# Invisibility to in-plane elastic waves

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SUMMARY.

In the present work we propose a cylindrical cloak for elastic waves, which has been designed for fully-coupled in-plane shear and pressure waves. The cloaking device is a functionally graded material where the elastic properties are deduced from coordinate transformation [A. Greenleaf et al. *Maths. Res. Lett.* **10**,1 (2003), Pendry et al., Science **312**, 1780 (2006)]. The main property of the proposed model lies in the fact that the chosen reparameterization is such that the elasticity equations maintain their initial form under the transformation, which is generally untrue in the elasticity case [Milton et al., New J. Phys. **8**, 248 (2006)].

Numerical checks that freely vibrating obstacles located inside the neutral region are cloaked disrespectful of the frequency and the polarization of the incoming elastic wave validate the model.

## 1 INTRODUCTION

*Invisibility* has been a source of fascination and an inspiration of myths, novels and films, from the mythical magical artifact Ring of Gyges mentioned by the philosopher Plato in *The Republic* to the Cheshire Cat from *Alice's Adventures in Wonderland* and the ships in the *Star Trek* universe equipped with hardware known as cloaking devices that conceal them from most varieties of scans. Recently invisibility turned from a device of fiction into a subject of science, in particular significant progress has been made on the control of acoustic and electromagnetic waves. Transformation based solutions to the conductivity and Maxwell's equations in curvilinear coordinate systems, subsequently reported by Greenleaf *et al.* [1] and then by Pendry *et al.* [2] and Leonhardt [3], enable one to bend electromagnetic waves around arbitrarily sized and shaped solids. More precisely, the electromagnetic invisibility cloak is a metamaterial which maps a concealment region into a surrounding shell: as a result of the coordinate transformation the permittivity and permeability are strongly heterogeneous and anisotropic within the cloak, yet fulfilling impedance matching with the surrounding vacuum. The cloak thus neither scatter waves nor induces a shadow in the transmitted field. In [4], a cylindrical electromagnetic cloak constructed using specially designed concentric arrays of split ring resonators, was shown to conceal a copper cylinder around $8.5$ GHz. Other routes to invisibility include reduction of backscatter [5] and cloaking through anomalous localized resonances, the latter one using negative refraction [6]. To date, a plethora of research papers has been published in the fast growing field of transformation optics.

In the case of elastodynamic waves and structural mechanics, transformation based invisibility cloaks received less attention, since the Navier equations are in general variant under geometric changes [7, 8]. For cylindrical geometries the problem is slightly simplified since out-of-plane shear waves decouple from in-plane waves; however, in-plane shear and pressure waves remain inherently coupled. Neutral elastic inclusions were proposed in the past using asymptotic and computational methods to find suitable material parameters for coated cylindrical inclusions [9]. The latter has proved successful in the elastostatic context in the case of anti-plane shear and in-plane coupled pressure and shear polarizations. However, neutrality breaks down for finite frequencies.

Restricting the analysis to acoustic waves in a fluid, the equations of motion undergo the same geometric transform as electromagnetic waves do and therefore retain their form [7, 10]. This result has been since then generalized to three-dimensional acoustic cloaks for pressure waves [11, 12]. Importantly, such cloaks require an anisotropic mass density which can be obtained via a homogenization approach, which presents the advantage to be broadband. Acoustic cloaking for linear surface water waves was chiefly achieved via the same mechanism in between $10$ and $15$ Hertz [13].

In the present work, we show that it is also possible to design a cylindrical cloak for in-plane coupled pressure and shear elastic waves. We demonstrate theoretically its unique mechanism and further perform finite element computations checked again analytical calculations of the Green's function for the Navier equations in transformed coordinates. The main difference with previous work [6] is that our elasticity tensor in the transformed coordinates is no longer symmetric, which is a necessary condition for the Navier equations to retain their form. Quite remarkably, we find that the density remains a scalar quantity in the transformed coordinates.

## 2 EQUATIONS OF MOTION

We consider time-harmonic propagation of in-plane elastic waves, the problem is described by the Navier equations

$$\nabla \cdot \mathbb{C} : \nabla \mathbf{u} + \rho \omega^2 \mathbf{u} + \mathbf{b} = \mathbf{0} \ , \tag{1}$$

where $\mathbf{u}$ is the displacement, $\rho$ the density, $\mathbb{C}$ the $4^{\text{th}}$-order constitutive tensor of the linear elastic material and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ represents the spatial distribution of a simple harmonic body force $\hat{\mathbf{b}}(\mathbf{x}, t) = \mathbf{b}(\mathbf{x}) \exp(i\omega t)$, with $\omega$ the wave-frequency and $t$ the time.

## 3 TRANSFORMED EQUATIONS OF MOTION

In fig. 1, we consider the following coordinate transformation $(r, \theta) \to (r', \theta')$ of [1, 2]

$$\begin{cases} r' = r_0 + \frac{r_1 - r_0}{r_1} r \ , \ \theta' = \theta \ , & \text{for } r \le r1 \\ r' = r \ , \ \theta' = \theta \ , & \text{for } r > r1. \end{cases} \tag{2}$$

The geometric transform is expressed in cylindrical coordinates $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = 2\text{atan}(x_2/(x_1 + \sqrt{x_1^2 + x_2^2}))$, with $r_0$ and $r_1$ the inner and outer radii of the circular cloak, respectively.

As a result of the transformation (2), in the region $r' \in [r_0, r_1]$ the Navier equations (1) are mapped into the equations

$$\nabla \cdot \mathbb{C}' : \nabla \mathbf{u} + \rho' \omega^2 \mathbf{u} = \mathbf{0} \ , \tag{3}$$

where the support of the body force is outside the ring. The stretched density is

$$\rho' = \frac{r - r_0}{r} \left(\frac{r_1}{r_1 - r_0}\right)^2 \rho, \tag{4}$$

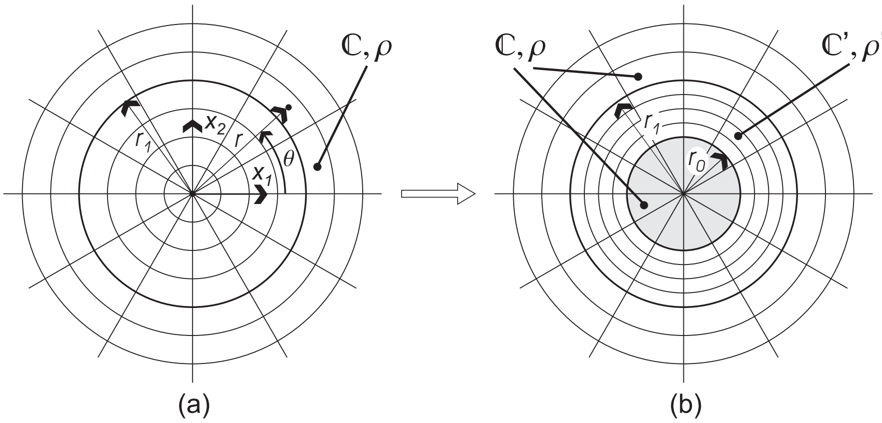


Figure 1: Geometric transform of eqn.(2) from $(r, \theta)$ (Fig.1(a)) to $(r', \theta')$ (Fig.1(b)); $r_0$ and $r_1$ are the inner and the outer radius of the cylindrical cloak, respectively. The elastic constitutive tensor and the density in the undeformed and in the deformed domains are denote by $\mathbb{C}$, $\rho$ and $\mathbb{C}'$, $\rho'$, respectively.

and the elasticity tensor $\mathbb{C}'$ has non zero cylindrical components

$$\begin{array}{ll} \mathbb{C}'_{rrrr} = \frac{r-r_0}{r}(\lambda + 2\mu), & \mathbb{C}'_{\theta\theta\theta\theta} = \frac{r}{r-r_0}(\lambda + 2\mu), \\ \mathbb{C}'_{rr\theta\theta} = \mathbb{C}'_{\theta\theta rr} = \lambda, & \mathbb{C}'_{r\theta\theta r} = \mathbb{C}'_{\theta rr\theta} = \mu, \\ \mathbb{C}'_{r\theta r\theta} = \frac{r-r_0}{r}\mu, & \mathbb{C}'_{\theta r\theta r} = \frac{r}{r-r_0}\mu, \end{array} \tag{5}$$

with $\lambda$ and $\mu$ the Lamé moduli characterizing the isotropic behavior described by $\mathbb{C}$. Note that $\mathbb{C}'$ has not the minor symmetries.

It is important to stress the fact that the transformation (2) preserves the isotropy of the density, which remains a scalar (yet spatially varying) quantity in (3), and the stress $\sigma = \mathbb{C}' : \nabla\mathbf{u}$ depends directly only on the displacement gradient and not on the displacement $\mathbf{u}$. This is a very unlikely situation for elastodynamic waves propagating in anisotropic heterogeneous media [7]. We also note that the proposed formulation poses no limitations on the applied $\omega$ ranging from low to high frequency, as the elasticity tensor and the density do not depend upon $\omega$.

## 4 MATCHING OF ELASTIC IMPEDANCE

At the cloak outer boundary, namely for $r = r_1$, the geometric transform (2) provides $r' = r_1$. Therefore on this interface the transformed density is

$$\rho' = \frac{r_1}{r_1 - r_0}\rho, \tag{6}$$

and the following transformed cylindrical components of the elasticity tensor are

$$\mathbb{C}'_{rrrr} = \frac{r_1 - r_0}{r_1}(\lambda + 2\mu), \qquad \mathbb{C}'_{r\theta r\theta} = \frac{r_1 - r_0}{r}\mu. \tag{7}$$

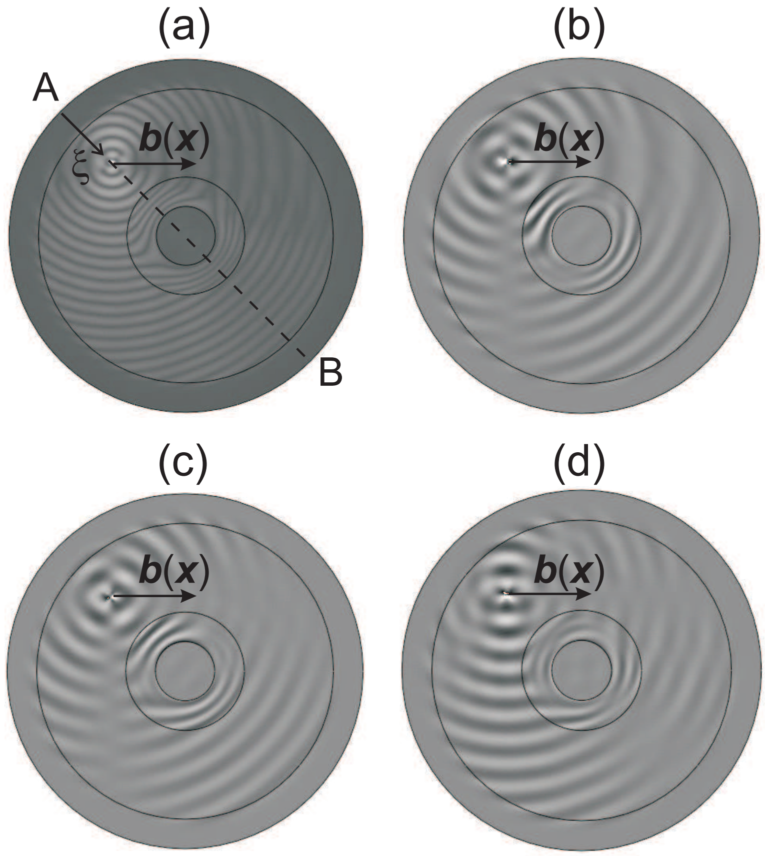


Figure 2: Elastic cloak in an elastic medium subjected to a concentrated load. (a) Displacement magnitude $u = \sqrt{u_1^2 + u_2^2}$; (b) deformation $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$; (c) deformation $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$; (d) deformation $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})$.

The elastic impedances

$$Z_p = \sqrt{\rho' \mathbb{C}'_{rrrr}}\ \omega = \sqrt{\rho(\lambda + \mu)}\ \omega, \qquad Z_s = \sqrt{\rho' \mathbb{C}'_{r\theta r\theta}}\ \omega = \sqrt{\rho\mu}\ \omega, \tag{8}$$

do not suffer any jump at $r = r_1$, avoiding any reflection.

## 5 NUMERICAL RESULTS

For the purpose of validation of the model we show in the following some finite element computations performed in the COMSOL multiphysics package. In particular, the elastic cloak of equation (5) is embedded in an isotropic elastic material with Lamé moduli $\lambda = 2.3$ and $\mu = 1$ and density $\rho = 1$, which correspond to normalized parameters for fused silica. The inner and outer radii of the elastic cloak are $r_0 = 0.2\,m$ and $r_1 = 0.4\,m$, respectively.

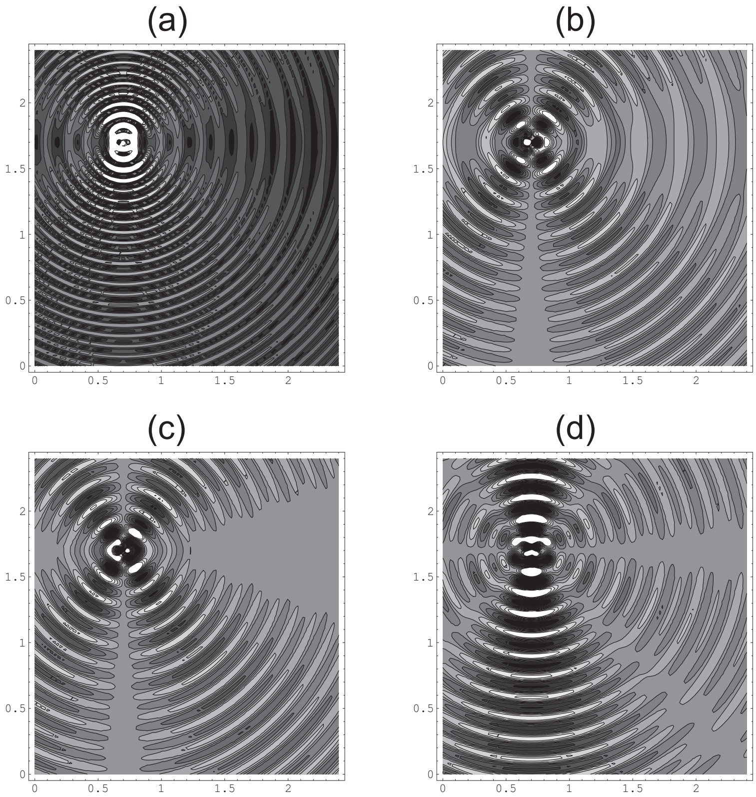


Figure 3: Harmonic Green's function in homogeneous elastic space. (a) Displacement magnitude $u$; (b) deformation $\varepsilon_{11}$; (c) deformation $\varepsilon_{22}$; (d) deformation $\varepsilon_{12}$.

We apply an harmonic unit concentrated force applied in direction $x_1$ and vibrating with angular frequency $\omega = 40$ Hz. In order to model the infinite elastic medium surrounding the cloak a perfectly matched cylindrical layer has been implemented (cf. outer ring on panels a, b, c and d of Fig.2); this has been obtained by application of the geometric transform [14],

$$r'' = r_2 + (1 - i)(r - r_2), \qquad \theta'' = \theta, \tag{9}$$

where $r_2 = 1\,m$ is the inner radius of the outer ring in Fig. 2.

It is shown in fig. 2 that the wave patterns of the displacement and deformations are smoothly bent around the central region within the cloak (where the magnitudes are nearly zero). Irrespectively of the fact that the coupling of shear and pressure waves generated by the concentrated force creates the optical illusion of interferences, the comparison with the analytic harmonic Green's function in homogeneous elastic space (see e.g. [15]) reported in Fig. 3 shows, at least qualitatively, that there is neither forward nor backward scattering. The absence of scattering is better detailed in Fig. 4, where results of Fig. 2 and Fig. 3 are compared: the perfect agreement of the displacement and deformation fields in the external matrix with and without the cloak is shown, the distortion being bounded to the central region delimited by the cloak. These are non-intuitive results, as the profiles of the horizontal and vertical displacements in Fig. 4 should display a visible phase shift, since the associated acoustic paths are different. More precisely, let us look at the expression of the elasticity tensor given in (5). On the inner boundary of the cloak, that is for $r = r_0$, its components $\mathbb{C}'_{rrrr}$ and $\mathbb{C}'_{r\theta r\theta}$ vanish, whereas its components $\mathbb{C}'_{\theta\theta\theta\theta}$ and $\mathbb{C}'_{\theta r\theta r}$ tend to infinity. This physically means that pressure and shear waves propagate with an infinite velocity in the azimuthal $\theta$-direction along the inner boundary, which results in a vanishing phase shift between a wave propagating in a homogeneous elastic space and another one propagating around the concealed region: this explains the superimposed profiles of horizontal and vertical displacements in Fig. 4.

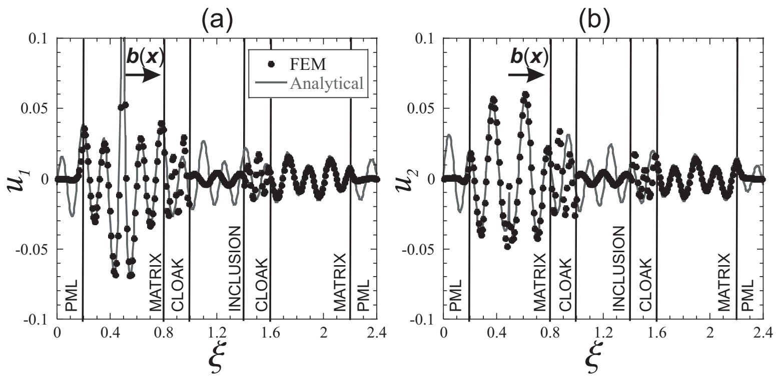


Figure 4: Comparison between numerical results in presence of the elastic cloak of Fig. 2 (black dots) and Green's function in homogeneous elastic space of Fig. 3 (grey lines). Results are given along the line $AB$ detailed in Fig. 2(a). (a) Horizontal displacement $u_1$; (b) Vertical displacement $u_2$.

In addition, we point out that the concentrated load propagates both shear and pressure components of elastic waves, which are inherently coupled; the displacement and deformation field distributions in Fig. 3 show that the inclusion is cloaked independently of the polarization.

Another property of the proposed model, which is important to note, is that the transformation does not affect the radian frequency $\omega$ and, therefore the formulation works for every applied $\omega$. This can be easily checked by comparing the Navier eqn. (1) with the modified Navier equation (2) and it is represented in Fig. 5 where the displacement magnitude $u$ distribution is plotted for $\omega = 20, 40, 60$.

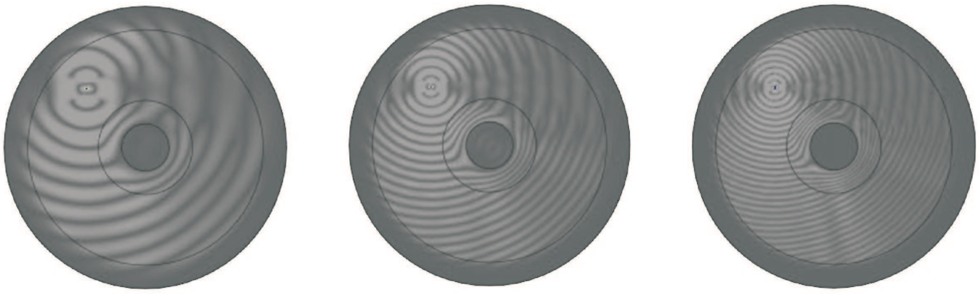


Figure 5: Elastic cloak in an elastic medium subjected to a concentrated load. The displacement magnitude $u$ is plotted for the same material parameters of Fig. 2 and for $\omega = 20, 40, 60$.

## 6 CONCLUSIONS

In the present work an elastic cloak bending the trajectory of in-plane coupled shear and pressure elastic waves around a cylindrical obstacle has been porposed. The device can be designed by the use of heterogeneous density and heterogeneous and anisotropic elastic stiffness; the distribution of the physical properties has been obtained with the introduction of stretched coordinates. Our results open new vistas in cloaking devices for elastodynamic waves in elastic media with non-symmetric constitutive tensor yet with an isotropic density. These can be fabricated with the introduction of metamaterials (in particular of structural interfaces), designed down to the microscopic level.

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