Some properties of the solutions of the equations of linear elastodynamics in unbounded domains

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SUMMARY. We prove that a solution of the system of linear elastodynamics in an unbounded domain, having a finite total initial energy with a suitable weight, decays at large distance with a rate depending on the weight, provided the acoustic tensor satisfies the *hyperbolicity condition*. Moreover, we show that this condition is necessary and sufficient for the equipartition in mean of the total energy.

1 INTRODUCTION

As is well–known, the motion of a linearly elastic body $\Omega \subset \mathbb{R}^3$ fixed at the boundary is governed by the differential system[1]

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}] &= \rho \ddot{\boldsymbol{u}} \quad \text{in } \Omega \times (0,+\infty), \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega \times (0,+\infty), \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{0\}, \\ \dot{\boldsymbol{u}} &= \dot{\boldsymbol{u}}_0 \quad \text{in } \Omega \times \{0\}, \end{aligned} \tag{1}$$

with $\mathbb{C}$ elasticity tensor, $i.e.$ a map from $\Omega \times \mathrm{Lin}$ to Sym, linear on Lin and such that $\mathbb{C}[\boldsymbol{W}] = \boldsymbol{0}$, for every $\boldsymbol{W} \in$ Skw, ρ mass density, $\boldsymbol{u}(x,t)$ (unknown) displacement field and $\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}_0$ initial conditions.

If Ω is a bounded domain and $\mathbb{C}$ is symmetric, a solution $\boldsymbol{u}$ of (1) satisfies the basic theorem of elastodynamics, as the conservation of the total energy, Graffi's reciprocity relation, uniqueness, etc.. Nevertheless, many problems of elastodynamics (as the wave propagation phenomena, the scattering theory [9], etc...) find their natural collocation in unbounded domains; the hypothesis making possible the extension of such theorems in unbounded domains is the so–called *hyperbolicity condition* on the acoustic tensor $\boldsymbol{A}$, introduced the first time in [2]. If $\mathbb{C}$ is symmetric and positive definite, then this condition assures, for instance, that a solution corresponding to initial data vanishing outside a bounded region has compact support at every instant, so that (what we naturally expect) it implies that a *finite perturbation* in the body Ω cannot reach the infinity in a finite time. It is of some interest to detect whether, assuming that the initial data have a total initial energy density $\varepsilon[\boldsymbol{u}](x,0)$ summable with a suitable weight, the (corresponding) solution decays at infinity with a rate depending on the weight. We prove that if $\mathbb{C}$ is symmetric and positive definite, $\boldsymbol{A}$ satisfies the hyperbolicity condition and $r^{\beta}\varepsilon[\boldsymbol{u}](x,0) \in L^1(\Omega)$ ($r = |x - o|$, with o the origin of a reference frame), then the mean of $|\boldsymbol{u}|^2(x,t)$ over the unit ball tends to zero at infinity as $r^{-(\beta+1)}$. Moreover, if $\mathbb{C}$ and ρ satisfy reasonable additional hypotheses, we show that $|\boldsymbol{u}|^2(x,t)$ decays at infinity as $r^{-\beta}$. For exterior domains and under the above assumptions on $\mathbb{C}$ and $\boldsymbol{A}$, in [1] the *equipartition in mean of the total*

[1]We essentially follow the notation of [7].

energy is proved: denoting by $\mathcal{K}[\boldsymbol{u}](t)$, $\mathcal{U}[\boldsymbol{u}](t)$ and $\mathcal{E}(0)$ respectively the kinetic and strain energies of a solution $\boldsymbol{u}$ at instant t and its total initial energy, it holds

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathcal{K}[\boldsymbol{u}](s)=\lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathcal{U}[\boldsymbol{u}](s)=\frac{1}{2}\mathcal{E}[\boldsymbol{u}](0). \tag{2}$$

We complete this result by extending it to arbitrary unbounded domains and by constructing a counterexample which shows that the hyperbolicity condition is also necessary for the validity of (2). Finally, we select a sharp subclass of solutions where (2) holds even if the acoustic tensor does not satisfy the hyperbolicity condition.

2 THE MAIN ASSUMPTIONS

Let $\Omega\subset\mathbb{R}^3$ be an unbounded domain of class C^∞ such that the Hardy inequality

$$\int_\Omega r^{\sigma-2}|\boldsymbol{u}|^2\le c\int_\Omega r^\sigma|\nabla\boldsymbol{u}|^2 \tag{3}$$

and the Korn inequality

$$\int_\Omega r^\sigma|\nabla\boldsymbol{u}|^2\le c\int_\Omega r^\sigma|\hat\nabla\boldsymbol{u}|^2, \tag{4}$$

hold for every field $\boldsymbol{u}$ vanishing on $\partial\Omega$, with $\hat\nabla\boldsymbol{u}=\operatorname{sym}\nabla\boldsymbol{u}\in L^2(\Omega)$. In (3), (4) $\sigma\ge 0$ and c is a positive constant depending only on Ω and σ.

We suppose $\mathbb{C},\rho,\boldsymbol{u}_0,\dot{\boldsymbol{u}}_0\in C^\infty(\overline{\Omega})$. Moreover, we assume that $\mathbb{C}$ is *symmetric*, $i.e.$,

$$\boldsymbol{L}\cdot\mathbb{C}[\boldsymbol{M}]=\boldsymbol{M}\cdot\mathbb{C}[\boldsymbol{L}]$$

for all $\boldsymbol{L},\boldsymbol{M}\in\mathrm{Lin}$, and *positive definite*, $i.e.$, there is a positive constant μ such that

$$\pi[\boldsymbol{L}]=\boldsymbol{L}\cdot\mathbb{C}[\boldsymbol{L}]\ge\mu|\operatorname{sym}\boldsymbol{L}|^2$$

for all $\boldsymbol{L}\in\mathrm{Lin}$. The above hypotheses imply the Cauchy inequality

$$\pm 2\boldsymbol{L}\cdot\mathbb{C}[\boldsymbol{M}]\le\pi[\boldsymbol{M}]+\pi[\boldsymbol{L}]. \tag{5}$$

The *acoustic tensor* in the direction $\boldsymbol{m}\in\mathrm{Unit}$ is the second–order tensor defined by

$$\boldsymbol{A}[x,\boldsymbol{m}]\boldsymbol{a}=\rho^{-1}\mathbb{C}[\boldsymbol{a}\otimes\boldsymbol{m}]\boldsymbol{m}$$

We say that $\boldsymbol{A}$ satisfies the *hyperbolicity condition* if there is a regular, increasing and unbounded function $p:(a,+\infty)\to(0,+\infty)$, $a>0$, such that

$$|\boldsymbol{A}[x,\boldsymbol{m}]|\le(p'(r))^{-2},\quad\forall x\in\Omega,\ \boldsymbol{m}\in\mathrm{Unit}. \tag{6}$$

Condition (6) is necessary and sufficient (see [2]) for the existence of a global (in time) solution to problem (1) and for the validity of the basic theorems of elastodynamics. In particular, denoting by

$$\varepsilon[\boldsymbol{u}](x,t)=\frac{1}{2}(\rho|\dot{\boldsymbol{u}}|^2+\pi[\nabla\boldsymbol{u}])(x,t)$$

the total energy density and setting

$$\mathcal{E}[\boldsymbol{u}](t)=\int_\Omega\varepsilon[\boldsymbol{u}](x,t),$$

the conservation of the total energy holds, $i.e.$

$$\mathcal{E}[\boldsymbol{u}](t) = \mathcal{E}[\boldsymbol{u}](0), \tag{7}$$

for all $t \geq 0$.

3 SPATIAL DECAY

Set

$$S_R = \{x \in \mathbb{R}^3 : |x - o| < R\}, \quad T_R = S_{2R} \setminus \overline{S_R}.$$

The following theorems hold.

Theorem 1. *Let Ω be an unbounded domain of $\mathbb{R}^3$, let $\boldsymbol{A}$ satisfy* (6)*, with $p(r) = \log r$, and let $\boldsymbol{u}$ be a solution of system* (1)*. If $r^\beta \varepsilon[\boldsymbol{u}](x,0) \in L^1(\Omega)$, $\beta \geq 0$, then*

$$\int_{\partial S_1} |\boldsymbol{u}|^2(R,\gamma,t) \leq \frac{c(t)}{R^{1+\beta}}. \tag{8}$$

PROOF – Let $w : \mathbb{R} \to [0,1]$ be a regular function vanishing in $(-\infty, 0]$ and equal to 1 in $[1, +\infty)$ and let

$$g(x,s) = w(\delta^{-1}(\log R - \log r + t - s)) \tag{9}$$

with δ and R positive constants such that $\delta < \log R$. It is simple to see that g vanishes outside the ball $S_{Re^{t-s}}$ and is equal to 1 in the ball $S_{Re^{t-s-\delta}}$. Moreover,

$$\dot{g} = -\delta^{-1} w'$$

and

$$\nabla g = -\delta^{-1} \frac{w'}{r} \boldsymbol{e}_r$$

with $\boldsymbol{e}_r = r^{-1}(x - o)$. Multiplying $(1)_1$ scalarly by $r^\beta g \dot{\boldsymbol{u}}$ and integrating over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega r^\beta g \varepsilon[\boldsymbol{u}](x,t) = \int_\Omega r^\beta \dot{g} \varepsilon[\boldsymbol{u}] - \int_\Omega \dot{\boldsymbol{u}} \cdot \mathbb{C}[\nabla \boldsymbol{u}] \nabla(r^\beta g). \tag{10}$$

Since by (6)

$$\begin{aligned}
2|\dot{\boldsymbol{u}} \cdot \mathbb{C}[\nabla \boldsymbol{u}] \nabla g| &\leq \delta^{-1} w' [\pi[\nabla \boldsymbol{u}] + \frac{1}{r^2} \dot{\boldsymbol{u}} \cdot \mathbb{C}[\dot{\boldsymbol{u}} \otimes \boldsymbol{e}_r] \boldsymbol{e}_r] \\
&= \delta^{-1} w' [\pi[\nabla \boldsymbol{u}] + \frac{1}{r^2} \rho \dot{\boldsymbol{u}} \cdot \boldsymbol{A}[\boldsymbol{e}_r] \dot{\boldsymbol{u}}] \\
&\leq 2\delta^{-1} w' \varepsilon[\boldsymbol{u}] = -2\dot{g} \varepsilon[\boldsymbol{u}],
\end{aligned}$$

$$2|\dot{\boldsymbol{u}} \cdot \mathbb{C}[\nabla \boldsymbol{u}] \nabla r^\beta| \leq \alpha r^\beta \pi[\nabla \boldsymbol{u}] + \alpha r^{\beta-2} \dot{\boldsymbol{u}} \cdot \mathbb{C}[\dot{\boldsymbol{u}} \otimes \boldsymbol{e}_r] \boldsymbol{e}_r \leq 2\alpha r^\beta \varepsilon[\boldsymbol{u}],$$

for some $\alpha > 0$, (10) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega r^\beta g \varepsilon[\boldsymbol{u}](x,t) \leq \alpha \int_\Omega r^\beta g \varepsilon[\boldsymbol{u}](x,t).$$

Hence it easily follows

$$\int_\Omega r^\beta g \varepsilon[\boldsymbol{u}](x,t) \leq e^{\alpha t} \int_\Omega r^\beta g \varepsilon[\boldsymbol{u}](x,0). \tag{11}$$

Since $r^\beta \varepsilon[\boldsymbol{u}](x,0) \in L^1(\Omega)$, we are allowed to let $R \to +\infty$ to obtain

$$\int_\Omega r^\beta \varepsilon[\boldsymbol{u}](x,t) \le e^{\alpha t} \int_\Omega r^\beta \varepsilon[\boldsymbol{u}](x,0). \tag{12}$$

Hence by (3) and (4), it follows

$$\int_\Omega r^{\beta-2}|\boldsymbol{u}|^2 \le c \int_\Omega r^\beta |\nabla \boldsymbol{u}|^2(x,t) \le c(t) \tag{13}$$

In virtue of the trace theorem [6] we have

$$\int_{\partial S_1} |\boldsymbol{u}|^2(R,\gamma) \le \frac{c}{R^2}\int_{T_R} |\boldsymbol{u}|^2 + c\int_{T_R} |\nabla \boldsymbol{u}|^2$$

so that by (13)

$$\lim_{R\to+\infty} \int_{\partial S_1} |\boldsymbol{u}|^2(R,\gamma) = 0.$$

Therefore, since

$$\int_{\partial S_1} |\boldsymbol{u}|^2(R,\gamma,t) = \int_{\partial S_1} \left(\int_R^{+\infty} \partial_r \boldsymbol{u}\right)^2 \le \int_{\complement S_R} |\nabla \boldsymbol{u}|^2 \int_R^{+\infty} \frac{1}{r^2},$$

it holds

$$\int_{\partial S_1} |\boldsymbol{u}|^2(R,\gamma,t) \le \frac{1}{R}\int_{\complement S_R} |\nabla \boldsymbol{u}|^2 \le \frac{c}{R^{\beta+1}} \int_{\complement S_R} r^\beta |\nabla \boldsymbol{u}|^2. \tag{14}$$

Hence the desired result follows, taking into account (13). □

Theorem 2. *Let Ω be an unbounded domain of $\mathbb{R}^3$, let $\boldsymbol{A}$ satisfy (6), let*

$$\rho,\ \rho^{-1},\ \mathbb{C},\ \nabla\mathbb{C} \in L^\infty(\Omega), \tag{15}$$

and let $\boldsymbol{u}$ be a solution of system (1). If $\boldsymbol{u}_0 \in L^2(\Omega)$ and $r^\beta \varepsilon[\boldsymbol{u}](x,0)$, $r^\beta \varepsilon[\dot{\boldsymbol{u}}](x,0) \in L^1(\Omega)$, then

$$|\boldsymbol{u}|^2(x,t) \le \frac{c(t)}{r^\beta} \tag{16}$$

for large r.

PROOF – Making use of (12) we obtain

$$\begin{aligned} \int_{\complement S_R} \varepsilon[\boldsymbol{u}](x,t) &\le \frac{e^{\alpha t}}{R^\beta} \int_\Omega r^\beta \varepsilon[\boldsymbol{u}](x,0), \\ \int_{\complement S_R} \varepsilon[\dot{\boldsymbol{u}}](x,t) &\le \frac{e^{\alpha t}}{R^\beta} \int_\Omega r^\beta \varepsilon[\dot{\boldsymbol{u}}](x,0). \end{aligned} \tag{17}$$

Since by the basic calculus

$$|\boldsymbol{u}|^2(x,t) = \left|\int_0^t \dot{\boldsymbol{u}}(x,s) + \boldsymbol{u}_0\right|^2 \le 2\left|\int_0^t \dot{\boldsymbol{u}}\right|^2 + 2|\boldsymbol{u}_0|^2 \le 2t\int_0^t |\dot{\boldsymbol{u}}|^2 + 2|\boldsymbol{u}_0|^2,$$

$(17)_1$ implies

$$\int_{\complement S_R} |\boldsymbol{u}|^2(x,t) \le \frac{c(t)}{R^\beta} \int_\Omega r^\beta \varepsilon[\boldsymbol{u}](x,0). \tag{18}$$

Consider now the equation

$$\operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}] = \rho \ddot{\boldsymbol{u}}$$

in the shell T_R. In virtue of well–known estimates about the solutions to an elliptic system, (see, $e.g.$ [6], Ch. 10), (15) assures that

$$\int_{T_R} |\nabla_2 \boldsymbol{u}|^2 \le c \left\{ \int_{S_{4R} \setminus S_{R/2}} |\nabla \boldsymbol{u}|^2 + |\rho \ddot{\boldsymbol{u}}|^2 \right\} \le \frac{c}{R^\beta} \left\{ \int_{S_{4R} \setminus S_{R/2}} r^\beta (|\nabla \boldsymbol{u}|^2 + |\rho \ddot{\boldsymbol{u}}|^2) \right\}.$$

so that

$$\int_{T_R} |\nabla_2 \boldsymbol{u}|^2(x,t) \le \frac{c}{R^\beta} \int_\Omega r^\beta (\varepsilon[\boldsymbol{u}] + \varepsilon[\dot{\boldsymbol{u}}])(x,t). \tag{19}$$

By Sobolev's lemma, (17), (18) and (19) we get

$$|\boldsymbol{u}|^2(x,t) \le c \|\boldsymbol{u}\|^2_{W^{2,2}(S_1(x))} \le \frac{c(t)}{R^\beta} \int_\Omega r^\beta (\varepsilon[\boldsymbol{u}] + \varepsilon[\dot{\boldsymbol{u}}])(x,0)$$

for all $x \in T_R$ such that $S_1(x) \in T_R$. Hence (16) follows. $\square$

4 THE EQUIPARTITION OF THE ENERGY AND COUNTEREXAMPLES

An interesting property of the solutions of system (1) is the so–called *equipartition in mean of the total energy*, first proved by W.D. Day [4] for bounded domain and then extended to exterior domain in [1]. In fact with few changes it is possible to prove the result for an arbitrary unbounded domain.

Let

$$\mathcal{K}[\boldsymbol{u}](t) = \frac{1}{2} \int_\Omega \rho |\dot{\boldsymbol{u}}|^2(x,t),$$
$$\mathcal{U}[\boldsymbol{u}](t) = \frac{1}{2} \int_\Omega \pi[\nabla \boldsymbol{u}](x,t)$$

denote the kinetic energy and the strain energy of $\mathcal{B}$ at instant t respectively.

The following theorem holds

Theorem 3. *Let Ω be an unbounded domain of $\mathbb{R}^3$, let $\boldsymbol{A}$ satisfy (6) and let $\mathbf{u}$ be a solution of system (1). If*

$$\rho|\boldsymbol{u}_0|^2,\ \varepsilon[\boldsymbol{u}](x,0) \in L^1(\Omega) \ \text{ and } \ r\rho\dot{\boldsymbol{u}}_0 \in L^2(\Omega), \tag{20}$$

then

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \mathcal{K}[\boldsymbol{u}](s) = \lim_{t\to+\infty} \frac{1}{t} \int_0^t \mathcal{U}[\boldsymbol{u}](s) = \frac{1}{2} \mathcal{E}[\boldsymbol{u}](0). \tag{21}$$

Open problem. Starting from (21) we can prove that along infinite sequences $\{s_k\}_{k\in\mathbb{N}}$, $s_k \to +\infty$,

$$\lim_{k\to+\infty} \mathcal{K}[\boldsymbol{u}](s_k) = \lim_{k\to+\infty} \mathcal{U}[\boldsymbol{u}](s_k) = \frac{1}{2} \mathcal{E}[\boldsymbol{u}](0). \tag{22}$$

But it is not known whether

$$\lim_{t\to+\infty}\mathcal{K}[\boldsymbol{u}](t)=\lim_{t\to+\infty}\mathcal{U}[\boldsymbol{u}](t)=\frac{1}{2}\mathcal{E}[\boldsymbol{u}](0). \tag{23}$$

$\diamond$

We aim at showing now that the hyperbolicity condition is also necessary to the validity of (21). To this end we use a minor modification of a famous example by De Giorgi (see [5], [10]) to construct an elastodynamic system for which the hyperbolicity condition is not met and whose solutions do not satisfy (21) [11]. Let consider the fourth–order tensor field $\mathbb{C}_0$ defined by

$$\mathbb{C}_0[\boldsymbol{L}]=(\boldsymbol{B}\otimes\boldsymbol{B})\boldsymbol{L}+\xi^2\mathrm{sym}\,\boldsymbol{L},\quad \xi\in\mathbb{R} \tag{24}$$

for all $\boldsymbol{L}\in\mathrm{Sym}$, where

$$\boldsymbol{B}=\mathbf{1}+3\boldsymbol{e}_r\otimes\boldsymbol{e}_r. \tag{25}$$

It is obvious that $\mathbb{C}_0\in L^\infty(\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3\setminus\{o\})$ is symmetric and positive definite. A simple computation (see [5]) shows that the elliptic system

$$\mathrm{div}\,\mathbb{C}_0[\nabla\boldsymbol{v}]=0 \tag{26}$$

admits the solutions

$$\boldsymbol{v}(x)=[c_1r^{-\epsilon-\frac{1}{2}}+c_2r^{\epsilon-\frac{1}{2}}]\boldsymbol{e}_r=\boldsymbol{v}_1+\boldsymbol{v}_2 \tag{27}$$

for every constant c_1 and c_2 and with

$$\epsilon=\frac{3}{2}\frac{|\xi|}{\sqrt{16+\xi^2}}. \tag{28}$$

Of course,

$$\lim_{\xi\to0}\epsilon=0,$$

and

$$\nabla\boldsymbol{v}_1\in L^2(\mathbb{R}^3\setminus S_R),\quad \nabla\boldsymbol{v}_2\notin L^2(\mathbb{R}^3\setminus S_R)$$

for all positive R. Moreover, we can choose the constants c_1 and c_2 in such a way that $\boldsymbol{v}$ satisfies the boundary condition

$$\boldsymbol{v}=\mathbf{0}$$

on ∂S_R. The fields

$$\boldsymbol{u}(x,t)=\boldsymbol{v}(x)t \tag{29}$$

are C^∞ solutions of the system

$$\rho\ddot{\boldsymbol{u}}=\mathrm{div}\mathbb{C}_0[\nabla\boldsymbol{u}] \tag{30}$$

in every domain Ω which does not contain the point o, *for every mass density* ρ. Moreover,

$$\int_\Omega\nabla\boldsymbol{u}\cdot\mathbb{C}_0[\nabla\boldsymbol{u}](x,0)=0 \tag{31}$$

and

$$\int_\Omega\nabla\boldsymbol{u}\cdot\mathbb{C}_0[\nabla\boldsymbol{u}](x,t)=+\infty \tag{32}$$

for all $t > 0$. Therefore, choosing

$$\rho(x) = \frac{1}{r^{2+\zeta}}, \quad \zeta > 2\epsilon, \tag{33}$$

we can see that the acoustic tensor associated to $\mathbb{C}_0$ and ρ does not satisfy the hyperbolicity condition and

$$\mathcal{U}[\boldsymbol{u}](0) = 0,\ \mathcal{K}[\boldsymbol{u}](t) < +\infty\ \forall t \geq 0 \text{ and } \mathcal{U}[\boldsymbol{u}](t) = +\infty\ \forall t > 0. \tag{34}$$

Then, by the arbitrariness of ζ and ϵ we can state

Theorem 4. *In an exterior domain of $\mathbb{R}^3$, the hyperbolicity condition* (6) *is necessary and sufficient for the validity of the conservation and the equipartition* $($*in mean*$)$ *of the energy.*

Now, following [3] we define a subclass of solutions in which we can state the classical theorems of linear elastodynamics, even if the hyperbolicity condition is not satisfied.

Let

$$\mathfrak{C} = \left\{ \boldsymbol{u} \in \mathbb{C}^\infty(\overline{\Omega} \times [0, +\infty)) : \sup_{\Omega} \left[p'(r) r^2 \rho \dot{\boldsymbol{u}} \cdot \boldsymbol{A}[\boldsymbol{m}] \dot{\boldsymbol{u}} \right] \leq c(t) \right\}, \tag{35}$$

for some positive and continuous function $c(t)$. We can prove the following relation

$$\begin{aligned} \int_{\Omega} (g^2 \varepsilon)[\boldsymbol{u}](x,t) = \int_{\Omega} (g^2 \varepsilon)[\boldsymbol{u}](x,0) \\ + 2\delta^{-1} \int_0^t \mathrm{d}s \int_{\Omega} g w' p'(r) \dot{\boldsymbol{u}} \cdot \mathbb{C}(\nabla \boldsymbol{u}) \boldsymbol{e}_r, \end{aligned} \tag{36}$$

where g is the function

$$g(x) = w(\delta^{-1}(p(R) - p(r))).$$

Since

$$2|\delta^{-1} g w' p'(r) \dot{\boldsymbol{u}} \cdot \mathbb{C}[\nabla \boldsymbol{u}] \boldsymbol{e}_r| \leq \alpha \pi [g \nabla \boldsymbol{u}] + \delta^{-2} (p'(r))^2 \rho \dot{\boldsymbol{u}} \cdot \boldsymbol{A}[\boldsymbol{e}_r] \dot{\boldsymbol{u}},$$

taking into account the properties of function g, (35) and choosing α suitably small, it follows that

$$\begin{aligned} \int_{\Omega_{R_\delta}} \varepsilon[\boldsymbol{u}](x,t) &\leq \int_{\Omega} \varepsilon[\boldsymbol{u}](x,0) + c\delta^{-2} \int_{\partial S_1} \int_{R_\delta}^{R} p'(r) \\ &\leq \int_{\Omega} \varepsilon[\boldsymbol{u}](x,0) + c\delta^{-1}. \end{aligned} \tag{37}$$

where $R_\delta = p^{-1}(p(R) - \delta)$. Letting $\delta \to +\infty$ in (37), we have that $\varepsilon[\boldsymbol{u}](x,t) \in L^1(\Omega)$, for all $t \geq 0$. As a consequence, letting $R \to +\infty$ in (37), we can state, in particular,

Theorem 5. *Let Ω be an unbounded domain of $\mathbb{R}^3$ and let $\boldsymbol{u} \in \mathfrak{C}$ be a solution of system* (1)*. If $\varepsilon[\boldsymbol{u}](x,0) \in L^1(\Omega)$, then the conservation of the energy holds. Moreover, if* (20) *holds, then the equipartition (in mean) of the energy holds.*

The above result is sharp. Indeed, if $\mathbb{C}$ is bounded and

$$\rho(x) = \frac{1}{r^{2+\zeta}},$$

for some positive ζ, then $\boldsymbol{A}$ does not satisfy the hyperbolicity condition and, for $p = \log\log\log r$, we have $\boldsymbol{u} \in \mathfrak{C} \Leftrightarrow |\dot{\boldsymbol{u}}|^2 = O(r^{-1} \log r \log\log r)$. Therefore, choosing $\zeta > 2\epsilon$, we see that the solution (29) does not belong to $\mathfrak{C}$ and does not satisfy (34).

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