Contact mechanics of functionally graded rough surfaces

Marco Paggi[1], Giorgio Zavarise[2]
[1]*Department of Structural and Geotechnical Engineering, Politecnico di Torino, Italy*
E-mail: *marco.paggi@polito.it*

[2]*Department of Innovation Engineering, University of Salento, Italy*
E-mail: *giorgio.zavarise@unisalento.it*

Keywords: Contact mechanics, rough surfaces, functionally graded materials.

SUMMARY. In this paper the Greenwood and Williamson contact theory for microscopically rough surfaces is generalized by considering a grading on the elastic modulus of the asperities. This situation can be representative of surfaces receiving superficial treatments or characterized by a chemical degradation. The effect of an exponential grading on the applied normal load, on the relationship between the real contact area and the load, on the plasticity index, as well as on the contact conductance is illustrated with detailed numerical examples.

1 INTRODUCTION

Greenwood and Williamson [1] proposed one of the first theories for the contact analysis of microscopically rough surfaces that accounts for the stochastic properties of interfaces. To simplify the properties of rough surfaces as detailed later also by Nayak [2], Greenwood and Williamson assumed in 1966 that the asperities have the same radius of curvature whatever their height and that the asperities are axisymmetric, so that they give circular contact areas. Considering the contact of each asperity as governed by the Hertzian theory, they obtained the expressions of the total real contact area, A, and of the load, P, as functions of the dimensionless mean plane separation, d/σ, where σ is the surface roughness. The derived equations were quite general, since they depend on three parameters: the asperity density, the mean radius of curvature of the asperities and the standard deviation of the asperity height Gaussian distribution. This made the Greenwood and Williamson contact theory prone to be generalized and extended. So far, most of the existing improvements and generalizations concern the description of the geometric properties of the surfaces, taking for instance into account a distribution of asperity curvatures in the calculation rather than using an average value, as originally proposed by Bush, Gibson and Thomas [3] and recently reconsidered by Greenwood himself [4] (see also [5,6] for a models overview). McCool [7] adopted a Weibull distribution of asperity heights and analyzed the effect of skewness and kurtosis on the predicted contact results. Another significant generalization of the original Greenwood and Williamson contact theory was proposed by Ciavarella, Greenwood and Paggi [8], taking into account the effect of asperity interaction which was neglected in the original formulation.

All of these approaches can be applied when contact involves at most two different materials. However, tribological contacts often involve surfaces covered by films of other materials. The bulk material presents also a different and thinner structure of the grains, which results into a deep variation of the mechanical characteristics. These films may be adventitious −for example, due to chemical reaction with the environment−, they may arise as a result of the rubbing process, or they may be deliberately formed using special techniques. Clearly, as the film thickness tends to zero, the behaviour is determined entirely by the properties of the substrate, whereas, when the film thickness becomes sufficiently large, the behaviour is determined entirely by the properties of the film. Between these two extremes, the behaviour is a function of the properties of both the film and the substrate materials. A classical example is the variation of the microhardness values for lead films of various thicknesses on mild steel substrates [9]. More importantly, the wear resistance of any surface is dependent on both the hardness and the elastic modulus of the contacting surface. A relatively thin surface film can modify both these properties and thus have a significant effect on its tribological properties. To deal with these important aspects, McCool [10,11] extended the capabilities of the original Greenwood and Willamson contact theory to the problem of elastic contact of coated rough surfaces, where a homogeneous rough interface of finite thickness is considered as mounted over a homogeneous elastic medium with different elastic properties.

To the knowledge of the authors, no general theories have been proposed so far in order to deal with the problem of contact mechanics of rough interfaces with functionally graded elastic properties. Theoretical models for contact mechanics of functionally graded materials (FGM) are in fact quite recent and only a few studies concerning smooth surfaces have been published in the literature. Among them, we mention the recent paper by Liu, Wang and Zhang [12], where the indentation of a half-space made of a FGM is analyzed with reference to spherical or conical indenters or even in the case of a flat elastic punch.

In this paper we propose a generalization of the Greenwood and Williamson contact theory to the case of functionally graded surfaces (FGS). To do so effectively, we consider a rough surface composed of asperities with a composite Young's modulus which is dependent on the *z* coordinate, according to a predetermined variation. This situation can be representative of surfaces receiving superficial treatments or characterized by a chemical degradation. In the former instance the higher asperities can be stiffer than the others. In the latter case an opposite behaviour can be observed as a result of chemical contamination. Under these conditions, the relationship between the real contact area and the mean plane separation still remains valid, since it does not depend on the elastic properties of the bodies in contact. On the contrary, the relationship between the load and the mean plane separation significantly changes. This has enormous implication for the slope of the real contact area-load relationship, for the normal stiffness of the joint, for the mode of deformation, as well as for the joint contact conductance. Useful diagrams will show the effect of the elastic grading on all of these parameters.

2 ADAPTATION OF THE GREENWOOD AND WILLIAMSON CONTACT THEORY TO FUNCTIONALLY GRADED SURFACES

Assuming axisymmetric asperities with the same radius of curvature whatever their height, the model considers the contact between a rough surface and an ideal flat rigid plane. However, the contact between two rough surfaces can be treated as the contact between an equivalent rough surface, defined by suitable composite geometrical parameters and mechanical properties, and an ideal flat rigid plane. Considering the contact of each asperity as governed by the Hertzian theory, the expressions of the real contact area, A, and of the load, P, as functions of the dimensionless mean plane separation, d/σ, is obtained in [1] as

$$A = \pi A_n \eta \sigma R\, F_1(d/\sigma) \tag{1a}$$

$$P = \frac{4}{3} A_n \eta E^* R^{1/2} \sigma^{3/2} F_{3/2}(d/\sigma) \tag{1b}$$

where σ is the r.m.s. of the asperity heights, R is the mean radius of curvatures of the asperities, η is the surface asperity density, $E^* = \left[\left(1-\nu_1^2\right)/E_1 + \left(1-\nu_2^2\right)/E_2\right]^{-1}$ is the composite Young's modulus of the materials 1 and 2 and A_n is the nominal (or apparent) contact area. The integrals $F_n(h)$ are given by the following expression, in the case of a Gaussian distribution of asperity heights

$$F_n(h) \equiv \frac{1}{\sqrt{2\pi}} \int_h^\infty (s-h)^n \exp(-0.5s^2) \mathrm{d}s \tag{2}$$

where $s = z/\sigma$ is the dimensionless height of a generic asperity.

It is important to note that Equations (1a) and (1b) are quite general, since they depend on three parameters: the asperity density, the mean radius of curvature of the asperities and the standard deviation of the asperity height Gaussian distribution. In order to generalize the Greenwood and Williamson contact theory in the case of a FGS, let us consider a rough surface composed of asperities with a composite Young's modulus dependent on the z coordinate, i.e., $E^*(z/\sigma) = E^*(s)$, according to a predetermined variation. This situation can be representative of surfaces receiving superficial treatments or characterized by a chemical degradation. In the former instance, the higher asperities can be stiffer than the others, whereas in the latter case an the opposite trend can be observed. Under these conditions, it is important to note that the relationship between the real contact area and the mean plane separation in Eq. (1a) still remains valid, since it does not depend on the elastic properties of the bodies in contact. On the contrary, the relationship between the load and the mean plane separation in Eq. (1b) is significantly affected by the presence of a nonuniform elastic modulus. Neglecting the effect of asperity interaction, as also in the original Greenwood and Williamson contact theory, Eq. (1b) becomes

$$P = \frac{4}{3} A_n \eta R^{1/2} \sigma^{3/2} \frac{1}{\sqrt{2\pi}} \int_{d/\sigma}^\infty (s - d/\sigma)^{3/2} \exp(-0.5s^2) E^*(s) \,\mathrm{d}s \tag{3}$$

where the Young's modulus is now included in the integral, being dependent on the variable s. Note that the exponent 3/2 in Eq. (3) is again valid, since we are considering each asperity with a given dimensionless height s as elastically uniform and in contact with a homogeneous half-space. So far, no restrictions to the Young's modulus profile, $E^*(s)$, have been made. As in the original Greenwood and Williamson theory with a Gaussian distribution of asperity heights, also in this case no closed-form solutions to Eq. (3) can be found and a numerical integration has to be performed for any prescribed elastic grading.

3 MODEL CHARACTERISTICS

In this section we focus our attention on an exponential variation of the Young's modulus with the asperity height

$$E^*(s) = E^* \left[\gamma - (\gamma - 1)\exp(-s)\right] \tag{4}$$

where γ is a constant and E^* is the composite elastic modulus of a conventional rough surface with homogeneous properties. By varying the parameter γ in Eq. (4), it is possible to model FGS with positive or negative gradings (see Fig. 1). In the former case, we have $\gamma > 1$ and the elastic modulus is an increasing function of *s* up to an asymptotic value attained for the highest asperities. In the latter case, $\gamma < 1$ and the elastic modulus is a decreasing function of *s*. The case with $\gamma = 1$ marks the transition between these two types of grading and corresponds to a homogeneous surface with $E^*(s) = E^*$.

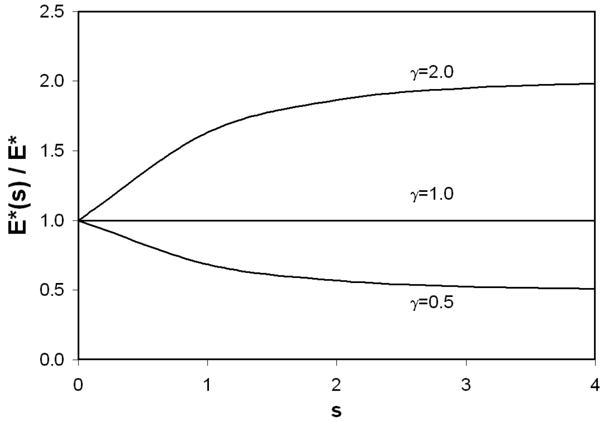


Figure 1: exponential grading on the composite Young's modulus vs. *s* for different values of γ.

Under these conditions, Eq. (3) becomes

$$P = \frac{4}{3} A_n \eta R^{1/2} \sigma^{3/2} \frac{E^*}{\sqrt{2\pi}} \int_{d/\sigma}^{\infty} (s - d/\sigma)^{3/2} \exp(-0.5 s^2) \left[\gamma - (\gamma - 1)\exp(-s)\right] \mathrm{d}s \qquad (5)$$

and, to shorten the notation, we shall write

$$P = \frac{4}{3} A_n \eta E^* R^{1/2} \sigma^{3/2} F_{3/2}^*(d/\sigma) \qquad (6)$$

where

$$F_{3/2}^*(s) = \frac{1}{\sqrt{2\pi}} \int_h^{\infty} (s - h)^{3/2} \exp(-0.5 s^2) \left[\gamma - (\gamma - 1)\exp(-s)\right] \mathrm{d}s \qquad (7)$$

To solve Eq. (7) numerically, we consider the change of variable $t = \exp(-s)$, obtaining $s = -\ln t$ and $\mathrm{d}s = -\mathrm{d}t/t$. In this way, the integration limits becomes finite and we get

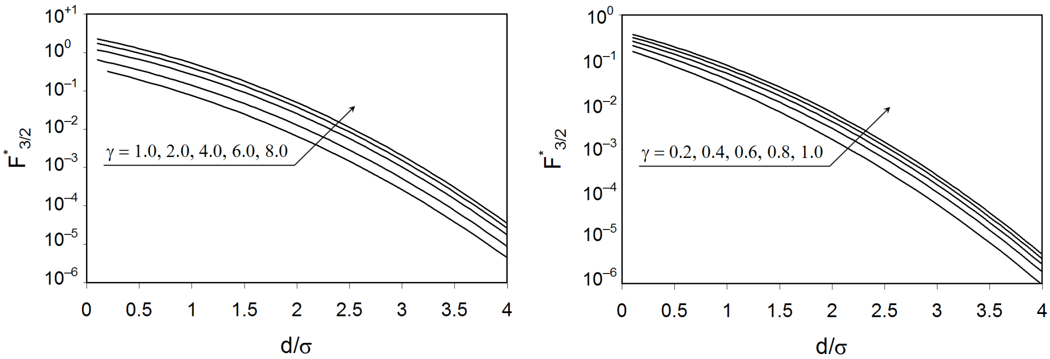
$$F_n^*(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\exp(-h)} (-\ln t - h)^n \exp[(-\ln t)^2/2] \frac{\gamma - (\gamma - 1)t}{t} \mathrm{d}t \qquad (8)$$

Now we apply the Legendre–Gauss' quadrature method [13] (see also [14] for a critical examination of the accuracy of the numerical scheme):

$$F_n^*(t) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{NGP} B_i^* \left(-\ln x_i^* - h\right)^n \exp\left[\left(-\ln x_i^*\right)^2 / 2\right] \frac{\gamma - (\gamma-1)x_i^*}{x_i^*} \tag{9}$$

where the summation is extended to all the Gauss Points NGP with abscissae $x_i^* = \exp(-h/2)\left(1+x_i\right)/2$ and weights $B_i^* = \exp(-h/2)B_i/2$.

The function $F_{3/2}^*$, which has the physical meaning of a dimensionless applied load (see Eq. (6)), is plotted vs. d/σ in Fig. 2 for positive and negative gradings. In these diagrams, the homogeneous case corresponds to the curve with $\gamma = 1$. In both cases, the higher the exponent γ, the higher the load for a given mean plane separation. Moreover, we note that the homogeneous case represents a lower bound to the load for positive gradings (see Fig. 2a), whereas it is an upper bound for negative gradings (see Fig. 2b).



(a) Positive grading: $\gamma > 1$ (b) Negative grading: $\gamma < 1$

Figure 2: the function ratio $F_{3/2}^*$ in Eq. (14) vs. d/σ for different values of γ.

To compare the contact predictions corresponding to a FGS with those of a homogenous interface, it is useful to consider the ratio between the real contact area and the applied load, which is a fundamental quantity in tribological applications. The variation of this quantity with separation is also a measure of the deviation from linearity of the real contact area-load curve. For FGS we have

$$\left(\frac{A}{P}\right)_{\text{FGS}} = \frac{3\pi R^{1/2}}{4\sigma^{1/2}E^*} \frac{F_1\ (d/\sigma)}{F_{3/2}^*(d/\sigma)} \tag{10}$$

whereas for the same interface but with homogeneous elastic properties we have

$$\left(\frac{A}{P}\right)_{\text{HOM}} = \frac{3\pi R^{1/2}}{4\sigma^{1/2}E^*} \frac{F_1\ (d/\sigma)}{F_{3/2}(d/\sigma)} \tag{11}$$

Hence, the ratio between Eqs. (10) and (11) can be considered as representative of the effect of the elastic grading

$$\left(\frac{A}{P}\right)_{\text{FGS}} \Bigg/ \left(\frac{A}{P}\right)_{\text{HOM}} = \frac{F_{3/2}(d/\sigma)}{F^*_{3/2}(d/\sigma)} \tag{12}$$

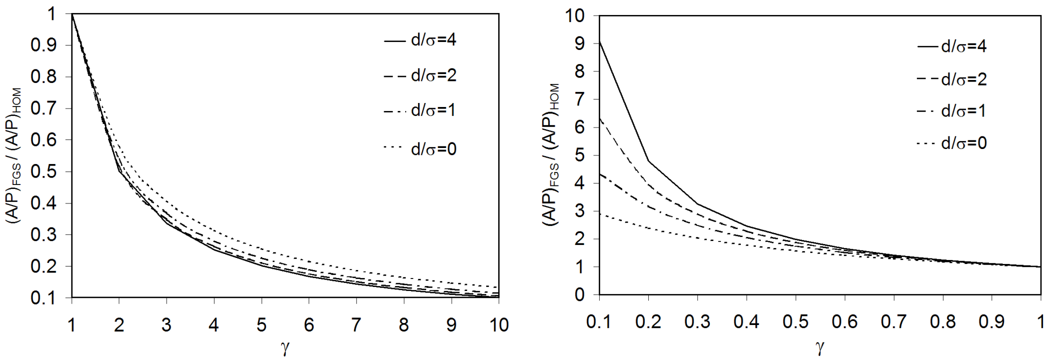
The value of this ratio is shown in Fig. 3a vs. γ for different mean plane separations in the case of positive grading $(\gamma > 1)$. In this case, the asperities above the mean summit height of the FGS are stiffer than those of the homogeneous surface. Hence, for a given mean plane separation or, equivalently, for a given real contact area, the load applied to the FGS has to be higher than that applied to the homogeneous surface. For high separations $(d/\sigma = 4)$, this effect is mainly governed by the asymptotic value of the function $E^*(s)/E^*$ and therefore we have $P_{\text{FGM}}/P_{\text{HOM}} \cong \gamma$. This trend is a little bit less pronounced for lower separations. In any case, these results show that the effect of a positive grading on the slope of the real contact area-load curve is particularly relevant. In fact, it is sufficient to have γ slightly higher than unity to significantly reduce such a slope.

The same ratio given by Eq. (12) is shown in Fig. 3b vs. γ in the case of negative grading $(\gamma < 1)$. In this instance, the trend is the opposite as before and, for a given real contact area, the load required to deform the FGS is lower than that applied to the homogeneous surface. Again, for high separations $(d/\sigma = 4)$, the asymptotic value of the function $E^*(s)/E^*$ characterizes the contact behaviour and we have $P_{\text{FGM}}/P_{\text{HOM}} \cong \gamma$. However, for intermediate or low separations, the effect of grading is much less pronounced and we should consider $\gamma < 0.5$ to see a significant increase of the slope as compared to the homogeneous case.

4 EFFECTS ON THE PLASTICITY INDEX

The results obtained in Section 3 have enormous implications on the deformation regime of an interface and on the plasticity index, ψ. By definition, the asperities experience elastic deformations when ψ is approximately less than unity, otherwise plastic deformations are expected. Several expressions available in the literature propose a plasticity index which is a function of both the surface statistical parameters and the mechanical properties of the contacting bodies. Sridhar and Yovanovich [15] proposed a plasticity index given by the ratio between two dimensionless quantities. One is the normal load divided by an equivalent elastic hardness, P/H_e, and the other is the normal load divided by the Vickers hardness, P/H_v. The equivalent elastic hardness is given by $H_e = E^* m/\sqrt{2}$, and is a function of the composite Young's modulus and of the mean absolute slope of the profiles, *m*. For homogeneous surfaces, this simply leads to $\psi_{\text{HOM}} = E^* m \Big/ \left(\sqrt{2} H_v\right)$. Generalizing this expression to FGS, we can replace the variable E^* with $E^*(s)$. In case of the exponential grading given in (4) we have

$$\psi_{\text{FGS}} = E^* \left[\gamma - (\gamma - 1)\exp(-s)\right] m \Big/ \left(\sqrt{2} H_v\right) = \left[\gamma - (\gamma - 1)\exp(-s)\right] \psi_{\text{HOM}} \tag{13}$$



(a) Positive grading: $\gamma > 1$ (b) Negative grading: $\gamma < 1$

Figure 3: the ratio predicted by Eq. (12) vs. γ for different values of d/σ.

Hence, we note that the ratio between the plasticity index of a FGS and the plasticity index of a homogenous surface is simply given by the ratio between their composite Young's moduli, i.e., $\psi_{\mathrm{FGS}} / \psi_{\mathrm{HOM}} = E^{*}(s) / E^{*}$, as already shown in Fig. 1. A closer inspection to this ratio is provided in Fig. 4 for $0.2 < \gamma < 2.0$ and 5 different values of d/σ. For light contacts (high separations), the plasticity index of a FGS can be significantly different from that of a homogeneous surfaces. For positive grading, the ratio $\psi_{\mathrm{FGS}} / \psi_{\mathrm{HOM}}$ is higher than unity and therefore plastic deformation should prevail over the elastic one. For instance, a typical microscopical surface of Zircalloy 4 with $\psi_{\mathrm{HOM}} = 0.86$ could attain a plasticity index up to $\psi_{\mathrm{FGS}} \cong 2\psi_{\mathrm{HOM}} = 1.8$ for a positive grading with $\gamma = 2.0$. Conversely, a negative grading allows to reduce the plasticity index and therefore a situation where almost all the asperities are elastically deformed is envisaged. As shown in Fig. 4, the effect of grading is much less pronounced for intermediate or high load levels (intermediate or low separations).

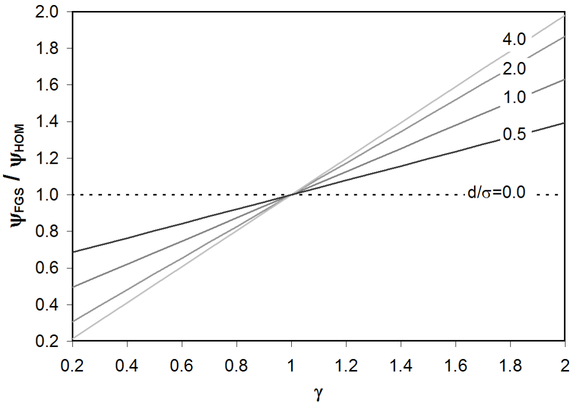


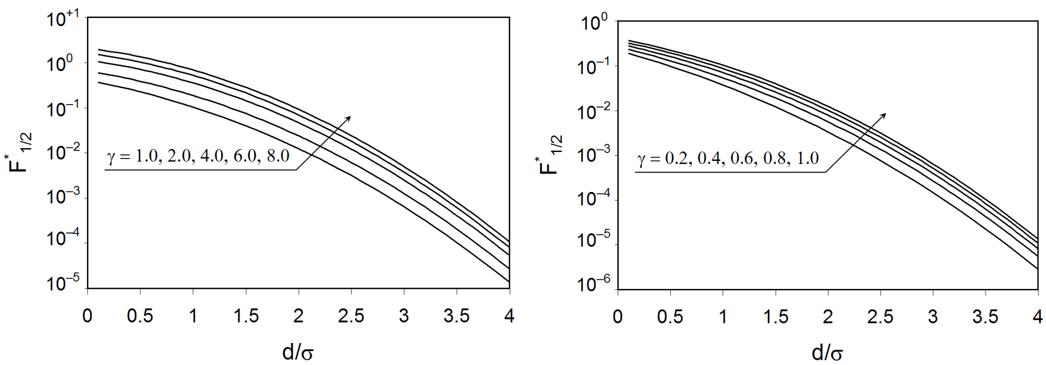
Figure 4: the ratio between the plasticity index of a FGS and that of a homogeneous surface vs. γ for different values of d/σ.

5 EFFECTS ON THE CONTACT CONDUCTANCE

The elastic grading also affects the joint contact conductance. This contact quantity can be determined by differentiating the applied load with respect to the mean plane separation distance, according to the analogy between contact stiffness and contact conductance proposed by Barber [16]

$$
\begin{aligned}
C &= -2\frac{\mathrm{d}(P)}{\mathrm{d}(d/\sigma)} = -\frac{8}{3}A_n\eta R^{1/2}\sigma^{3/2}E^*\frac{1}{\sqrt{2\pi}}\frac{\mathrm{d}\left(F_{3/2}^*\right)}{\mathrm{d}(d/\sigma)} = \\
&= -4A_n\eta R^{1/2}\sigma^{3/2}E^*\frac{1}{\sqrt{2\pi}}\int_{d/\sigma}^{\infty}(s-d/\sigma)^{1/2}\exp(-0.5s^2)\left[\gamma-(\gamma-1)\exp(-s)\right]\mathrm{d}s = \\
&= -\frac{8}{3}A_n\eta R^{1/2}\sigma^{3/2}E^*F_{1/2}^*
\end{aligned}
\tag{14}
$$

The function $F_{1/2}^*(h)$, which would correspond to a dimensionless contact conductance, is plotted vs. d/σ in Fig. 5 for positive and negative gradings. In these diagrams, the homogeneous case corresponds to the curve with $\gamma=1$. The higher the exponent γ, the higher the conductance for a given mean plane separation. Moreover, we note that the homogeneous case represents a lower bound to the conductance for positive gradings (see Fig. 5a), whereas it is an upper bound for negative gradings (see Fig. 5b).



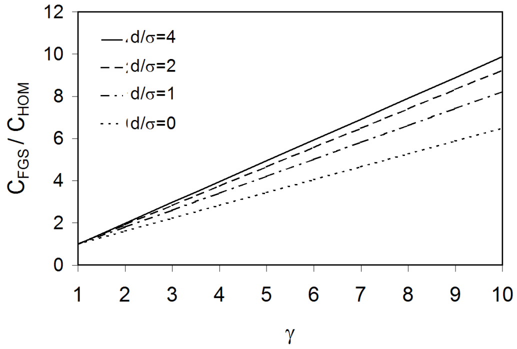
(a) Positive grading: $\gamma > 1$ (b) Negative grading: $\gamma < 1$

Figure 5: the function $F_{1/2}^*$ in Eq. (14) vs. d/σ for different values of γ.

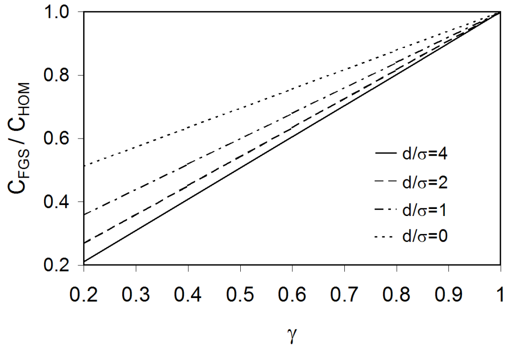
Again, to compare the contact predictions for a FGS with those for a homogenous interface, it is useful to consider the ratio between the contact conductance of a FGS surface and the same quantity of the homogeneous surface

$$
\frac{C_{\mathrm{FGS}}}{C_{\mathrm{HOM}}} = \frac{F_{1/2}^*}{F_{1/2}} \tag{15}
$$

This ratio is plotted in Fig. 6a vs. γ for different mean plane separations in the case of positive grading $(\gamma > 1)$.



(a) Positive grading: $\gamma > 1$



(b) Negative grading: $\gamma < 1$

Figure 6: the conductance ratio predicted by Eq. (15) vs. γ for different values of d/σ.

For a given mean plane separation or, equivalently, for a given real contact area, the contact conductance is predicted to be higher than that corresponding to the homogeneous surface. For high separations $(d/\sigma = 4)$, this effect is mainly governed by the asymptotic value of the function $E^*(s)/E^*$ and therefore we have approximately $C_{\mathrm{FGM}}/C_{\mathrm{HOM}} \cong \gamma$, as for the load. This trend is less pronounced for lower separations. The same ratio given by Eq. (15) is shown in Fig. 6b vs. γ in the case of negative grading $(\gamma < 1)$. In this instance, the trend is the opposite as before and, for a given real contact area, the contact conductance of the FGS is lower than that of the homogeneous surface. Again, for high separations $(d/\sigma = 4)$, the asymptotic value of the function $E^*(s)/E^*$ characterizes the contact behaviour and we have $C_{\mathrm{FGM}}/C_{\mathrm{HOM}} \cong \gamma$. However, for intermediate or low separations, the effect of grading is much less pronounced.

6 CONCLUSIONS

In this paper the Greenwood and Williamson contact theory for microscopically rough surfaces has been generalized by considering a grading on the elastic modulus of the asperities. This permits to model surfaces receiving superficial treatments or characterized by a chemical degradation. In the special case of an exponential grading, the numerical examples have shown that the dimensionless functions $F^*_{1/2}(h)$ and $F^*_{3/2}(h)$ increase by increasing the grading parameter γ . Correspondingly, the elastic grading has a significant effect on the applied load, on the slope of the real contact area-load relationship, on the plasticity index, as well as on the joint contact conductance. Treated surfaces with the highest asperities stiffer than the others (positive grading) are expected to have a higher load, a lower slope of the real contact area-load relationship and a higher contact conductance than homogeneous surfaces. Exactly the opposite trend is predicted in the case of negative grading, a situation that could be representative of the phenomenon of superficial chemical degradation.

References

[1] Greenwood J.A., Williamson J.B.P., "The contact of nominally flat surfaces", *Proc. R. Soc. London A*, **295**, 300-319, (1966).

[2] Nayak P.R., "Random process model of rough surfaces", *ASME J. Lubr. Technol.*, 93, 398-407, (1971).

[3] Bush A.W., Gibson R.D., Thomas T.R., "The elastic contact of a rough surface", *Wear*, **35**, 87-111, (1975).

[4] Greenwood J.A., "A simplified elliptic model of rough surface contact", *Wear*, **261**, 191-200, (2006).

[5] Zavarise G., Borri-Brunetto M., Paggi M., "On the reliability of microscopical contact models", *Wear*, **257**, 229-245, (2004).

[6] Zavarise G., Borri-Brunetto M., Paggi M., "On the resolution dependence of micromechanical contact models", *Wear*, **262**, 42-54, (2007).

[7] McCool J., "Nongaussian effects in microcontact", *Int. J. Mach. Tools Manufact.*, **32**, 115-123 (1992).

[8] Ciavarella M., Greenwood J.A., Paggi M., "Inclusion of "interaction" in the Greenwood & Williamson contact theory", *Wear*, **265**, 729-734, (2008).

[9] Hegazy A.A.H., "Thermal joint conductance of conforming rough surfaces: effect of surface micro-hardness variation", Ph.D. Thesis, University of Waterloo, (1985).

[10] McCool J., "Elastic contact of coated rough surfaces", In *Proc. Leeds-Lyon Symposium on Mechanics of Coatings,* **16**, 157-165, (1990).

[11] McCool J., "Extending the capability of the Greenwood Williamson microcontact model", *ASME J. Trib.*, **122**, 496-502, (2000).

[12] Liu T.-J., Wang Y.-S., Zhang C., "Axisymmetric frictionless contact of functionally graded materials", *Arch. Appl. Mech.*, **78**, 267-282, (2008).

[13] Zavarise G., Schrefler B.A., "Numerical analysis of microscopically elastic contact problems", in: J.J. Moreau, M. Raous, M. Jean (Eds.), *Contact Mechanics*, Plenum Press, New York, 305-312, (1995).

[14] Morandi Cecchi M., Pirozzi E., Zavarise G., "Numerical evaluation of special integrals with application to contact problems", Proc. Advanced Mathematical Tools in Metrology II, (P. Ciarlini, M.G. Cox, F. Pavese, D. Richter Eds.), 182-193, Oxford, UK, (1995).

[15] Sridhar M.R., Yovanovich M.M., "Review of elastic and plastic contact conductance models: comparison with experiments", *J. Thermophys. Heat Trans.*, **8**, 633-640, (1994).

[16] Barber J.R., "Bounds on the electrical resistance between contacting elastic rough bodies", *Proc. Roy. Soc. London A*, **459**, 53-66, (2003).