A direct evaluation of the Fabric Tensor in anisotropic porous media

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SUMMARY. The paper investigates the possibility of using Inertia-based Fabric Tensors for describing elliptic orthotropy in heterogeneous materials characterized by a solid skeleton depleted by voids. Example applications are finally illustrated and discussed, to stress the helpfulness of the proposal.

1 INTRODUCTION

Advances in soil mechanics, tissue engineering, biomechanics, composites, porous and granular media and micro/nano-structured materials have motivated a significant and growing interest in the analysis of heterogeneous solids, that often exhibit inhomogeneous and anisotropic mechanical features [1]. As a consequence, recently, homogenization techniques and micro-mechanical approaches are widely presented in Literature, making reference to the *volume fraction* – as stereological measure of the inhomogeneities (voids, fraction of constituents in composites, isotropic damage variables) – and to second-order (or higher) *Fabric Tensors*, in order to take into account void (or matrix) orientations revealed at the micro-structure level of a selected Representative Volume Element (RVE) material, with the aim of estimating its overall mechanical anisotropy [2, 3].

Despite the use of Fabric Tensor approaches constitutes a mechanically consistent and useful way for determining the influence of RVE material microstructure in many engineering problems, several disadvantages and difficulties limit the actual use of this strategy and inhibit its implementation in some fields, such as design of new materials or prediction of biological tissue remodelling phenomena [4, 5].

In particular, the above mentioned disadvantages can be identified in two classes. The first one collects several aspects, say of practical order, related to the difficulty of estimating the quantities characterizing the Fabric point-by-point for a prescribed heterogeneous media, even if it presents periodical microstructure [6]. Indeed – for example in a porous material – without considering the volume fraction, at least three second-order Fabric Tensor eigenvalues and the corresponding three eigenvectors (or equivalently principal axis Euler angles) have to be carried out for a selected RVE. This number significantly grows if the solid under analysis shows a non periodical structure.

Also, being the Fabric classically deduced by first constructing the Mean Intercept Length (MIL), the final interpolation of the density *rose diagrams* with an ellipse (or ellipsoid in 3D cases) requires other computational efforts and – at the authors knowledge – no automatic software or standard protocols exist in Literature for ensuring that two distinct operators obtain the same results.

The second class of disadvantages in the use of Fabric Tensor approaches can be traceable in the theoretical assessment of the method. Indeed, a shared response is not yet furnished in Literature with reference to a rational and geometrically consistent definition – for example in porous materials – of (RVE matrix or void) *orientation*. Also, the proof that the MIL, as well as other minor techniques, are actually able to trace anisotropy of a RVE with any oriented microstructure remains an open issue.

2 THE MODEL

2.1. Physic-geometrical definition of "orientation"

It is possible to naturally derive an indicator of the microstructural morphology starting from the meaning of *oriented* microstructure. To better clarify this idea, some preliminary observations are necessary.

First, let us consider a spherical and an ellipsoidal mass distributions, tacitly assuming that the first one is *not oriented* while the second one is *oriented*. The difference between the two bodies is here investigated in the behaviour of the two bodies when they rotate around a fixed axes. In particular, it is observed that the angular velocity and the angular momentum vectors are coaxial for every rotation axis in case of the sphere, while for only three axis of rotation (principal axis of inertia) in case of the ellipse. The *orientation* is therefore linked to the coaxiality between the angular velocity and the angular momentum. By remembering that these vectors are related by the Inertia Tensor through the well-known law

$$\boldsymbol{l} = \mathbf{J}\boldsymbol{\omega}, \qquad \mathbf{J} = \int_{\Omega} \left[tr(\boldsymbol{x} \otimes \boldsymbol{x})\mathbf{I} - \boldsymbol{x} \otimes \boldsymbol{x} \right] d\Omega \tag{1}$$

where Ω is the domain occupied by the mass within the RVE, $\boldsymbol{x} \in \Omega$ is the generic point, $\mathbf{J}$ is the inertia tensor, while $\boldsymbol{l}$ and $\boldsymbol{\omega}$ are the angular momentum and the angular velocity, the Inertia Tensor appears as a natural indicator of the orientation of a mass.

A procedure to establish if a mass distribution inside an RVE is oriented or not can be then identified by invoking the Gyroscopic Principle. Indeed, by imaging of imprisoning the RVE in a spherical black-box and verify if the angular velocity and angular momentum are coaxial for every axis of rotation, it is possible to establish if the mass inside the RVE is oriented or not. The same result may be more easily achieved by finding the eigenvectors of the inertia tensor of the RVE and verifying if the corresponding eigenvalues are equal or different. In fact, if the three eigenvalues are all different, the microstructure is oriented – case of the ellipsoid; if two of the eigenvalues are equal, the microstructure is not oriented in a plane – case of an ellipsoid with a circular cross section; finally, if the eigenvalues of the inertia tensor are all equal, the microstructure is not oriented – case of the sphere. The natural consequence of these observations consists in recognizing that the orientation – and that the fabric tensor - is directly related to the Inertia Tensor.

Successively, the problem of choosing a inertia-related Fabric Tensor is approached, with the intent of both making it obeying to that which we will call *periodicity condition* and rendering the measure unique and as simple as possible respect to a practical in-field estimate (for example having in mind the reading of X-ray attenuation data given in output by a QCT).

Finally, before to furnish the definitive form of the elasticity tensor for a porous anisotropic (elliptical orthotropic) inhomogeneous material, the limit cases of Low Volume Fraction (LVF) and High Volume Fraction (HVF) are discussed, with reference to the Eshelby results for ellipsoidal dilute voids [7] and to the extended orthotropic Flugge model [8], respectively, showing how the corresponding estimates of the overall elastic moduli confirm the consistency of the present proposal.
Thanks to a consolidated analytical solution, constitutive relations based on inertia are derived following an heuristic micromechanical approach, that means choosing *a priori* the inertia of the mass distribution inside the RVE to describe the microstructural anisotropy.
By using the Flugge solution (Flugge, 1972), the overall elasticity tensor for an idealized porous material, consisting in many isotropic and homogeneous thin walls of different thickness, say $t_i \ll a$, parallel to the coordinate planes of a Cartesian system $\{x_1, x_2, x_3\}$ can be written in the form

$$\mathbb{C} = \begin{bmatrix} \frac{E(t_2+t_3)}{a(1-\nu^2)} & \frac{\nu E t_3}{a(1-\nu^2)} & \frac{\nu E t_2}{a(1-\nu^2)} & 0 & 0 & 0 \\ \frac{\nu E t_3}{a(1-\nu^2)} & \frac{E(t_1+t_3)}{a(1-\nu^2)} & \frac{\nu E t_1}{a(1-\nu^2)} & 0 & 0 & 0 \\ \frac{\nu E t_2}{a(1-\nu^2)} & \frac{\nu E t_1}{a(1-\nu^2)} & \frac{E(t_1+t_2)}{a(1-\nu^2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E t_1}{a(1+\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{E t_2}{a(1+\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E t_3}{a(1+\nu)} \end{bmatrix} \quad (1)$$

where t_i represents the thickness of the wall whose unit normal vector is coaxial with the x_i -axis, *a* is the size of the regularized cell, E and ν are the Young modulus and the Poisson ratio of the isotropic and homogeneous matrix.
The overall elasticity tensor of the real RVE is expressed as function of the microstructural parameters – the volume fraction and the inertia tensor – as well as of the matrix elasticity in the form

$$\begin{aligned} C_{iiii} &= \frac{2E}{(1-\nu^2)}\,\gamma - \frac{4E}{(1-\nu^2)}\gamma\frac{J_{ii}}{\mathrm{Tr}\,\mathbf{J}} \\ C_{iijj} &= \frac{3E\nu}{(1-\nu^2)}\,\gamma - \frac{4E\nu}{(1-\nu^2)}\gamma\left(\frac{J_{ii}+J_{jj}}{\mathrm{Tr}\,\mathbf{J}}\right) \\ C_{ijij} &= \frac{3E}{(1+\nu)}\,\gamma - \frac{4E}{(1+\nu)}\gamma\left(\frac{J_{ii}+J_{jj}}{\mathrm{Tr}\,\mathbf{J}}\right) \end{aligned} \quad (2)$$

where no summation is performed on the repeated index.
The (2) allows to express the orthotropic elasticity tensor as function of the inertia tensor for LVF materials. The eigenvalues of the inertia tensor normalized respect to the trace of the inertia tensor could be assumed as fabric tensor components.
Analogously, it can be demonstrated that the elastic moduli of RVEs characterized by a dilute distribution of voids can be related to the inertia-based fabric tensor measurements.

3 NUMERICAL RESULTS

Some applications related to 2D case of porous RVEs can be constructed. One of the most interesting is represented by the comparison between the traditional method adopted to measure the orientation – in particular the *star length distribution* (SLD) method – and the new Inertia-based strategy. This example would illustrate how, in some cases, despite an evident orientation of the microstructure, SLD can obscure the anisotropy while the Inertia does not. Also, it is possible to show that when the SLD method detects the anisotropy, the Inertia preserves the same directions of anisotropy.

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4 CONCLUSIONS

The effectiveness of the choice of an Inertia-based Fabric Tensor can be summarized in the three following points:

1) The matrix (or equivalently void) orientation of a porous RVE structure can be deduced by means of a rigorous geometrical and physical way;
2) The choice of a Fabric proportional to the Inertia Tensor excludes the necessity of making reference to interpolating ellipsoids for representing as a second-order tensor the RVE microstructure, classically measured by means of *rose* diagrams when MIL or other strategies are involved;
3) The use of inertia as measure of the RVE matrix (or void) orientation is of extreme convenience and practical interest, being all the vector-based codes (CAD, FE programs, QCT software) able to give in output – for a selected RVE – the volume fraction, as well as the whole set of the so-called *mass-properties*, included the inertia tensor components respect to an arbitrary reference frame.

The obtained results may be applied in a wide range of mechanical problems involving anisotropic heterogeneous materials, for example the characterization of granular media (soils and rocks), cellular and ceramic materials, biomaterials and biological tissues. Moreover, to study damage and remodelling phenomena in living tissues, as well as for solving optimization problems for designing engineered materials, the present model constitutes an useful tool.

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