{z}) = \alpha + \alpha_0 \hat{z} + \sum_{k=1}^{n} \alpha_k (\hat{z} - \hat{z}_k) \log_{P_{\hat{z}_k}} (\hat{z} - \hat{z}_k) \tag{12}$$

where $\alpha_k = a_k + ib_k \quad (a_k, b_k \in C)$ are complex unknown coefficients. There are different versions of CVBEM, but the widely used is the latter one. Moreover, let us suppose that only boundary values for the imaginary part $\varphi(\hat{z})$ of the complex potential are known. Then the function $\varphi(\hat{z})$ can be expressed, by (12), as:

$$\breve{\varphi}(\hat{z}) = b + a_0 \operatorname{Im}[\hat{z}] + b_0 \operatorname{Re}[\hat{z}] + \sum_{k=1}^{n} \left(a_k \operatorname{Im}\left[f_{\hat{z}_k}(\hat{z}) \right] + b_k \operatorname{Re}\left[f_{\hat{z}_k}(\hat{z}) \right] \right) \tag{13}$$

The unknown coefficients a, a_k and b_k may be found solving a system obtained imposing the known boundary values (11 b) $\breve{\varphi}(\hat{z}_j) = \varphi_j$, with $j = 1, 2, ..., 3n+5$, where the evaluation points $\hat{z}_j$ are a new set of equally spaced points chosen on the contour of the domain.

It is worth to note that, although the CVBEM belongs to the category of the boundary element methods, the complete solution of the torsion problem can be achieved only by solving a domain integral.

3.2 *The CPM for torsion problem*

The Complex Polynomial Method has been proposed by Hromadka and Guymon [3] for the evaluation, in a simple connected domain A, of a complex analytic potential function, whose values on the boundary C are known. In particular, the torsion problem for a simply connected domain can be solved defining the potential function in eq. (8) that can be expressed in polynomial form. Taking into account only its imaginary part and using the polar coordinates $x = \rho \cos \vartheta$ and $y = \rho \sin \vartheta$, the function $\varphi(x, y)$ can be written in the following form:

$$\varphi(\rho, \vartheta) = b_0 + \sum_{k=1}^{r} a_k \rho^k \sin k\vartheta + b_k \rho^k \cos k\vartheta \tag{14}$$

where a_k and b_k may be determined imposing the known associate boundary condition, in terms $\varphi(\rho,\vartheta)=\rho^2/2$. Analogously to the CVBEM case, the torsional stiffness factor may be evaluated only by means of the domain integral, that is also for this method boundary conditions alone are not sufficient to complete determine the torsion problem solution.

3.3 *The LEM for torsion problem*

Recently Di Paola et al [4] introduced the Line Element-less Method (LEM), where the potential function defined in (8), being analytic in all the domain A, may be expanded in the double-ended Laurent series as:

$$U(\hat{z})=\sum_{k=-\infty}^{+\infty}\alpha_k\left(\hat{z}-\hat{z}_0\right)^k;\quad \alpha_k,\hat{z}_0\in C \tag{15}$$

where $\left(\hat{z}-\hat{z}_0\right)^k$ with $k>0$ is analytic in the whole domain, while for $k<0$ it is analytic with exception of the point $\hat{z}_0$, that is called a pole of order k. In eq.(15) the series $\sum_{k=0}^{+\infty}\alpha_k\left(\hat{z}-\hat{z}_0\right)^k$ is called *regular part* and it is capable to express any analytic function everywhere. While the summation $\sum_{k=-\infty}^{-1}\alpha_k\left(\hat{z}-\hat{z}_0\right)^k$ is called *principal part* and accounts for singularities in $\hat{z}_0$. It follows that if the function $U(\hat{z})$ is analytic everywhere (including the boundary and $\hat{z}_0$) then only the regular part will be accounted for. Generally powers $(\hat{z})^k$ are denoted as P_k+iQ_k where P_k and Q_k are the so-called harmonic polynomials $(\nabla^2P_k=0\,,\nabla^2Q_k=0\quad\forall k)$, defined as follows:

$$P_k(x,y)=\mathrm{Re}(x+iy)^k;\quad Q_k(x,y)=\mathrm{Im}(x+iy)^k \tag{16}$$

and they may be evaluated recursively as

$$P_k(x,y)=P_{k-1}x-Q_{k-1}y;\quad Q_k(x,y)=Q_{k-1}x+P_{k-1}y;\quad k>0 \tag{17}$$

$$P_{-k}(x,y)=\frac{P_k(x,y)}{P_k^2(x,y)+Q_k^2(x,y)};\,Q_{-k}(x,y)=-\frac{Q_k(x,y)}{P_k^2(x,y)+Q_k^2(x,y)}\quad k>0 \tag{18}$$

with $P_0=1, Q_0=0, P_1=x, Q_1=y.$
The derivatives of the harmonic polynomials are ruled by the following recursive properties:

$$\frac{\partial P_k}{\partial x}=kP_{k-1};\quad \frac{\partial P_k}{\partial y}=-kQ_{k-1};\quad \frac{\partial Q_k}{\partial x}=kQ_{k-1};\quad \frac{\partial Q_k}{\partial y}=kP_{k-1};\quad\forall k \tag{19}$$

By letting $\alpha_k=a_k+ib_k\quad(a_k,b_k\in C)$ in eq.(15) and assuming $\hat{z}_0=0$ the complex potential function is now expanded in terms of harmonic polynomials in the form:

$$U(\hat{z}) = \omega(x,y) + i\varphi(x,y) = \sum_{k=-\infty}^{+\infty} (a_k P_k - b_k Q_k) + i \sum_{k=-\infty}^{+\infty} (a_k Q_k + b_k P_k) \tag{20}$$

If the initial point $\hat{z}_0$ of the Laurent series is selected as different from zero, then it is enough to define P_k and Q_k as $P_k = \mathrm{Re}\left[(x-x_0)+i(y-y_0)\right]^k$ and $Q_k = \mathrm{Im}\left[(x-x_0)+i(y-y_0)\right]^k$ and the ensuing derivations don't change.

This means that, taking into account the expression (8), the Prandtl function $\psi(x,y)$, being defined unless a constant, can be written in terms of harmonic polynomials as:

$$\psi(x,y) = G\theta \left[\sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} (a_k Q_k + b_k P_k) - \frac{1}{2}(x^2+y^2) + a_0 \right] \tag{21}$$

Due to the relations (6) between $\boldsymbol{\tau}(x,y)$ and $\psi(x,y)$ an approximation for the shear stresses $\tau_{zx}(x,y)$ and $\tau_{zy}(x,y)$ can be expressed in terms of truncated Laurent series of harmonic polynomials as follows:

$$\tau_{zx} = G\theta \left[\sum_{\substack{k=-r_1 \\ k\neq 0}}^{+r_2} (a_k k P_{k-1} - b_k k Q_{k-1}) - y \right]; \tau_{zy} = -G\theta \left[\sum_{\substack{k=-r_1 \\ k\neq 0}}^{+r_2} (a_k k Q_{k-1} + b_k k P_{k-1}) - x \right] \tag{22}$$

By letting:

$$\mathbf{p}(x,y) = \begin{vmatrix} -r_1 P_{-r_1-1}(x,y) \\ \vdots \\ -P_{-2}(x,y) \\ P_0(x,y) \\ \vdots \\ r_2 P_{r_2-1}(x,y) \end{vmatrix}; \quad \mathbf{q}(x,y) = \begin{vmatrix} -r_1 Q_{-r_1-1}(x,y) \\ \vdots \\ -Q_{-2}(x,y) \\ Q_0(x,y) \\ \vdots \\ r_2 Q_{r_2-1}(x,y) \end{vmatrix}; \quad \mathbf{a} = \begin{vmatrix} a_{-r_1} \\ \vdots \\ a_{-1} \\ a_1 \\ \vdots \\ a_{r_2} \end{vmatrix}; \quad \mathbf{b} = \begin{vmatrix} b_{-r_1} \\ \vdots \\ b_{-1} \\ b_1 \\ \vdots \\ b_{r_2} \end{vmatrix}; \tag{23}$$

Eqs.(22) can be expressed as:

$$\boldsymbol{\tau}(x,y) = G\theta \left[\mathbf{D}(x,y) + \mathbf{g} \right] \tag{24}$$

where:

$$\mathbf{D}(x,y) = \begin{vmatrix} \mathbf{p}^T & -\mathbf{q}^T \\ -\mathbf{q}^T & -\mathbf{p}^T \end{vmatrix}; \quad \mathbf{w} = \begin{vmatrix} \mathbf{a} \\ \mathbf{b} \end{vmatrix}; \quad \mathbf{g} = \begin{vmatrix} -y \\ x \end{vmatrix}; \tag{25}$$

As apparent eq.(24) completely fulfils equilibrium and compatibility equations in the domain, but, since the series in equations (22) are truncated retaining the first $r_1 + r_2$ terms, equations (24) may not fulfil in each point of the contour C the free-stress boundary conditions (3).

In order to satisfy the latter, it has been proposed that the unknown coefficients $\mathbf{w}$ are evaluated by introducing the following functional:

$$\Im(\mathbf{w},\theta)=\oint_C\left(\boldsymbol{\tau}^T\mathbf{n}\right)^2 dC=\oint_C\left(\tau_{zx}n_x+\tau_{zy}n_y\right)^2 dC \tag{26}$$

that is the squared value net flux of the shear stress vector $\boldsymbol{\tau}$ through the boundary of the domain, and minimizing it with respect to $\mathbf{w}$ and θ under the static equivalence condition (7):

$$\Im(\mathbf{w},\theta)=\oint_C\left(\boldsymbol{\tau}^T\mathbf{n}\right)^2 dC=\min_{\mathbf{w},\theta} \quad \text{subjected to} \quad \int_A \boldsymbol{\tau}^T\mathbf{g}dA=M_z \tag{27}$$

At this step another novel point, introduced in [4], is that surface integrals in equation (27) are converted into line integrals avoiding any discretization of the inner domain. In order to do this, taking into account relations (19) between harmonic polynomials and their derivatives, the static equivalence condition may be rewritten in terms of harmonic polynomials as:

$$G\theta\sum_{\substack{k=-r_1\\k\neq 0}}^{k=r_2}\left[a_k\int_A\left(\frac{\partial P_k}{\partial y}x-\frac{\partial P_k}{\partial x}y\right)dA+b_k\int_A\left(-\frac{\partial Q_k}{\partial y}x+\frac{\partial Q_k}{\partial x}y\right)dA\right]+G\theta I_p=M_z \tag{28}$$

where I_p represents the polar inertia moment. It is straightforward that equation (28) may be written in the form:

$$G\theta\sum_{\substack{k=-r_1\\k\neq 0}}^{k=r_2}\left[a_k\int_A \operatorname{div}\mathbf{u}_k dA+b_k\int_A \operatorname{div}\mathbf{v}_k dA\right]+G\theta I_p=M_z \tag{29}$$

where $\mathbf{u}_k$ and $\mathbf{v}_k$ are given as:

$$\mathbf{u}_k=\begin{vmatrix}-yP_k(x,y)\\ xP_k(x,y)\end{vmatrix};\quad \mathbf{v}_k=\begin{vmatrix}yQ_k(x,y)\\ -xQ_k(x,y)\end{vmatrix}; \tag{30}$$

Then, according to the Green lemma, the integrals in equation (29) may be performed as line boundary integrals as follows:

$$G\theta\sum_{\substack{k=-r_1\\k\neq 0}}^{k=r_2}\left[a_kc_k+b_kd_k\right]+G\theta I_p=M_z \tag{31}$$

where: $c_k=\oint_C \mathbf{u}_k^T\mathbf{n}dC;\quad d_k=\oint_C \mathbf{v}_k^T\mathbf{n}dC$. By collecting c_k and d_k in the following vector $\mathbf{h}$ as:

$$\mathbf{h}^T=\begin{vmatrix}c_{-r_1} & \cdots & c_{-2} & c_0 & \cdots & c_{r_2} & d_{-r_1} & \cdots & d_{-2} & d_0 & \cdots & d_{r_2}\end{vmatrix} \tag{32}$$

equation (31) can be expressed in a compact form as follows: $G\theta\mathbf{h}^T\mathbf{w}+G\theta I_p=M_z$

Then the constrained minimum problem described in equations (27) may be solved by using the Lagrange multiplier method by performing the minimization of the free enlarged functional expressed in the following form:

$$\Im(\mathbf{w},\theta,\lambda)=\oint_C\left(\boldsymbol{\tau}^T\mathbf{n}\right)^2 dC+\lambda\left(G\theta\mathbf{h}^T\mathbf{w}+G\theta I_p-M_z\right)=\min_{\mathbf{w},\theta,\lambda} \tag{33}$$

where λ is the Lagrange multiplier. Solving the system provided by the minimization, the unknown series coefficients $\mathbf{w}$,as well as the twist angle θ may be evaluated, that is a complete torsion problem is determined in terms of Prandtl function from equation (21), shear stress field from equation (24), and warping function and its harmonic conjugate as real and imaginary part of the potential function from equation (20). For simply connected regions the contour C is the external one and the series starts from $k=1$, that is only the regular part of the Laurent series is accounted for. For multiply-connected regions the various line integrals are simply performed on the summation of the external and internal contours; hence, as previously pointed out, also the principal part of the Laurent series is needed and the poles $\hat{z}_0$ must be opportunely selected.

It is worth to note that, in spite of CPM and CVBEM, the LEM returns directly the value of the twist angle θ, so that the torsional stiffness, defined in equation (7), may be simply evaluated as:

$J_t = M_z/G\theta$ without performing any integral on the domain as required from the previous two methods. Moreover it has to be emphasized that the torsion problem has been solved without requiring any discretization of the boundary or the domain, so that the LEM can be really considered a truly no-mesh method.

4 NUMERICAL APPLICATION

In this section some applications on simply connected cross-sections will be reported in order to compare accuracy and efficiency of the three previously described methods. Proper algorithms based on the theories described in the previous sections have been developed using Fortran and Mathematica 6.0 environments. The cases whose exact solution is known [5], namely circular, elliptic and equilateral triangular domains have been performed. Firstly, results provided by CPM and CVBEM are reported in terms of torsional stiffness factor J_t (Table 1) for different set of nodes. Then results by LEM on the same cross-sections are reported in Table 2, specifying the terms of unknown series coefficients, Lagrange multiplier λ, shear stresses distributions τ_{zx} and τ_{zy} , Prandtl $\psi(x,y)$ and warping $\omega(x,y)$ functions and torsional stiffness factor J_t for the elliptic and equilateral triangular cross-sections. As shown in Table 2, all the Laurent series coefficients are equal to zero (no matter the value of r), except for a_2 for the elliptical cross-section and a_3 for the triangular one. It is worth to note that for the circular cross-section all the series coefficients are equal to zero, then according to equation (22) $\tau_{zx}=-G\theta y$ and $\tau_{zy}=G\theta x$ with $\theta=M_z/GI_p$.

It has to be emphasized that the method is robust in the sense that it exactly reproduces solutions for all the cases in which the analytical solution is already known without requiring any discretization neither on the domain nor its boundary, and in this case the boundary condition is satisfied continuously on the contour domain.

Table 1) Computed values of torsional stiffness factor J_t for various values of number n of nodes

n	circle (R=1)		ellipse (a=2, b=1)		triangle (a=1)	
	CPM	CVBEM	CPM	CVBEM	CPM	CVBEM
15	1.5708	1.5708	5.02656	5.02907	3.11769	3.11885
33	1.5708	1.5708	5.02656	5.02701	3.11769	3.11883
75	1.5708	1.5708	5.02656	5.02667	3.11769	3.11736

Table 2) Results by LEM for shear stresses for elliptical and equilateral triangular cross-sections

Cross-section	ellipse	triangle
Series coefficients $\neq 0$	$a_2 = \frac{\left(a^2 - b^2\right)}{2\left(a^2 + b^2\right)}$	$a_3 = \frac{1}{6a}$
Lagrange multiplier λ	0	0
Unitary twist angle θ	$\frac{\left(a^2 + b^2\right)M_z}{\pi G a^3 b^3}$	$\frac{5M_z}{9\sqrt{3}a^4 G}$
Shear stresses $\tau_{zx}\left(x,y\right)$	$-\frac{2M_z}{\pi a b^3}y$	$\frac{G\theta}{2a}\left(x^2 - y^2 - 2ay\right)$
Shear stresses $\tau_{zy}\left(x,y\right)$	$\frac{2M_z}{\pi a^3 b}x$	$\frac{G\theta}{a}\left(a - y\right)x$
Prandtl function $\psi\left(x,y\right)$	$-\frac{\left(x^2 b^2 + y^2 a^2\right)M_z}{\pi a^3 b^3} + const$	$\frac{G\theta}{6a}\left(3x^2 y - y^3 - 3ax^2 - 3ay^2\right) + const$
Warping function $\omega\left(x,y\right)$	$\frac{-a^2 + b^2}{a^2 + b^2}xy$	$\frac{x^3 - 3xy^2}{6a}$
Torsional stiffness factor J_t	$\frac{\pi a^3 b^3}{a^2 + b^2}$	$\frac{9\sqrt{3}a^4}{5}$

Moreover, regarding a case whose exact solution is not known, a cross-sections with the form of hypocycloid is reported. In Fig. 1a results in terms of stress lines corresponding to the contour lines of the Prandtl function $\psi\left(x,y\right)$ obtained by LEM with r=18 are depicted, while in Fig. 1b results obtained by CVBEM, that are almost the same with those obtained by CPM, are reported in terms of $\psi\left(x,y\right)/G\theta$ having considered a boundary discretization using 50 nodes.

5 CONCLUSIONS

In this paper a comparison among three numerical methods for the solution of torsion problems in Saint-Venant cylinder of an arbitrary, but uniform, cross section, has been proposed. In this context the Complex Polynomial Method (CPM), the Complex Variable Boundary Element Method (CVBEM) and the Line Element-less Method (LEM) can be considered efficient tools for the numerical analysis of a torsion problem, improving computational facility by use of complex analysis. However, LEM overcomes the limits of the CPM, in the sense that while the CPM can be applied only to convex simply connected domains, LEM can be applied both for simply and multi-connected domains, with the introduction in the last case of the principal part of the Laurent series. Moreover, the LEM allows to set the order of truncation of the Laurent series a priori, while for the CPM the same order is related to the number of nodes, that is to the boundary discretization.

The applicability to multi-connected domains result to be a limit for the CVBEM as well, that has the only advantage to be able to solve problems for cross-sections with reentrant angles. For these cases the LEM can still be applied replacing the reentrant angles with circular joints, and introducing opportunely singularity points in the complex potential function.

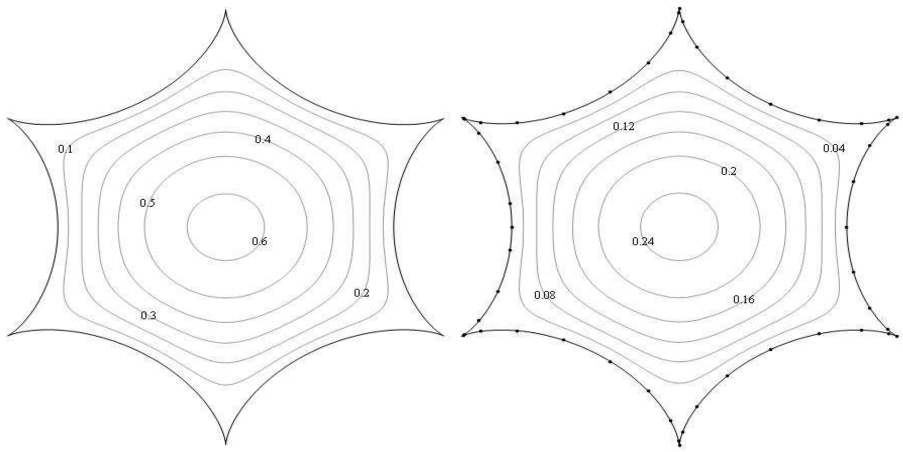

Figure 1:a) contour lines of the Prandtl function $\psi(x, y)$ obtained by LEM with r=18; b) $\psi(x, y)/G\theta$ by CVBEM and by CPM using 50 nodes.

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