A unified mathematical formulation for the asymptotic analysis of singular elastic and electromagnetic fields

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Keywords: Singularities, Linear Elasticity, Electromagnetic fields.

SUMMARY. In the present contribution, the mathematical analogy existing between the singular stress field in elasticity due to antiplane loading and the singular electromagnetic fields in electromagnetism is derived with reference to the problem of isotropic multi-material wedges. These configurations, where dissimilar sectors converge to the same vertex, are very commonly observed in composite materials and may lead to stress-singularities. The proposed analogy permits to extend several elastic solutions already available in the literature to the analogous electromagnetic problems, without the need of performing new calculations.

1 INTRODUCTION

Interfaces between two materials are defined as bounding surfaces where a discontinuity of some kind occurs. In general, the interface is a surface through which material characteristics, such as concentration of an element, crystal structure, elastic properties, density, as well as dielectric permittivity and magnetic permeability, change abruptly from one side to another. This mismatch in the material properties is the reason for the occurrence of singularities.

Singular stress states can exist in several boundary value problems of linear elasticity where different materials are present (see [1–4] for a wide overview). In this context, the problems of multi-material wedges or junctions have received a great attention from the scientific community, since they are very commonly observed in composite materials. From the terminology point of view, multi-material wedges correspond to the situation where two or more different elastic wedges are joined together with a total wedge angle less than 2π. On the contrary, multi-material junctions imply that the total wedge angle formed by the material regions equals 2π, i.e. the whole plane is occupied by the materials without any voids.

In linear elasticity, the problem of bi-material wedges subjected to in-plane loading was firstly analyzed by Bogy [5] and by Hein and Erdogan [6] in 1971. Bi-material junctions were addressed by Bogy and Wang [7] in the same year and the general mathematical treatment of multi-material junctions was proposed by Theocaris [8] in 1974. Pageau et al. [9] and Carpinteri and Paggi [10] analyzed several configurations involving tri-material junctions with perfectly bonded or debonded interfaces, whereas Inoue and Koguchi [11] proposed a detailed study on tri-material wedges. In these contributions, three different mathematical techniques were used for the characterization of the singular stress field and a demonstration of their equivalence has been recently provided by Paggi and Carpinteri [4].

Most of the efforts, including those previously mentioned, have been directed to the characterization of stress-singularities for in-plane loading, where the problem is governed by a biharmonic equation. The out-of-plane loading, also referred to as antiplane shear problem, is governed by a simpler harmonic equation. In spite of that, it has received a minor attention as compared to the in-plane problem. From the historical point of view, stress-singularities due to antiplane loading were firstly addressed by Rao [12] in 1971. In his study, a general procedure for the identification of stress-singularities at the intersection of two or more interfaces in domains governed by harmonic equations was presented. Afterwards, Fenner [13] examined the Mode III loading problem of a crack meeting a bi-material interface using the eigenfunction expansion method proposed by Williams [14]. More recently, Ma and Hour [15, 16] analyzed bi-material wedges using the Mellin transform technique and Pageau et al. [17] investigated the singular stress field of bonded and debonded tri-material junctions according to the eigenfunction expansion method.

The mathematical problems characterized by biharmonic or harmonic type of equations where the stress-field is singular present several analogies with other engineering problems. Regarding singular biharmonic problems, the analysis of the stress-singularities at the vertex of a multi-material wedge or junction has its analogous counterpart in the analysis of the Stokes flow of dissimilar immiscible fluids, as recently pointed out by Paggi and Carpinteri [4]. As far as the harmonic problems are concerned, the mathematical analogy between the steady-state heat transfer and the antiplane loading of composite regions was firstly recognized by Sinclair [18] in 1980. Very recently, Paggi and Carpinteri [4] put into evidence the analogy between antiplane loading and the St. Venant torsion of composite bars. To the knowledge of the present authors, it seems that the analogy between elasticity and electromagnetism has been overlooked. In the solution of diffraction problems, in fact, Bouwkamp [19] and Meixner [20, 21] found that the electromagnetic field vectors may become infinite at the sharp edges of a diffracting obstacle. As for the problem of re-entrant corners in elasticity, the order of the singularity is determined by the imposition of the boundary conditions (BCs) near the singular point. As it will be shown in the sequel, this mathematical problem is governed by the Helmholtz equation (see also [22] for a detailed overview). This partial differential equation admits a separable variable form solution, as for the antiplane problem in elasticity governed by the Laplace equation. Moreover, as far as the asymptotic analysis of the singular electromagnetic fields is concerned, the eigenfunction expansion method can be used, in close analogy with the well-known method proposed by Williams [14] in elasticity.

In the present paper, the mathematical analogy existing between the singular stress field in elasticity due to antiplane loading and the singular electromagnetic fields in electromagnetism is derived with reference to the problem of isotropic multi-material wedges. As a main outcome, the order of the stress-singularities of various geometrical and mechanical configurations already determined in the literature can be adopted for the analogous electromagnetic problems, without the need of performing new calculations.

2 STRESS-SINGULARITIES IN ELASTICITY DUE TO ANTIPLANE LOADING

The geometry of a plane elastostatic problem consisting of $n-1$ dissimilar isotropic, homogeneous sectors of arbitrary angles perfectly bonded along their interfaces converging to the same vertex O is shown in Fig. 1. Each of the material regions is denoted by Ω_i with $i = 1, \dots, n-1$, and it is comprised between the interfaces Γ_i and Γ_{i+1}.

Out-of-plane loading due to antiplane shear (Mode III) on composite wedges can lead to stresses that can be unbounded at the junction vertex O. When out-of-plane deformations only exist, the following displacements in cylindrical coordinates can be considered with the origin at the vertex O:

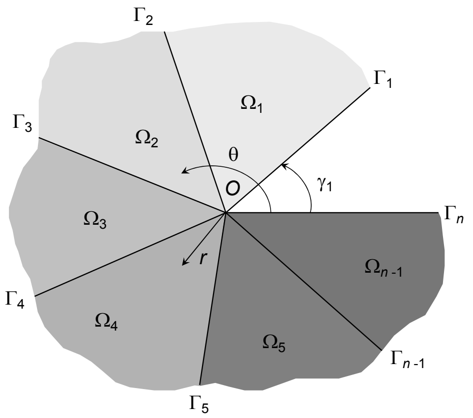

Figure 1: Scheme of a multi-material wedge.

$$u_r = 0, \tag{1a}$$
$$u_\theta = 0, \tag{1b}$$
$$u_z = u_z(r, \theta), \tag{1c}$$

where u_z is the out-of-plane displacement, which depends on r and θ. For such a system of displacements, the strain field components become

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = 0, \tag{2a}$$
$$\gamma_{rz} = \frac{\partial u_z}{\partial r}, \tag{2b}$$
$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta}. \tag{2c}$$

From the application of the Hooke's law, the stress field components are given by:

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = 0, \tag{3a}$$
$$\tau_{rz} = G_i \gamma_{rz} = G_i \frac{\partial u_z}{\partial r}, \tag{3b}$$
$$\tau_{\theta z} = G_i \gamma_{\theta z} = \frac{G_i}{r}\frac{\partial u_z}{\partial \theta}, \tag{3c}$$

where G_i is the shear modulus of the i-th material region. The equilibrium equations in absence of body forces reduce to a single relationship between the tangential stresses:

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{1}{r}\tau_{rz} = 0, \quad \forall (r, \theta) \in \Omega_i. \tag{4}$$

Introducing Eqs. (3) into Eq. (4), the harmonic condition upon u_z is derived:

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} = \nabla^2 u_z = 0,\ \forall (r, \theta) \in \Omega_i. \tag{5}$$

In the framework of the eigenfunction expansion method [14], the following separable variable form for the longitudinal displacement u_z can be adopted ($\forall(r,\theta) \in \Omega_i$):

$$u_z(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}\,(\theta,\lambda_j) + r^{\lambda_j+1} h_{i,j}\,(\theta,\lambda_j) + r^{\lambda_j+2} l_{i,j}\,(\theta,\lambda_j) + \dots, \tag{6}$$

where λ_j are the eigenvalues of the problem, whereas $f_{i,j}$, $h_{i,j}$ and $l_{i,j}$ are the eigenfunctions. The summation with respect to the subscript j is introduced in Eq. (6), since it is possible to have more than one eigenvalue and the Superposition Principle can be applied.

Introducing Eq. (6) into Eq. (5), we find the following relationship holding for each eigenvalue λ_j:

$$\begin{aligned} & r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2} + \lambda_j^2 f_{i,j}\right) + r^{\lambda_j-1}\left(\frac{\mathrm{d}^2 h_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+1)^2 h_{i,j}\right) + \\ & + r^{\lambda_j}\left(\frac{\mathrm{d}^2 l_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+2)^2 l_{i,j}\right) + \dots = 0. \end{aligned} \tag{7}$$

Hence, the coefficients of the term in r^{λ_j-2} must vanish, implying that the eigenfunctions $f_{i,j}$ are a linear combination of trigonometric functions:

$$f_{i,j}(\theta,\lambda_j) = A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta). \tag{8}$$

The eigenfunctions $f_{i,j}$ are particularly important, since they enter the first term of the series expansion (6), which is responsible for the singular behaviour of the stress field components for $r \to 0$. In fact, if we truncate the series expansion (6) to this first term and we introduce it into Eq. (3), the longitudinal displacement and the tangential stresses can be expressed in terms of the eigenfunction and its first derivative:

$$u_z = r^{\lambda_j} f_{i,j} = r^{\lambda_j}\left[A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta)\right] \tag{9a}$$

$$\tau_{rz} = G_i\lambda_j r^{\lambda_j-1} f_{i,j} = G_i\lambda_j r^{\lambda_j-1}\left[A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta)\right] \tag{9b}$$

$$\tau_{\theta z} = G_i r^{\lambda_j-1} f'_{i,j} = G_i\lambda_j r^{\lambda_j-1}\left[A_{i,j}\cos(\lambda_j\theta) - B_{i,j}\sin(\lambda_j\theta)\right]. \tag{9c}$$

The determination of the power of the stress-singularity, $\lambda_j - 1$, can be performed by imposing the boundary conditions (BCs) along the edges Γ_1 and Γ_n and at the bi-material interfaces Γ_i, with $i = 2, \dots, n-1$. Along the edges Γ_1 and Γ_n, defined by the angles γ_1 and γ_n, we consider two possibilities: one corresponding to unrestrained stress-free edges

$$\tau_{\theta z}(r,\gamma_1) = 0, \tag{10a}$$

$$\tau_{\theta z}(r,\gamma_n) = 0, \tag{10b}$$

and the other for fully restrained (clamped) edges

$$u_z^i(r,\gamma_1) = 0, \tag{11a}$$

$$u_z^i(r,\gamma_n) = 0. \tag{11b}$$

At the interfaces, the following continuity conditions of displacements and stresses have to be imposed ($i = 1, \dots, n-2$):

$$u_z^i(r,\gamma_{i+1}) = u_z^{i+1}(r,\gamma_{i+1}), \tag{12a}$$

$$\tau_{\theta z}^i(r,\gamma_{i+1}) = \tau_{\theta z}^{i+1}(r,\gamma_{i+1}). \tag{12b}$$

In this way, a set of $2n$ homogeneous equations in the $2n+1$ unknowns $A_{i,j}$, $B_{i,j}$, and λ_j can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{v} = \mathbf{0}, \tag{13}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue, and $\mathbf{v}$ represents the vector that collects the unknowns $A_{i,j}$ and $B_{i,j}$. More specifically, the coefficient matrix entering Eq. (13) is characterized by a sparse structure:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{N}^1_{\gamma_1} & & & & & & & \\ \mathbf{M}^1_{\gamma_2} & -\mathbf{M}^2_{\gamma_2} & & & & & & \\ & \mathbf{M}^2_{\gamma_3} & -\mathbf{M}^3_{\gamma_3} & & & & & \\ & & \cdots & \cdots & & & & \\ & & & \mathbf{M}^{i-1}_{\gamma_i} & -\mathbf{M}^{i}_{\gamma_i} & & & \\ & & & & \cdots & \cdots & & \\ & & & & & \mathbf{M}^{n-2}_{\gamma_{n-1}} & -\mathbf{M}^{n-1}_{\gamma_{n-1}} \\ & & & & & & \mathbf{N}^{\mathrm{n}-1}_{\gamma_\mathrm{n}} \end{bmatrix} \tag{14}$$

where the non null elementary matrix $\mathbf{M}^i_\theta$ related to the interface BCs is given by:

$$\mathbf{M}^i_\theta = \begin{bmatrix} \sin(\lambda_j\theta) & \cos(\lambda_j\theta) \\ G_i\cos(\lambda_j\theta) & -G_i\sin(\lambda_j\theta) \end{bmatrix} \tag{15}$$

and the components of the vector $\mathbf{v}$ are:

$$\mathbf{v} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^i, \ldots, \mathbf{v}^{n-2}, \mathbf{v}^{n-1}\}^T, \tag{16}$$

with $\mathbf{v}^i = \{A_{i,j}, B_{i,j}\}^T$. The term $\mathbf{N}^i_\theta$ depends on the BCs along the edges Γ_1 and Γ_n. For stress-free edges we have:

$$\mathbf{N}^i_\theta = \{G_i\cos(\lambda_j\theta), -G_i\sin(\lambda_j\theta)\}, \tag{17}$$

whereas for clamped edges it is given by

$$\mathbf{N}^i_\theta = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}. \tag{18}$$

A nontrivial solution of the equation system (13) exists if and only if the determinant of the coefficient matrix vanishes. This condition yields an eigenequation which has to be solved for the eigenvalues λ_j that, in the most general case, do depend on the elastic properties of the materials.

3 SINGULARITIES IN THE ELECTRO-MAGNETIC FIELDS

Let us consider the multi-material wedge shown in Fig. 2. Each material is isotropic and has a dielectric permittivity ϵ_i and a magnetic permeability μ_i. We also admit the presence of a perfect electric conductor (PEC) in the region 1 defined by the interfaces Γ_1 and Γ_n. For periodic fields with circular frequency ω, the Maxwell's equations for each homogeneous angular domain read [21]:

$$\mathrm{i}\omega\epsilon_i\mathbf{E} = \nabla\times\mathbf{H}, \tag{19a}$$

$$-\mathrm{i}\omega\mu_i\mathbf{H} = \nabla\times\mathbf{E}, \tag{19b}$$

where the symbol i stands for the imaginary unit.

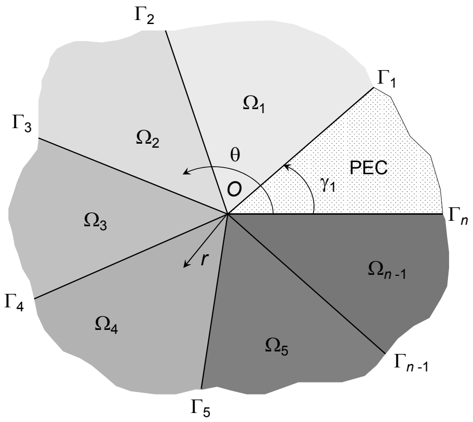

Figure 2: Scheme of a multi-material wedge with a PEC material.

In cylindrical coordinates r, θ, z, with the z axis perpendicular to the plane of the wedge, and considering electromagnetic fields independent of z, the Maxwell's equations reduce to the following conditions upon the components of the electric and magnetic fields:

$$\mathrm{i}\omega\epsilon_i E_r = \frac{1}{r}\frac{\partial H_z}{\partial \theta}, \tag{20a}$$

$$\mathrm{i}\omega\epsilon_i E_\theta = -\frac{\partial H_z}{\partial r}, \tag{20b}$$

$$\mathrm{i}\omega\epsilon_i E_z = \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta}, \tag{20c}$$

$$-\mathrm{i}\omega\mu_i H_r = \frac{1}{r}\frac{\partial E_z}{\partial \theta}, \tag{20d}$$

$$-\mathrm{i}\omega\mu_i H_\theta = -\frac{\partial E_z}{\partial r}, \tag{20e}$$

$$-\mathrm{i}\omega\mu_i H_z = \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}. \tag{20f}$$

It is easy to verify that the E_z and H_z components satisfy the Helmholtz equation:

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + k_i^2 E_z = \nabla^2 E_z + k_i^2 E_z = 0, \tag{21a}$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2} + k_i^2 H_z = \nabla^2 H_z + k_i^2 H_z = 0, \tag{21b}$$

where $k_i = \omega^2\epsilon_i\mu_i$.

In close analogy with the antiplane problem in linear elasticity, the following separable form for E_z and H_z can be postulated ($\forall\ (r,\theta) \in \Omega_i$):

$$E_z(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}\ (\theta,\lambda_j) + r^{\lambda_j+1} h_{i,j}\ (\theta,\lambda_j) + r^{\lambda_j+2} l_{i,j}\ (\theta,\lambda_j) + \dots \tag{22a}$$

$$H_z(r,\theta) = \sum_j r^{\lambda_j} F_{i,j}\ (\theta,\lambda_j) + r^{\lambda_j+1} H_{i,j}\ (\theta,\lambda_j) + r^{\lambda_j+2} L_{i,j}\ (\theta,\lambda_j) + \dots \tag{22b}$$

where λ_j are the eigenvalues, whereas $f_{i,j}$, $h_{i,j}$, $l_{i,j}$, $F_{i,j}$, $H_{i,j}$ and $L_{i,j}$ are the eigenfunctions.

We can introduce Eq. (22) into Eq. (21), obtaining the following equalities:

$$\begin{aligned} r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2}+\lambda_j^2 f_{i,j}\right)+r^{\lambda_j-1}\left(\frac{\mathrm{d}^2 h_{i,j}}{\mathrm{d}\theta^2}+(\lambda_j+1)^2 h_{i,j}\right)+ \\ +r^{\lambda_j}\left(\frac{\mathrm{d}^2 l_{i,j}}{\mathrm{d}\theta^2}+(\lambda_j+2)^2 l_{i,j}+k_i^2 f_{i,j}\right)+\cdots=0 \end{aligned} \tag{23a}$$

$$\begin{aligned} r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 F_{i,j}}{\mathrm{d}\theta^2}+\lambda_j^2 F_{i,j}\right)+r^{\lambda_j-1}\left(\frac{\mathrm{d}^2 H_{i,j}}{\mathrm{d}\theta^2}+(\lambda_j+1)^2 H_{i,j}\right)+ \\ +r^{\lambda_j}\left(\frac{\mathrm{d}^2 L_{i,j}}{\mathrm{d}\theta^2}+(\lambda_j+2)^2 L_{i,j}+k_i^2 F_{i,j}\right)+\cdots=0 \end{aligned} \tag{23b}$$

The coefficients of the term in r^{λ_j-2} must vanish, implying that the eigenfunctions $f_{i,j}$ and $F_{i,j}$ are a linear combination of trigonometric functions, in perfect analogy with the eigenfunction $f_{i,j}$ in antiplane elasticity (see Eq. (8)):

$$f_{i,j}(\theta,\lambda_j)=A_i\sin(\lambda_j\theta)+B_i\cos(\lambda_j\theta), \tag{24a}$$

$$F_{i,j}(\theta,\lambda_j)=C_i\sin(\lambda_j\theta)+D_i\cos(\lambda_j\theta). \tag{24b}$$

These eigenfunctions are particularly important, since they enter the first terms of the series expansions (22a) and (22b) and are responsible for the singular behaviour of the components E_r, E_θ, H_r and H_θ of the electric and magnetic fields near the wedge apex. In particular, from Eq. (20), we observe that:

$$E_r=\frac{1}{r\mathrm{i}\omega\epsilon_i}\frac{\partial H_z}{\partial\theta}=\frac{1}{\mathrm{i}\omega\epsilon_i}\sum_j\lambda_j r^{\lambda_j-1}F'_{i,j}+\cdots\sim O(r^{\lambda_j-1}), \tag{25a}$$

$$E_\theta=-\frac{1}{\mathrm{i}\omega\epsilon_i}\frac{\partial H_z}{\partial r}=-\frac{1}{\mathrm{i}\omega\epsilon_i}\sum_j\lambda_j r^{\lambda_j-1}F_{i,j}+\cdots\sim O(r^{\lambda_j-1}), \tag{25b}$$

$$H_r=-\frac{1}{r\mathrm{i}\omega\mu_i}\frac{\partial E_z}{\partial\theta}=-\frac{1}{\mathrm{i}\omega\mu_i}\sum_j\lambda_j r^{\lambda_j-1}f'_{i,j}+\cdots\sim O(r^{\lambda_j-1}), \tag{25c}$$

$$H_\theta=\frac{1}{\mathrm{i}\omega\mu_i}\frac{\partial E_z}{\partial r}=\frac{1}{\mathrm{i}\omega\mu_i}\sum_j\lambda_j r^{\lambda_j-1}f'_{i,j}+\cdots\sim O(r^{\lambda_j-1}). \tag{25d}$$

Hence, $E_z\sim O(r^{\lambda_j})$ and $H_z\sim O(r^{\lambda_j})$ are the analogous counterparts of u_z and remain finite for $r\to 0$. On the contrary, the radial components of the electric and magnetic fields, E_r and H_r, are analogous to $\tau_{\theta z}$ and the circumferential components, E_θ and H_θ, are analogous to τ_{rz}. All of these components diverge when $r\to 0$ with a power-law singularity of order $-1<(\lambda_j-1)<0$.

The following BCs hold along the edges Γ_1 and Γ_n of the PEC:

$$E_z^1(r,\gamma_1)=0, \tag{26a}$$

$$E_z^{n-1}(r,\gamma_n)=0, \tag{26b}$$

$$E_r^1(r,\gamma_1)=0, \tag{26c}$$

$$E_r^{n-1}(r,\gamma_n)=0, \tag{26d}$$

whereas continuity BCs have to be imposed along the bi-material interfaces ($i = 1, \ldots, n-2$):

$$E_z^i(r, \gamma_{i+1}) = E_z^{i+1}(r, \gamma_{i+1}), \tag{27a}$$

$$E_r^i(r, \gamma_{i+1}) = E_r^{i+1}(r, \gamma_{i+1}), \tag{27b}$$

$$H_z^i(r, \gamma_{i+1}) = H_z^{i+1}(r, \gamma_{i+1}), \tag{27c}$$

$$H_r^i(r, \gamma_{i+1}) = H_r^{i+1}(r, \gamma_{i+1}). \tag{27d}$$

Using Eqs. (25), the BCs (26) become:

$$E_z^1(r, \gamma_1) = 0, \tag{28a}$$

$$E_z^{n-1}(r, \gamma_n) = 0, \tag{28b}$$

$$\frac{\partial H_z^1}{\partial \theta}(r, \gamma_1) = 0, \tag{28c}$$

$$\frac{\partial H_z^{n-1}}{\partial \theta}(r, \gamma_n) = 0, \tag{28d}$$

whereas those defined by Eq. (27) become ($i = 1, \ldots, n-2$):

$$E_z^i(r, \gamma_{i+1}) = E_z^{i+1}(r, \gamma_{i+1}), \tag{29a}$$

$$\frac{1}{\epsilon_i}\frac{\partial H_z^i}{\partial \theta}(r, \gamma_{i+1}) = \frac{1}{\epsilon_{i+1}}\frac{\partial H_z^{i+1}}{\partial \theta}(r, \gamma_{i+1}) \tag{29b}$$

$$H_z^i(r, \gamma_{i+1}) = H_z^{i+1}(r, \gamma_{i+1}), \tag{29c}$$

$$\frac{1}{\mu_i}\frac{\partial E_z^i}{\partial \theta}(r, \gamma_{i+1}) = \frac{1}{\mu_{i+1}}\frac{\partial E_z^{i+1}}{\partial \theta}(r, \gamma_{i+1}). \tag{29d}$$

It is interesting to note that Eqs. (21), (28) and (29) can be separated into two sets of equations, one involving only H_z and another involving only E_z. They correspond to the so-called Transverse Electric (TE) and Transverse Magnetic (TM) fields, respectively.

Considering the series expansion for E_z and H_z truncated at the first term, along with the expressions for the eigenfunctions $f_{i,j}$ and $F_{i,j}$, the boundary value problem consists of two sets of $2n$ equations in $2n+1$ unknowns, one for E_z and another for H_z. The former equation set (TM case) involves the coefficients $A_{i,j}$, $B_{i,j}$ and λ_j and can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{v} = \mathbf{0}, \tag{30}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue and $\mathbf{v}$ represents the vector which collects the unknowns $A_{i,j}$ and $B_{i,j}$. The coefficient matrix in Eq. (30) has exactly the same structure as that for the elasticity problem in Eq. (13), provided that we consider $\mathbf{N}_\theta^i = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}$ and we set $G_i = 1/\mu_i$.

The latter equation set (TE case) involves the coefficients $C_{i,j}$, $D_{i,j}$ and λ_j and can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{w} = \mathbf{0}, \tag{31}$$

where $\mathbf{\Lambda}$ is the coefficient matrix which depends on the eigenvalue and $\mathbf{w}$ represents the vector which collects the unknowns $C_{i,j}$ and $D_{i,j}$. Again, the coefficient matrix in Eq. (31) has exactly the same structure as that for the elasticity problem in Eq. (13), provided that we consider $\mathbf{N}_\theta^i = \{\cos(\lambda_j\theta), -\sin(\lambda_j\theta)\}$ and we set $G_i = 1/\epsilon_i$.

For the existence of nontrivial solutions, the determinants of the coefficient matrices must vanish, yielding two eigenequations that, for given values of ϵ_i and μ_i, determine the eigenvalues λ_j^{TE} and λ_j^{TM}. Hence, this proves that the analysis of the singularities of the electro-magnetic field is mathematically analogous to that for the elastic field due to antiplane loading. For a given multi-material wedge problem, in elasticity we can distinguish between singularities due to either stress-free or clamped edges, depending on the BCs specified along the edges defined by the interfaces Γ_1 and Γ_n. In electromagnetism, these BCs are both present when a multi-material wedge includes a PEC material. In this instance, the singularities related to the homogeneous equation system (30) correspond to those obtained from the analogous elastic problem with $G_i = 1/\mu_i$ and with clamped edges Γ_1 and Γ_n. On the other hand, the singularities related to the homogeneous equation system (31) correspond to those obtained from the analogous elastic problem with $G_i = 1/\epsilon_i$ and with stress-free edges Γ_1 and Γ_n.

A notable limit case is represented by a PEC embedded into a single homogeneous material, say Ω_1 (see Fig. 1). In this case, both E_z and H_z have the same singularity, whose power is independent of the material properties of Ω_1:

$$\lambda^{TE} = \lambda^{TH} = m\frac{\pi}{2\pi - \gamma_1}, \tag{32}$$

where m is a natural number. The minimum eigenvalue is equal to $1/2$ for $\gamma_1 = 0$. In elasticity, this situation corresponds to a crack (when stress-free BCs are imposed) or to a rigid line inclusion or anti-crack (when clamped BCs are imposed). For higher values of γ_1, $(1 - \lambda)$ diminishes and vanishes for a half-plane ($\gamma_1 = \pi$). For $\gamma_1 > \pi$, the electromagnetic fields are no longer singular.

4 CONCLUSIONS

In the present paper, we have demonstrated that the asymptotic analysis of the stress-singularities at the vertex of multi-material wedges and junctions in antiplane elasticity is analogous to the corresponding problem in electromagnetism. In particular, when an isotropic multi-material wedge with PEC boundaries is considered, we have shown that two independent problems can be defined, one for TE fields, associated to an eigenequation for H_z, and one for TM fields, associated to an eigenequation for E_z. The eigenequation for E_z corresponds exactly to that obtained for the same geometrical configuration in antiplane elasticity by setting $G_i = 1/\mu_i$ and replacing the PEC region with an infinitely stiff material leading to clamped edge BCs along Γ_1 and Γ_n. Similarly, the other eigenequation for H_z can be obtained in antiplane elasticity for the same geometrical configuration by setting $G_i = 1/\epsilon_i$ and replacing the PEC region with an infinitely soft material leading to stress-free BCs along Γ_1 and Γ_n.

References

[1] A.H. England. On stress singularities in linear elasticity. *Int. J. Engng. Sci.*, 9:571–585, 1971.

[2] G.B. Sinclair. Stress singularities in classical elasticity-I: removal, interpretation, and analysis. *Appl. Mech. Rev.*, 57:251–297, 2004.

[3] G.B. Sinclair. Stress singularities in classical elasticity-II: asymptotic identification. *Appl. Mech. Rev.*, 57:385–439, 2004.

[4] M. Paggi and A. Carpinteri. On the stress singularities at multimaterial interfaces and related analogies with fluid dynamics and diffusion. *Appl. Mech. Rev.*, 61:1–22, 2008.

[5] D.B. Bogy. Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. *J. Appl. Mech.*, 38:377–386, 1971.

[6] V.L. Hein and F. Erdogan. Stress singularities in a two-material wedge. *Int. J. Fract. Mech.*, 7:317–330, 1971.

[7] D.B. Bogy and K.C. Wang. Stress singularities at interface corners in bonded dissimilar isotropic elastic materials. *Int. J. Sol. Struct.*, 7:993–1005, 1971.

[8] P.S. Theocaris. The order of singularity at a multi-wedge corner of a composite plate. *Int. J. Engng. Sci.*, 12:107–120, 1974.

[9] S.S. Pageau, P.F. Joseph, and S.B. Jr. Biggers. The order of stress singularities for bonded and disbonded three-material junctions. *J. Appl. Mech.*, 31:2979–2997, 1994.

[10] A. Carpinteri and M. Paggi. Analytical study of the singularities arising at multi-material interfaces in 2D linear elastic problems. *Engng. Fract. Mech.*, 74:59–74, 2007.

[11] T. Inoue and H. Koguchi. Influence of the intermediate material on the order of stress singularity in three-phase bonded structure. *Int. J. Sol. Struct.*, 33:399–417, 1996.

[12] A.K. Rao. Stress concentrations and singularities at interface corners. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, 51:395–406, 1971.

[13] D.N. Fenner. Stress singularities in composite materials with an arbitrarily oriented crack meeting an interface. *Int. J. Fract.*, 12:705–712, 1976.

[14] M.L. Williams. Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.*, 74:526–528, 1952.

[15] C.C. Ma and B.L. Hour. Analysis of dissimilar anisotropic wedges subjected to antiplane shear deformation. *Int. J. Sol. Struct.*, 25:1295–1308, 1989.

[16] C.C. Ma and B.L. Hour. Antiplane problems in composite materials with an inclined crack terminating at a bi-material interface. *Int. J. Sol. Struct.*, 26:1387–1400, 1990.

[17] S.S. Pageau, P.F. Joseph, and S.B. Jr. Biggers. Finite element evaluation of free-edge singular stress fields in anisotropic materials. *Int. J. Num. Meth. Engng.*, 38:2225–2239, 1995.

[18] G.B. Sinclair. On the singular eigenfunctions for plane harmonic problems in composite regions. *J. Appl. Mech.*, 47:87–92, 1980.

[19] C. Bouwkamp. A note on singularities occurring at sharp edges in electromagnetic diffraction theory. *Physica*, 12:467, 1946.

[20] J. Meixner. Die kentenbedingung in der theorie der beugung electromagnetischer wellen an vollkommen leitenden ebenen schirmen. *Ann. Phys.*, 6:1–9, 1949.

[21] J. Meixner. The behavior of electromagnetic fields at edges. *IEEE Trans. Antennas Propag.*, AP20:442–446, 1972.

[22] J. van Bladel. *Singular Electromagnetic Fields and Sources.* Clarendon Press, Oxford, UK, 1991.