On the Contact Problem of Linear Elasticity

Antonio Russo[1], Alfonsina Tartaglione[1]
[1]*Dipartimento di Matematica, Seconda Università di Napoli, Italy*
E-mail: a.russo@unina2.it, alfonsina.tartaglione@unina2.it

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SUMMARY. We prove sharp uniqueness theorems for the contact problem of homogeneous and isotropic linear elastostatics in domains exterior to convex and regular regions of $\mathbb{R}^3$, provided the elasticity tensor is strongly elliptic.

1 INTRODUCTION

It is well–known that the existence and uniqueness theorem for the displacement problem of homogeneous and isotropic linear elastostatics requires the elasticity tensor to be only strongly elliptic [3], [6]:

$$\mu > 0,\ \lambda + 2\mu > 0, \tag{1}$$

where λ and μ are the constant Lamé moduli. For the traction problem (1) is not longer sufficient for the uniqueness of a solution modulo an arbitrary additive infinitesimal rigid displacement, as can be showed by classical counter–examples of W.S. Edelstein and R.I. Fosdick [2], and has to be replaced by the more restrictive assumption

$$\mu > 0,\ 3\lambda + 2\mu > 0. \tag{2}$$

In a recent paper, R. Fosdick, M.D. Piccioni and G. Puglisi [5] proved that (1) is also sufficient to the uniqueness of a solution of a mixed boundary value problem in a bounded domain which is a combination of the *displacement*, *contact* and its *dual* problems. Precisely, one decomposes $\partial\Omega$ in three portions $\{\partial_i\Omega\}_{i=1,2,3}$ and assigns the displacement on $\partial_1\Omega$, the tangential component of the displacement and the normal component of the traction on $\partial_2\Omega$, the normal component of the displacement and the tangential component of the traction on $\partial_3\Omega$. Denoted by

$$\mathcal{K}(\xi) = -\nabla_\sigma \boldsymbol{n}(\xi) \tag{3}$$

the Weingarten tensor on $\partial\Omega$, where ∇_σ stands for the gradient over $\partial\Omega$ and $\boldsymbol{n}$ is the exterior (with respect to Ω) unit normal to $\partial\Omega$, they need one of the following hypotheses:

(a) $\operatorname{tr}\mathcal{K}$ is positive on $\partial_2\Omega$ and $\mathcal{K}$ is positive definite over $\partial_3\Omega$;

(b) Ω is simply connected, $\operatorname{tr}\mathcal{K}$ is nonnegative on $\partial_2\Omega$, $\mathcal{K}$ is positive semi–definite over $\partial_3\Omega$.

The above assumptions require a particular "geometry" to $\partial_2\Omega$ and $\partial_3\Omega$ (see Section 4 of [5]). In particular, for the contact problem in which $\partial_3\Omega = \partial\Omega$ [6] Ω cannot be convex. We observe that this restriction is not present for domains exterior to convex regions, where nevertheless other problems appear, as for instance the choice of a condition at infinity assuring uniqueness in sharp function classes.

The purpose of this note is to prove that under hypothesis (1) the contact problem and its dual in domains exterior to convex bounded and regular regions admit sharp uniqueness theorems.

2 THE MODEL

Let $\mathcal{B}$ be a homogeneous and isotropic linearly elastic body identified with the exterior domain

$$\Omega = \mathbb{R}^3 \setminus \overline{\Omega'} \tag{4}$$

which it occupies in a reference configuration. In (4) Ω' is a bounded domain, we assume for simplicity of class C^∞ and with connected boundary. The elastic properties of $\mathcal{B}$ are expressed by the elasticity tensor[1]

$$\mathbb{C}[\boldsymbol{E}] = 2\mu\,\mathrm{sym}\boldsymbol{E} + \lambda(\mathrm{tr}\boldsymbol{E})\mathbf{1}, \qquad \forall \boldsymbol{E} \in \mathrm{Lin},$$

with λ and μ constant Lamé moduli.

As is well–known [6],

$$\mathbb{C} \text{ is positive definite } \Leftrightarrow\ \mu > 0,\ 3\lambda + 2\mu > 0$$

and

$$\mathbb{C} \text{ is strongly elliptic } \Leftrightarrow\ \mu > 0,\ \lambda + 2\mu > 0.$$

Let

$$s,\ \boldsymbol{a} \in C^\infty(\partial\Omega)$$

[1] We use a standard vector notation as, $e.g.$, in [6]. Moreover, x and ξ denote generic points of Ω and of $\partial\Omega$, respectively; $r = |x-o|$, where $o \in \mathbb{R}^3\setminus\overline{\Omega}$ is the origin of the reference frame in $\mathbb{R}^3$; $\boldsymbol{e}_r = (x-o)/r$; $S_R(x_0)$ is the ball of radius R centered at x_0; $\Omega_R(x_0) = \Omega \cap S_R(x_0)$; $S_R = S_R(x_0)$; $\Omega_R = \Omega_R(o)$. We denote by $\boldsymbol{n}$ the exterior (with respect to Ω) unit normal to $\partial\Omega$. If $\boldsymbol{w}$ is a vector field on $\partial\Omega$, we set $\boldsymbol{w}_\tau = \boldsymbol{w} - (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{n}$. The symbol c stands for a positive constant whose numerical value is not essential to our purposes.

be assigned fields on $\partial\Omega$ and let $\boldsymbol{u}_0$ be a constant vector. The contact problem of linear elastostatics[2] is to find a solution of the equations (see [6] p. 129)

$$\begin{aligned} \operatorname{div}\mathbb{C}[\nabla\boldsymbol{u}] &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= a \quad \text{on } \partial\Omega, \\ [\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{n}]_\tau &= \boldsymbol{s} \quad \text{on } \partial\Omega, \\ \lim_{r\to+\infty}\boldsymbol{u} &= \boldsymbol{u}_0. \end{aligned} \tag{5}$$

It is well–known that, if $\mathbb{C}$ is positive definite, then system (5) has a unique solution $\boldsymbol{u}\in C^\infty(\overline{\Omega})$ [7]. As far as we are aware, if $\mathbb{C}$ is strongly elliptic not too much is known about existence and uniqueness of solutions of problem (5).

Set

$$\mathbb{S}[\boldsymbol{E}] = (\lambda+2\mu)(\operatorname{tr}\boldsymbol{E})\boldsymbol{1} + 2\mu\operatorname{skw}\boldsymbol{E}, \qquad \forall\boldsymbol{E}\in\ \mathrm{Lin}$$

and

$$\begin{aligned} \varpi[\nabla\boldsymbol{u}] &= (\lambda+2\mu)(\operatorname{div}\boldsymbol{u})^2 + \mu|\operatorname{curl}\boldsymbol{u}|^2, \\ \gamma[\nabla\boldsymbol{u}] &= \lambda(\operatorname{div}\boldsymbol{u})^2 + 2\mu|\operatorname{sym}\nabla\boldsymbol{u}|^2. \end{aligned}$$

Of course,

$$\mathbb{C}[\boldsymbol{E}] = \mathbb{S}[\boldsymbol{E}] + 2\mu\big[\boldsymbol{E}^{\mathrm{T}} - (\operatorname{tr}\boldsymbol{E})\boldsymbol{1}\big]. \tag{6}$$

Observe that $\boldsymbol{u}$ is a solution of $(5)_1$ if and only if

$$\operatorname{div}\mathbb{S}[\nabla\boldsymbol{u}] = \boldsymbol{0} \quad \text{in } \Omega \tag{7}$$

and the following vector identity holds

$$\Delta\boldsymbol{u} = \nabla\operatorname{div}\boldsymbol{u} - \operatorname{curl}\operatorname{curl}\boldsymbol{u}. \tag{8}$$

Let $\mathcal{K}$ be the Weingarten tensor defined by (3). We say that $\mathcal{K}$ is positive definite if

$$\boldsymbol{w}\cdot\mathcal{K}[\boldsymbol{w}] > 0$$

for all nonzero vectors $\boldsymbol{w}$ and positive semi–definite if

$$\boldsymbol{w}\cdot\mathcal{K}[\boldsymbol{w}] \geq 0$$

for all vectors $\boldsymbol{w}$.

Starting from (6) and recalling that $\mathcal{K}$ is defined by (3), one can prove the following three lemmas.

[2] Also known as the third problem of elastostatics [7].

Lemma 1. [4] *If $\boldsymbol{u}$ is a solution of equations $(5)_1$, then*

$$\boldsymbol{\sigma}[\boldsymbol{u}] = \int_{\partial\Omega} \mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n} = \int_{\partial\Omega} \mathbb{S}[\nabla \boldsymbol{u}]\boldsymbol{n}. \tag{9}$$

Lemma 2. [5] *If $\boldsymbol{\varphi}$ and $\boldsymbol{u}$ are fields on $\partial\Omega$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$, then*

$$\int_{\partial\Omega} \boldsymbol{\varphi}\cdot\mathbb{S}[\nabla \boldsymbol{u}]\boldsymbol{n} = \int_{\partial\Omega} \boldsymbol{\varphi}_\tau \cdot \big\{(\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n})_\tau - 2\mu \mathcal{K}[\boldsymbol{u}_\tau]\big\}.$$

Lemma 3. [5] *If $\boldsymbol{\varphi}$ and $\boldsymbol{u}$ are fields on $\partial\Omega$ such that $\boldsymbol{\varphi}_\tau = \boldsymbol{u}_\tau = \boldsymbol{0}$, then*

$$\int_{\partial\Omega} \boldsymbol{\varphi}\cdot\mathbb{S}[\nabla \boldsymbol{u}]\boldsymbol{n} = \int_{\partial\Omega} (\boldsymbol{\varphi}\cdot\boldsymbol{n})\big\{\boldsymbol{n}\cdot\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n} - 2\mu \mathrm{tr}\,\mathcal{K}(\boldsymbol{u}\cdot\boldsymbol{n})\big\}.$$

The following two lemmas are well–known (see, $e.g.$, [1], [10]).

Lemma 4. [1] *Let φ be a harmonic function in Ω. If* [3]

$$\nabla\varphi = o(1),$$

then there is a constant φ_0 and a regular function ψ such that for all $x \in \Omega$

$$\varphi(x) = \varphi_0 + \frac{1}{4\pi r}\int_{\partial\Omega} \partial_n \varphi + \psi(x),$$

with

$$\nabla_k \psi(x) = O(r^{-2-k})$$

for every nonnegative integer k.

Lemma 5. [1] *Let $\boldsymbol{u}$ be a solution of equations $(5)_1$. If*

$$\boldsymbol{u} = o(r), \tag{10}$$

then there is a constant vector $\boldsymbol{\kappa}$ and a regular function $\boldsymbol{\psi}$ such that for all $x \in \Omega$

$$\boldsymbol{u}(x) = \boldsymbol{\kappa} + \boldsymbol{U}(x)\boldsymbol{\sigma}[\boldsymbol{u}] + \boldsymbol{\psi}(x), \tag{11}$$

where $\boldsymbol{\sigma}[\boldsymbol{u}]$ is defined by (9),

$$\boldsymbol{U}(x) = \frac{1}{16\pi\mu(1-\nu)r}\big[(3-4\nu)\boldsymbol{1} + \boldsymbol{e}_r\otimes\boldsymbol{e}_r\big],$$

$\nu = \lambda/2(\lambda+\mu)$ and

$$\nabla_k \boldsymbol{\psi}(x) = O(r^{-2-k})$$

for every nonnegative integer k.

[3] If f is a scalar, vector or tensor field on Ω $f = o(g)$ and $f = O(g)$ mean respectively that $\lim_{r\to+\infty} |f(x)|/g(r) = 0$ and $|f(x)| \le cg(r)$ for some positive c. Moreover, we set $\nabla_k f = \underbrace{\nabla\ldots\nabla}_{k \text{ times}} f$.

The dual of the contact problem of elastostatics[4] is to find a solution of the equations

$$
\begin{aligned}
\operatorname{div}\mathbb{C}[\nabla \boldsymbol{u}] &= \mathbf{0} \quad \text{in } \Omega,\\
\boldsymbol{u}_\tau &= \boldsymbol{a} \quad \text{on } \partial\Omega,\\
\boldsymbol{n}\cdot[\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n}] &= s \quad \text{on } \partial\Omega,\\
\lim_{r\to+\infty} \boldsymbol{u} &= \boldsymbol{u}_0,
\end{aligned}
\tag{12}
$$

where $\boldsymbol{a}$ and s are assigned fields on $\partial\Omega$.

Let $\boldsymbol{u} = o(1)$ be a solution to system $(5)_1$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Multiplying $(5)_1$ scalarly by $\boldsymbol{u}$ and integrating on Ω_R, we get

$$
\begin{aligned}
&\int_{\Omega_R} \varpi[\nabla \boldsymbol{u}] + 2\mu \int_{\partial\Omega} \boldsymbol{u}_\tau \cdot \mathcal{K}[\boldsymbol{u}_\tau] - \int_{\partial S_R} \big[(\lambda+2\mu)(\operatorname{div}\ \boldsymbol{u})\boldsymbol{u} + \mu \boldsymbol{u}\times \operatorname{curl}\boldsymbol{u}\big]\cdot \boldsymbol{e}_R\\
&\qquad = \int_{\Omega_R} \gamma[\nabla \boldsymbol{u}] - \int_{\partial S_R} \boldsymbol{u}\cdot \big[2\mu \operatorname{sym}\nabla \boldsymbol{u} + \lambda(\operatorname{div}\boldsymbol{u})\mathbf{1}\big]\boldsymbol{e}_R = \int_{\partial\Omega} \boldsymbol{u}\cdot\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n}.
\end{aligned}
$$

Hence, letting $R\to+\infty$ and taking into account Lemma 5, it follows

$$
\int_\Omega \gamma[\nabla \boldsymbol{u}] = \int_\Omega \varpi[\nabla \boldsymbol{u}] + 2\mu \int_{\partial\Omega} \boldsymbol{u}_\tau\cdot\mathcal{K}[\boldsymbol{u}_\tau] = \int_{\partial\Omega} \boldsymbol{u}\cdot\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n}. \tag{13}
$$

Likewise, if $\boldsymbol{u} = o(1)$ is a solution to $(5)_1$ such that $\boldsymbol{u}_\tau = \mathbf{0}$, it holds

$$
\int_\Omega \gamma[\nabla \boldsymbol{u}] = \int_\Omega \varpi[\nabla \boldsymbol{u}] + 2\mu \int_{\partial\Omega} (\boldsymbol{u}\cdot\boldsymbol{n})^2 \operatorname{tr}\mathcal{K} = \int_{\partial\Omega} \boldsymbol{u}\cdot\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n} \tag{14}
$$

Relations (13), (14) are the starting point for our analysis on the uniqueness of a solution to problems (5) and (12). Also, as it will be clear from the argument we use in Section 3, if Ω is simply connected and $\mathcal{K}$ is positive semi–definite[5], then strong ellipticity implies that the strain energy $\mathcal{U}[\boldsymbol{u}]$, corresponding to a deformation of $\mathcal{B}$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, is positive. Moreover, if Ω is simply connected, $\operatorname{tr}\mathcal{K} \geq 0$ and $\boldsymbol{u}_\tau = \mathbf{0}$, then $\mathcal{U}[\boldsymbol{u}] > 0$.

We shall need the following existence theorems that are proved in [9], by making use of (13), (14) and well–known techniques of functional analysis.

Theorem 1. *If $\mathbb{C}$ is strongly elliptic and $\mathcal{K}$ is positive definite, then system* (5) *has a solution $\boldsymbol{u} \in C^\infty(\overline{\Omega})$.*

Theorem 2. *If $\mathbb{C}$ is strongly elliptic and* $\operatorname{tr}\mathcal{K}$ *is positive, then system* (12) *has a solution $\boldsymbol{u} \in C^\infty(\overline{\Omega})$.*

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[4] Also known as the fourth problem of elastostatics [7].

[5] This occurs for instance if Ω' is convex.

Denote by $\mathfrak{C}$ the linear space of all solutions of the system

$$\begin{aligned} \operatorname{div}\mathbb{C}[\nabla\boldsymbol{u}] &= \mathbf{0} \quad \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega,\\ [\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{n}]_\tau &= \mathbf{0} \quad \text{on } \partial\Omega,\\ \boldsymbol{u}(x) &= o(r). \end{aligned} \tag{15}$$

Lemma 6. *If $\mathbb{C}$ is strongly elliptic and $\mathcal{K}$ is positive definite, then*

$$\dim\mathfrak{C} = 3. \tag{16}$$

PROOF – If $\boldsymbol{u}\,(\neq \mathbf{0}) \in \mathfrak{C}$, then $\boldsymbol{\sigma}[\boldsymbol{u}] \neq \mathbf{0}$. Otherwise, an integration by parts yields

$$\int_{\Omega_R} \varpi[\nabla\boldsymbol{u}] = \int_{\partial\Omega} (\boldsymbol{u}-\boldsymbol{\kappa})\cdot\mathbb{S}[\nabla\boldsymbol{u}]\boldsymbol{n} + \int_{\partial S_R} (\boldsymbol{u}-\boldsymbol{\kappa})\cdot\mathbb{S}[\nabla\boldsymbol{u}]\boldsymbol{e}_R,$$

for large R. Hence, letting $R\to+\infty$ and taking into account that by Lemma 5

$$(\boldsymbol{u}-\boldsymbol{\kappa})\cdot\mathbb{S}[\nabla\boldsymbol{u}]\boldsymbol{e}_R = O(R^{-3}),$$

it follows

$$\int_{\Omega} \varpi[\nabla\boldsymbol{u}] = \int_{\partial\Omega} \boldsymbol{u}\cdot\mathbb{S}[\nabla\boldsymbol{u}]\boldsymbol{n}. \tag{17}$$

Using Lemma 2 in (17) we have

$$\begin{aligned} \operatorname{curl}\boldsymbol{u} &= \mathbf{0} \quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{u}_\tau &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{18}$$

Therefore, by (8) $\boldsymbol{u}$ is a solution of the system

$$\begin{aligned} \Delta\boldsymbol{u} &= \mathbf{0} \quad \text{in } \Omega,\\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\Omega,\\ \boldsymbol{u}(x) &= o(r). \end{aligned} \tag{19}$$

Since by $(18)_{1,2}$

$$\mathbf{0} = \boldsymbol{\sigma}[\boldsymbol{u}] = \int_{\partial\Omega} [2\mu\partial_n\boldsymbol{u} + \mu\boldsymbol{n}\times\operatorname{curl}\boldsymbol{u} + \lambda(\operatorname{div}\boldsymbol{u})\boldsymbol{n}] = 2\mu\int_{\partial\Omega}\partial_n\boldsymbol{u},$$

an integration by parts gives

$$\int_{\Omega_R} |\nabla \boldsymbol{u}|^2 = \int_{\partial\Omega} (\boldsymbol{u}-\boldsymbol{\kappa})\cdot\partial_n \boldsymbol{u} + \int_{\partial S_R} (\boldsymbol{u}-\boldsymbol{\kappa})\cdot\partial_r \boldsymbol{u} = \int_{\partial S_R} (\boldsymbol{u}-\boldsymbol{\kappa})\cdot\partial_r \boldsymbol{u}.$$

Hence, letting $R \to +\infty$, it follows that $\nabla \boldsymbol{u} = \mathbf{0}$. Therefore, since Ω is connected and $\boldsymbol{u}$ vanishes on $\partial\Omega$ we have the absurd $\boldsymbol{u} = \mathbf{0}$.

Let $\{\boldsymbol{u}_i\}_{\imath=1,\dots,4} \subset \mathfrak{C}$. Of course, the system $\{\boldsymbol{\sigma}[\boldsymbol{u}_i]\}_{\imath=1,\dots,4}$ is linearly dependent. Therefore, there are nonzero scalars α_i such that

$$\boldsymbol{\sigma}[\alpha_i \boldsymbol{u}_i] = \alpha_i \boldsymbol{\sigma}[\boldsymbol{u}_i] = \mathbf{0}.$$

Then, repeating the steps in the above argument we see that

$$\alpha_i \boldsymbol{u}_i = \mathbf{0}$$

so that $\{\boldsymbol{u}_i\}_{\imath=1,\dots,4}$ is linearly dependent. Hence $\dim \mathfrak{C} \le 3$. On the other hand, if $\{\boldsymbol{e}_i\}_{i=1,2,3}$ is the canonical basis of $\mathbb{R}^3$, Theorem 1 assures that the problem

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}] &= \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega, \\ [\mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n}]_\tau &= \mathbf{0} \quad \text{on } \partial\Omega, \\ \boldsymbol{u}(x) - \boldsymbol{e}_i &= o(1), \end{aligned} \tag{20}$$

has a unique variational solution $\boldsymbol{u}_i$. Hence the desired result follows, taking into account that the system $\{\boldsymbol{u}_i\}_{i=1,2,3}$ is linearly independent. $\square$

We are in a position to prove the following uniqueness theorems.

Theorem 3. *Let Ω be an exterior domain of $\mathbb{R}^3$, let $\mathbb{C}$ be strongly elliptic and let $\mathcal{K}$ be positive definite. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of $(5)_{1,2,3}$ such that*

$$\boldsymbol{u}_1(x) - \boldsymbol{u}_2(x) = o(r),$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ modulo a field in $\mathfrak{C}$. Moreover, if

$$\boldsymbol{u}_1(x) - \boldsymbol{u}_2(x) = o(1), \tag{21}$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

PROOF – The field $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ satisfies equations (15). Then the first part of the theorem follows from Lemma 6. If $\boldsymbol{u}$ satisfies (21), then in virtue of (17) and the positive definiteness of $\mathcal{K}$, $\boldsymbol{u}$ is a solution of the system

$$\begin{aligned} \Delta \boldsymbol{u} &= \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \\ \boldsymbol{u} &= o(1). \end{aligned} \tag{22}$$

Hence it follows that $\boldsymbol{u} = \mathbf{0}$. $\square$

Theorem 4. *Let Ω be a simply connected exterior domain of $\mathbb{R}^3$, let $\mathbb{C}$ be strongly elliptic and let $\mathcal{K}$ be positive semi–definite. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of $(5)_{1,2,3}$ such that*

$$\boldsymbol{u}_1(x) - \boldsymbol{u}_2(x) = o(1),$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

PROOF – Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, from $(18)_1$ it follows that there is a regular function φ such that $\boldsymbol{u} = \nabla\varphi$ in Ω. Moreover, $(18)_2$, $(15)_2$ and (21) imply that φ is a solution of the Neumann problem

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in } \Omega, \\ \partial_n\varphi &= 0 \quad \text{on } \partial\Omega, \\ \nabla\varphi &= o(1). \end{aligned} \tag{23}$$

Integrating by parts, we get

$$\int_{\Omega_R} |\nabla\varphi|^2 = \int_{\partial S_R} (\varphi - \varphi_0)\partial_r\varphi.$$

Hence, letting $R \to +\infty$ and taking into account Lemma 4, it follows that $\boldsymbol{u} = \mathbf{0}$. $\square$

Let us pass to consider the uniqueness of a solution of problem (12). Denote by $\mathfrak{M}$ the linear space of all solutions of the system

$$\begin{aligned} \operatorname{div}\mathbb{C}[\nabla\boldsymbol{u}] &= \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{u}_\tau &= \mathbf{0} \quad \text{on } \partial\Omega, \\ \boldsymbol{n}\cdot\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega, \\ \boldsymbol{u}(x) &= o(r). \end{aligned}$$

It is not difficult to see that the reasoning we used to prove Lemma 6 works as well to show that

$$\dim\mathfrak{M} = 3.$$

Therefore, we can state

Theorem 5. *Let Ω be an exterior domain of $\mathbb{R}^3$, let $\mathbb{C}$ be strongly elliptic and let $\operatorname{tr}\mathcal{K}$ be positive. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of system $(12)_{1,2,3}$ such that*

$$\boldsymbol{u}_1(x) - \boldsymbol{u}_2(x) = o(r),$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ modulo a field in $\mathfrak{M}$. Moreover, if

$$\boldsymbol{u}_1(x) - \boldsymbol{u}_2(x) = o(1), \tag{24}$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Moreover, we have

Theorem 6. *Let Ω be a simply connected exterior domain of $\mathbb{R}^3$, let $\mathbb{C}$ be strongly elliptic and let* $\operatorname{tr}\mathcal{K}$ *be nonnegative. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of $(12)_{1,2,3}$ such that*

$$\boldsymbol{u}_1(x) - \boldsymbol{u}_2(x) = o(1),$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

PROOF – Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and using Lemma 3 in (17) we have

$$\begin{aligned} \operatorname{curl}\boldsymbol{u} &= \mathbf{0} \quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega. \end{aligned}$$

Hence, taking into account that Ω is simply connected, it follows that there are a function φ and a vector field $\boldsymbol{h}$ such that

$$\boldsymbol{u} = \nabla\varphi = \operatorname{curl}\boldsymbol{h}. \tag{25}$$

The function φ is a solution of the system

$$\begin{aligned} \Delta\varphi &= 0 && \text{in } \Omega, \\ \varphi &= \text{const.} && \text{on } \partial\Omega, \\ \nabla\varphi &= o(1). \end{aligned} \tag{26}$$

Moreover by Stokes' theorem (25) implies

$$\int_{\partial S_R} \partial_r\varphi = \int_{\partial S_R} \boldsymbol{e}_r \cdot \operatorname{curl}\boldsymbol{h} = 0. \tag{27}$$

Integrating by parts, taking into account (27) and that φ is constant on $\partial\Omega$, we have

$$\int_{\Omega_R} |\nabla\varphi|^2 = \int_{\partial\Omega} (\varphi - \varphi_0)\partial_n\varphi + \int_{\partial S_R} (\varphi - \varphi_0)\partial_r\varphi = \int_{\partial S_R} (\varphi - \varphi_0)\partial_r\varphi.$$

Hence, letting $R \to +\infty$ and using Lemma 4, it follows that φ is constant in Ω so that $\boldsymbol{u} = \mathbf{0}$. □

We aim at concluding by observing that in virtue of Theorem 1, 2, our results are sharp in the sense that *small o* cannot be replaced by *capital O*.

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