Numerical Methods for the Evaluation of The Shakedown and Limit Loads

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SUMMARY. The paper exploits the similarity in the numerical strategies used to evaluate limit or shakedown load safety factors. While theoretically different, the interior point methods and strain driven path–following algorithms, are similar from a computational point of view. In this work a comparison of these methods is performed in order to show analogies and numerical performances. The aim is to reuse the great amount of research carried out to develop efficient interior point algorithms to improve the performance of the strain–driven based formulations and viceversa.

1 INTRODUCTION

The static and kinematic shakedown theorems, including the limit analysis as a special case, furnish in a direct and elegant fashion, a reliable safety factor against plastic collapse, loss in functionality due to excessive deformation (ratcheting) or collapse due to fatigue (plastic shakedown) [1]. Based on these theorems the so called *direct methods* evaluate the safety factor solving convex optimization problems, that for real structures discretized by means of finite elements, usually require the solution of large dimension convex programming problems. In contrast the structure safety factor can be evaluated by means of the complete reconstruction of the equilibrium path, using standard *strain driven* strategies (see [2] for extension to shakedown).

Alternatively hand the complete elasto–plastic path–following analysis, independently of its use as a tool to evaluate the limit load, gives important additional mechanical information about the structural behaviour. The extremal paths theory of Ponter and Martin for example gives a coherent justification of the use of a return mapping algorithm based on the closest point projection and a mechanical sense to the numerical evaluated equilibrium path, so making it possible to use all the information obtained, regarding both the static and kinematic behaviour of the structure, even for equilibrium configurations other than the collapse one. Although still not completely analyzed, also in the case of shakedown analysis strain driven type methods furnish further information regarding the mechanical behavior of the structure (see for example section 4.6 of [2]) that could be used in the design process. However an extension of the interior point methods also for the evaluation of the elasto-plastic equilibrium path is, possible and has been recently proposed by Krabbenhøft et al. [7].

A comparison of the convenience of a series of aspects, regarding efficiency, robustness and accuracy between these two different strategies of analysis, will be investigated. In particular the apparently large theoretical difference between these two strategies is not so great in the numerical implementation, where the two methods closely resemble each other. The similarities between the two approaches also show the convenience and disadvantages of the two formulations also making it possible to improve the performance of each method using the experience from the other.

The comparison is performed with an improved version of the strain driven algorithm for shakedown and limit analysis described in [2] that will be presented here for the first time. The algorithm is reformulated expressing shakedown load domain in terms of a suitable reference load that directly fulfills, from a theoretical point of view, the convergence requirements presented in [3]. The problems so reformulated directly reduce to the classic strain driven limit analysis case when a single proportional load is present and, more than the original proposal in [2], appear to be easy to implement in an existing code that already performs limit analysis. Also the performance of the algorithm can take advantage of this reformulation. Furthermore the use of nonlinear yield function gives a better performance with respect to both effectiveness and accuracy. With a small modification in the elastic modula used to evaluate the trial stress we obtain a simple expression for the closest point correction that reduces to the classical radial return mapping when a single yield function is present, as usually occurs for the limit analysis case.

The proposed modified method proves to be competitive with respect to interior point methods as regards both efficiency and robustness. A series of results regarding accuracy, robustness and efficiency will be presented.

2 THE DISCRETE FEM MODEL

In the following limit and shakedown problems are reformulated in terms of finite element algebraic equations to have immediately clarify how the problem at hand can be framed in the general context of a convex nonlinear optimization problem.

Using a symbolic notation, we assume that the displacement $\boldsymbol{u}[\boldsymbol{x}]$ and the stress $\boldsymbol{\sigma}[\boldsymbol{x}]$ of a point $\boldsymbol{x}$ of the body domain $\mathcal{B}$ are expressed, by means of a finite number of interpolation variables using a mixed finite element format. In particular we assume the discrete FEM model be expressed in terms of the values that stresses and displacements assume in a number of stress and displacement nodes (or control points):

$$\boldsymbol{\sigma}_g = \boldsymbol{\sigma}[\boldsymbol{x}_g] \quad , \quad \boldsymbol{u}_i = \boldsymbol{u}[\boldsymbol{x}_i] \tag{1}$$

with $g = 1 \ldots N_\sigma$ and $i = 1 \ldots N_u$ so that the vector displacements and stress fields can be defined as:

$$\boldsymbol{\sigma}[\boldsymbol{x}] = \boldsymbol{N}_\sigma[\boldsymbol{x}]\mathbf{t} \quad , \quad \boldsymbol{u}[\boldsymbol{x}] = \boldsymbol{N}_u[\boldsymbol{x}]\boldsymbol{d} \tag{2}$$

where $\boldsymbol{N}_\sigma[\boldsymbol{x}]$ and $\boldsymbol{N}_u[\boldsymbol{x}]$ are the matrices collecting the interpolation functions and the global displacements $\boldsymbol{d} \in \mathbb{R}^{N_d}$ and stress $\mathbf{t} \in \mathbb{R}^{N_t}$ vectors collects all the n_σ stress nodes and the n_u displacement node vectors as:

$$\mathbf{t} = \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \vdots \\ \boldsymbol{\sigma}_{N_\sigma} \end{bmatrix} \quad , \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{n_u} \end{bmatrix}$$

The discrete forms of the kinematical relationship between the strain $\boldsymbol{\varepsilon}[\boldsymbol{x}]$ and displacements $\boldsymbol{u}[\boldsymbol{x}]$ written as:

$$\boldsymbol{\varepsilon}[\boldsymbol{x}] = \boldsymbol{N}_\epsilon[\boldsymbol{x}]\boldsymbol{d} \quad , \quad \boldsymbol{N}_\epsilon = \boldsymbol{D}[\boldsymbol{x}]\boldsymbol{N}_u[\boldsymbol{x}]$$

where $\boldsymbol{D}[\boldsymbol{x}]$ is the kinematical operator while the discrete finite element representation of the equilibrium equations becomes

$$\boldsymbol{Q}^T\mathbf{t} = \lambda\boldsymbol{p} \quad , \quad \boldsymbol{Q}^T \equiv \int_B \boldsymbol{N}_\varepsilon[\boldsymbol{x}]^T\boldsymbol{N}_\sigma[\boldsymbol{x}]\; dV \tag{3}$$

where the external forces vector, when only mechanical actions are considered, is

$$\boldsymbol{p} = \int_{\mathcal{B}} \boldsymbol{N}_u^T\mathbf{b}[\boldsymbol{x}]dV + \int_{\partial_f\mathcal{B}} \boldsymbol{N}_u^T\boldsymbol{f}[\boldsymbol{x}]dA$$

where $\mathbf{b}[\boldsymbol{x}]$ represent the external body forces and $\boldsymbol{f}[\boldsymbol{x}]$ the surface force on the boundary $\partial_f \mathcal{B}$. Finally we also have the discrete form of the compatibility condition as

$$\boldsymbol{\varrho} = \boldsymbol{Q}\boldsymbol{d} \tag{4}$$

where $\boldsymbol{\varrho}^T = [\boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_{N_\sigma}]$ collects the discrete strain conjugate, in the virtual work sense, to $\mathbf{t}$.

When a linear elastic constitutive law defined by the compliance operator $\mathbf{E}$ is adopted

$$\boldsymbol{\sigma}[\boldsymbol{x}] = \mathbf{E}\boldsymbol{\varepsilon}[\boldsymbol{x}] \tag{5}$$

and a compatible interpolation is used we have:

$$\boldsymbol{K}_e \boldsymbol{d} = \lambda \boldsymbol{p} \ \ , \ \ \boldsymbol{K}_e \equiv \int_B \boldsymbol{N}_\varepsilon[\boldsymbol{x}]^T \mathbf{E} \boldsymbol{N}_\varepsilon[\boldsymbol{x}] \, dV \tag{6}$$

From now on the dependence of quantities on $\boldsymbol{x}$ will be omitted for an easier reading.

2.1 The elastic envelope of the stresses

We assume that the external actions $\boldsymbol{p}$ can be expressed as a combination of basic actions $\boldsymbol{p}_i$ with $i = 1 \ldots p$ belonging to the admissible closed and convex *load domain*

$$\mathbb{P} := \left\{ \boldsymbol{p} \equiv \sum_{i=1}^{p} a_i \boldsymbol{p}_i : \ a_i^{min} \leq a_i \leq a_i^{max} \right\} \tag{7}$$

Denoting with $\mathbf{t}_{ei}$ the stress elastic solution for $\boldsymbol{p}_i$ it is possible to define the *elastic envelope* $\mathbb{S}_e$ of the elastic stresses $\mathbf{t}_e$:

$$\mathbb{S}_e := \left\{ \mathbf{t}_e \equiv \sum_{i=1}^{p} \alpha_i \mathbf{t}_{ei} : \ a_i^{min} \leq a_i \leq a_i^{max} \right\} \tag{8}$$

that define, the set of the elastic stresses produced by each load path contained in the load domain.

By construction $\mathbb{S}_e$ and $\mathbb{P}$ are convex polytopes defined by N_v vertexes and each $\mathbf{t}_e \in \mathbb{S}_e$ can be expressed as a convex combination of the elastic envelope vertexes $\mathbf{t}^{E\alpha}$ that can be usefully referred to a reference stress $\mathbf{t}^{E0}$ so obtaining:

$$\mathbf{t}_e = \mathbf{t}^{E0} + \sum_{\alpha=1}^{N_v} s_\alpha \mathbf{t}^{E\alpha} \qquad s_\alpha \geq 0 \qquad \sum_{\alpha=1}^{N_v} s_\alpha = 1 \tag{9}$$

If elastic stresses (external loads) are amplified by a real number λ the amplified elastic envelope $\lambda\mathbb{S}_e := \{\lambda \mathbf{t}_e : \ \mathbf{t}_e \in \mathbb{S}_e\}$ is obtained from the original one $\mathbb{S}_e$ by a translation defined by the translation of the reference stress $(\lambda - 1)\mathbf{t}^{E0}$ and by an expansion (contraction if $\lambda < 1$) defined by the motion $(\lambda - 1)\mathbf{t}^{E\alpha}$ of the vertexes with respect to $\mathbf{t}^{E0}$.

Note also that $\mathbb{S}_e$ is polar symmetric with respect to its center, that is the value of $\mathbf{t}_e \in \mathbb{S}_e$ defined by the following combination

$$a_\alpha^0 = \frac{a_\alpha^{max} + a_\alpha^{min}}{2} \quad \ \alpha = 1 \ldots N_p \tag{10}$$

2.2 The shakedown elastic domain

Assuming elastic perfectly plastic Drucker material the stress field will be *plastically admissible* if

$$f[\boldsymbol{\sigma}[\boldsymbol{x}]] \equiv \phi[\boldsymbol{\sigma}[\boldsymbol{x}]] - \sigma_{gy} \leq 0 \quad \forall \boldsymbol{x} \in \mathcal{B} \tag{11}$$

where the convex yield function f is the sum of the homogeneous function ϕ and of the yield stress $\sigma_{gy} \in \mathbb{R}$. In a FEM context of analysis the previous condition could be expressed in a weighted sense as proposed as an example in [8] or tested in a finite series of points. To simplify the notation, we control plastic admissibility in the N_σ stress control nodes $\boldsymbol{x}_g$ so that $\mathbf{t}$ will be plastically admissible if each $\boldsymbol{\sigma}_g$ is contained in the elastic domain $\mathbb{E}_g$:

$$\mathbb{E}_g := \{\boldsymbol{\sigma}_g : f[\boldsymbol{\sigma}_g] \leq 0\} \quad , \quad f[\boldsymbol{\sigma}_g] \equiv \phi[\boldsymbol{\sigma}_g] - \sigma_{gy} \tag{12}$$

Introducing the global yield function

$$\boldsymbol{f}[\mathbf{t}]^T = \begin{bmatrix} f[\boldsymbol{\sigma}_1] & f[\boldsymbol{\sigma}_2] & \ldots & f[\boldsymbol{\sigma}_{N_\sigma}] \end{bmatrix} \tag{13}$$

the plastically admissible condition for the global vector $\mathbf{t}$ becomes

$$\mathbf{t} \in \mathbb{E} \quad \text{with} \quad \mathbb{E} := \{\mathbf{t} : \boldsymbol{f}[\mathbf{t}] \leq 0\}$$

where, from now on, vector inequality will be considered in a componentwise fashion that is:

$$\boldsymbol{f}[\mathbf{t}] \leq \mathbf{0} \iff f[\boldsymbol{\sigma}_g] \leq 0 \; \forall g = 1 \ldots N_\sigma$$

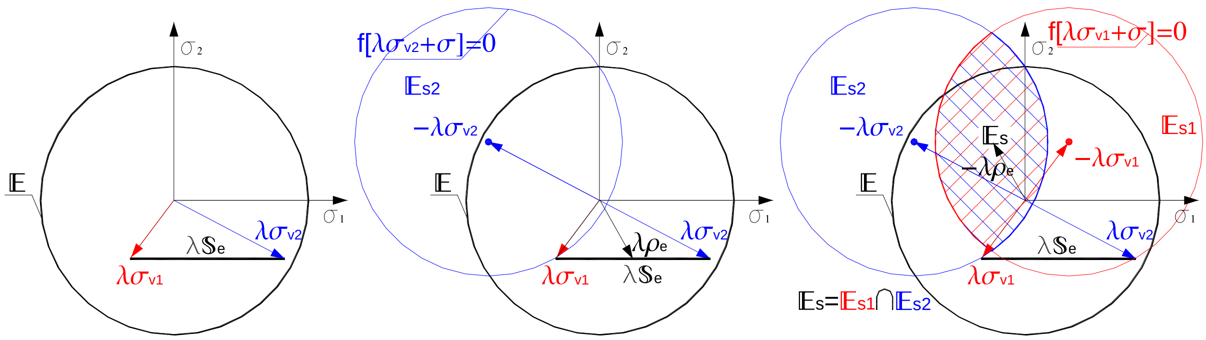


Figure 1: Classic definition of the shakedown domains

Finally the plastically admissible condition for all the stresses contained in the amplified elastic envelope $\lambda\mathbb{S}_e$ and translated by a fixed stress $\bar{\mathbf{t}}$, due to the convexity of $\mathbb{E}$, require the plastic admissibility of all vertex stresses $\mathbf{t}_\alpha = \lambda(\mathbf{t}^{E\alpha} + \mathbf{t}^{E0}) + \bar{\mathbf{t}}$

$$\boldsymbol{f}[\lambda\mathbf{t}_e + \bar{\mathbf{t}}] \leq \mathbf{0}, \; \forall \mathbf{t}_e \in \mathbb{S}_e \iff \boldsymbol{f}[\mathbf{t}_\alpha] \leq \mathbf{0} \quad \alpha = 1 \ldots N_v \tag{14}$$

3 FORMULATION OF LIMIT ANALYSIS AND SHAKEDOWN PROBLEMS

The aim of this section is to give a general frame for the evaluation of the larger multiplier λ_a, we call *shakedown safety factor*, used for amplifying the load domain $\mathbb{P}$ and allowing for the shakedown of the structure using both direct methods based on shakedown theorems or strain driven pseudo–elasto plastic analysis as proposed in [2]. Note that, even if in the following we always refers to shakedown analysis and shakedown safety factor, the limit analysis case is trivially obtained as a subcase of the shakedown ones when the elastic envelope collapse in a single point.

3.1 Shakedown theorems

Sufficient and necessary conditions for shakedown are given in the classic Bleich–Melan static theorem and Koiter's kinematic theorem that can be rewritten using the definition of elastic envelope.

3.1.1 Static theorem, safe multipliers and multiplier bounds

For a given $\lambda_s \in \mathbb{R}$ the classic form of static shakedown theorems demands the existence of a time–independent self-equilibrated stress field $\bar{\mathbf{t}}$ so that the total stress will be plastically admissible and equilibrated with the external load for each load in $\lambda_s\mathbb{P}$. λ_a can be evaluated as the maximum of the safe multipliers using the static theorems rewritten in terms of the reference stress $\mathbf{t}^{E0}$ using a formulation usually adopted when using direct methods (see [9]) such as

$$\begin{aligned} \text{maximize} \quad & \lambda_s \\ \text{subject to} \quad & \boldsymbol{Q}^T\mathbf{t} = \lambda_s\boldsymbol{p}_0 \\ & \mathbf{t}_\alpha = \mathbf{t}_0 + \lambda_s\mathbf{t}^{E\alpha} \quad \alpha = 1\dots N_v \\ & \boldsymbol{f}[\mathbf{t}_\alpha] \leq \mathbf{0}, \qquad\quad \alpha = 1\dots N_v \end{aligned} \tag{15}$$

with

$$\boldsymbol{p}_0 \equiv \boldsymbol{Q}^T\mathbf{t}^{E0}$$

The N_v new variables $\mathbf{t}_\alpha$ representing vertex stresses and $\mathbf{t}^{E0}$, without any loss in generality, could be selected as a vertex stress $\mathbf{t}^{E0} = \mathbf{t}^{E1}$ for example. Finally is $\mathbf{t}^{E0} = \mathbf{0}$ we have the classic form of the theorem in terms of the self–equilibred stress.

Note that when the external load domain collapsed in a single point ($\alpha_i^{min} = \alpha_i^{max}$) problem (15) directly transforms into the standard form of the static theorem of limit analysis if $\mathbf{t} \in \mathbb{S}$. We also have that the λ_a will be the lesser of the minimum values of the limit load multiplier obtained for a generic $\boldsymbol{p}_0 \in \mathbb{P}$, and so also lesser than or equal to each limit load obtained for a single vertex load.

3.1.2 The dual problem: kinematical theorem

Static theorem of shakedown has the form of a primal, convex nonlinear optimization problem. Starting from eq. (15) we can evaluate the Lagrangian:

$$\mathcal{L}[\lambda, \mathbf{t}, \mathbf{t}_\alpha, \boldsymbol{\mu}_\alpha, \boldsymbol{u}, \boldsymbol{\varrho}_\alpha] = \lambda + \boldsymbol{u}^T(\boldsymbol{Q}^T\mathbf{t} - \lambda\boldsymbol{p}_0) + \sum_{\alpha=1}^{N_v} \boldsymbol{\varrho}_\alpha^T(\mathbf{t}_\alpha - \mathbf{t} - \lambda\mathbf{t}^{E\alpha}) - \sum_{\alpha=1}^{N_v} \boldsymbol{\mu}_\alpha^T\boldsymbol{f}[\mathbf{t}_\alpha] \tag{16}$$

where each $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\sigma}_y$ are vectors of dimension N_σ collecting the Lagrange multipliers

$$\boldsymbol{\mu}_\alpha^T = \begin{bmatrix} \mu_{\alpha,1} & \mu_{\alpha,2} & \dots & \mu_{\alpha,N_\sigma} \end{bmatrix}$$

and the yield stresses

$$\boldsymbol{\sigma}_y = \begin{bmatrix}\sigma_{y1} & \sigma_{y2} & \dots & \sigma_{yN_\sigma}\end{bmatrix}$$

In the optimal values the Lagrangian has a saddle point [10, 11], that is the optimal value for the solution of the following condition

$$\lambda_a = \min_{(\boldsymbol{\mu}_\alpha, \boldsymbol{u})} \max_{(\lambda, \mathbf{t})} \mathcal{L}[\lambda, \mathbf{t}, \boldsymbol{\mu}, \boldsymbol{u}]$$

that is, evaluating the max respect to primal variable and after substitution we have the following dual problems coincident with the Koiter kinematical theorem of shakedown:

$$\begin{aligned} \text{minimize} \quad & \lambda_c \equiv \boldsymbol{\sigma}_y^T \sum_\alpha \boldsymbol{\mu}_\alpha \\ \text{subject to} \quad & \boldsymbol{\mu}_\alpha \geq 0 \\ & \boldsymbol{u}^T \boldsymbol{p}_0 + \sum_{\alpha=1}^{N_v} \boldsymbol{\varrho}_\alpha^T \mathbf{t}^{E\alpha} = 1 \\ & \boldsymbol{Q}\boldsymbol{u} = \sum_{\alpha=1}^{N_v} \boldsymbol{\varrho}_\alpha \quad , \quad \boldsymbol{\varrho}_\alpha = \boldsymbol{A}[\mathbf{t}_\alpha]\boldsymbol{\mu}_\alpha \end{aligned} \tag{17}$$

where

$$\boldsymbol{A}_{\sigma_\alpha} := \left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{t}}\right|_{\mathbf{t} \equiv \mathbf{t}_\alpha}^T$$

In previous equation the Euler theorem for the homogeneous functions $\boldsymbol{\phi}[\mathbf{t}_\alpha]$ has been used. Note as, the first equality constraint in (17) represent the normalization conditions of the power spent by he stress $\mathbf{t}_e \in \mathbb{S}_e$ for the kinematical mechanism, and in the limit analysis case simply becomes

$$\boldsymbol{u}^T \boldsymbol{p}_0 = 1$$

The saddle point properties of the Lagrangian show, that the maximum problem is concave and the minimum problem in convex such that both problem have the same optimal value $\lambda_a = \max \lambda_s = \min \lambda_c$ when the primal problem has an admissible solution, that is when $\lambda_e \neq 0$. Because of the convexity of the problem the obtained optimum is global such that the shakedown (limit) load factor is unique.

3.2 Finite step in pseudo-elastoplastic analysis

Limit and shakedown multipliers can also be obtained by evaluating the equilibrium path in the case of limit analysis or performing a pseudo elasto–plastic analysis as proposed in [2, 4] for the shakedown. As presented in [7] the single finite step of elasto-plasticity can be expressed as an optimization problem, which when extended for the shakedown becomes

$$\begin{aligned} \text{maximize} \quad & \lambda^{(k)} - \frac{1}{2}\sum_{i=0}^{N_v} (\Delta \mathbf{t}_i^{(k)})^T \mathbf{F} \Delta \mathbf{t}_i^{(k)} \\ \text{subject to:} \quad & \boldsymbol{Q}^T \mathbf{t}^{(k)} = \lambda^{(k)} \boldsymbol{p}_0 \\ & \mathbf{t}_\alpha^{(k)} = \mathbf{t}^{(k)} + \lambda^{(k)} \mathbf{t}^{E\alpha} \qquad \alpha = 1 \dots N_v \\ & \boldsymbol{f}[\mathbf{t}_\alpha^{(k)}] \leq \mathbf{0}, \qquad \alpha = 1 \dots N_v \end{aligned} \tag{18}$$

obtained from the static theorems (15) by adding the elastic definite positive quantities

$$\frac{1}{2}\sum_{i=0}^{N_v}(\Delta\mathbf{t}_i^{(k)})^T\mathbf{F}\Delta\mathbf{t}_i^{(k)} \text{ being } \Delta\mathbf{t}_i^{(k)} = \mathbf{t}_i^{(k)} - \mathbf{t}_i^{(k-1)}$$

where $a^(k)$ denotes the quantities a in the kth step of the elasoplasic analysis and

$$\Delta(\cdot) = (\cdot)^{(k+1)} - \Delta(\cdot)^{(k)}$$

The first order (Khun-Tucker) condition for eq.(20) gives in fact the same equation obtained using standard strain–driven elastoplastic analysis with a closest–point projection (a backward Euler integration scheme for the strain driven scheme):

$$\begin{array}{ll}
\text{Normalization:} & \Delta\boldsymbol{u}^T\boldsymbol{p}_0 + \sum_{\alpha=1}^{N_v}\Delta\boldsymbol{\varrho}_\alpha^T\mathbf{t}^{E\alpha} = 1 \\
\text{Kinematical compatibility:} & \boldsymbol{Q}\Delta\boldsymbol{u} = \mathbf{F}\Delta\mathbf{t} + \sum_{\alpha=1}^{N_v}\Delta\boldsymbol{\varrho}_\alpha \\
\text{Strain definition:} & \Delta\boldsymbol{\varrho}_\alpha = \mathbf{F}_\alpha\Delta\mathbf{t}_\alpha + \boldsymbol{A}[\mathbf{t}_\alpha^{(k+1)}]\boldsymbol{\mu}_\alpha^{(k+1)} \\
\text{Equilibrium:} & \boldsymbol{Q}^T\mathbf{t}^{(k+1)} = \lambda^{(k+1)}\boldsymbol{p}_0 \\
\text{Stress definition:} & \mathbf{t}_\alpha^{(k+1)} = \mathbf{t}^{(k+1)} + \lambda^{(k+1)}\mathbf{t}^{E\alpha} \\
\text{Yielding:} & \boldsymbol{f}[\mathbf{t}_\alpha^{(k+1)}] \le 0 \\
\text{Consistency:} & \boldsymbol{\mu}_\alpha^{(k+1)T}\boldsymbol{f}[\mathbf{t}_\alpha^{(k+1)}] = 0 \\
\text{Dual feasibility} & \boldsymbol{\mu}_\alpha \ge 0
\end{array} \tag{19}$$

that is the finite step of elasto-plastic or shakedown analysis.

Note that when $\mathbf{F} \to \mathbf{0}$ we obtain the first order condition for the static and kinematic theorems. This also suggests that for increasing values of $\mathbf{F}^{-1}$ with a single step of eq. (19) we have a suitable estimate of the shakedown safety factor.

3.3 INTERIOR POINT AND STRAIN DRIVEN METHODS IN ELASTO-PLASTICITY

In the sequel we describe the two approaches to the evaluation of the equilibrium path of structures subjected to a single proportional load, that is in the standard elasto-plastic case. In this way we focus the discussion on the difference between the two solution approaches avoiding useless complications in the writing. The evaluation of the limit load directly from eq.(15) could be seen as a single step of the interior point solution when the elastic modula in $\mathbf{F}$ are set to zero (or to small quantities).

3.4 Path–following Interior point methods

The elasto-plastic analysis, as formulated in (19), can be easily treated using the standard primal dual interior point formulation, introducing the logarithmic barrier and the positive slack variables for each inequality constraint. Assuming as reference stress $\mathbf{t} \equiv \mathbf{t}_0$ the unique vertex of the elastic domain now being $N_v = 1$ we have:

$$\begin{aligned} &\text{maximize} && \lambda - \frac{1}{2}\Delta\mathbf{t}^T\mathbf{F}\Delta\mathbf{t} + \omega_j \sum_{g=1}^{N_\sigma} \log(s_g) \\ &\text{subject to:} && \boldsymbol{Q}^T\mathbf{t} = \lambda\boldsymbol{p} \\ &&& \boldsymbol{f}[\mathbf{t}] + \boldsymbol{s} = \mathbf{0} \\ &&& \boldsymbol{s} \geq \mathbf{0} \end{aligned} \tag{20}$$

where the dependence on the kth step has been omitted for an easy reading and ω_j is a positive real quantity.

The Lagrangian associated with this problem will be:

$$\mathcal{L}[\boldsymbol{z}] = \lambda + \omega_j \sum_{g=1}^{N_\sigma} \log(s_g) + \Delta\boldsymbol{u}^T(\boldsymbol{Q}^T\mathbf{t} - \lambda\boldsymbol{p}) - \frac{1}{2}\Delta\mathbf{t}^T\mathbf{F}\Delta\mathbf{t} - \boldsymbol{\mu}^T(\boldsymbol{f}[\mathbf{t}] + \boldsymbol{s}) \tag{21}$$

where $\boldsymbol{z} = (\lambda, \mathbf{t}, \Delta\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{s})$ is the vector collecting all the unknowns of the problem.

There are a series of refinements of the strategies used to solve convex programming problems using Interior Point methods but the basic idea is to solve a sequence of linearized Newton problems derived as the first order condition of the nonlinear eq.(21) for $\omega_1 > \omega_2 > \cdots > \omega_j > 0$ with the positive constraints on the Lagrange multiplier $\boldsymbol{\mu}$ and $\boldsymbol{s}$ imposed by the logarithmic barrier. Starting from a primal dual feasible point $\boldsymbol{z}_j$ and given values of $\omega_{j+1} < \omega_j$, each subproblem, furnishes a search direction $\dot{\boldsymbol{z}}_j$ suitable to give a new primal dual feasible point $\boldsymbol{z}_{j+1}$ following the so called central path:

$$\mathbf{T}_j\dot{\boldsymbol{z}}_j = -\boldsymbol{r}_j \quad , \quad \boldsymbol{z}_{j+1} = \boldsymbol{z}_j + \theta_j\dot{\boldsymbol{z}}_j \tag{22}$$

where

$$\mathbf{r}_j = \left.\frac{\partial\mathcal{L}}{\partial\boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{z}_j} \text{ and } \mathbf{T}_j = \left.\frac{\partial^2\mathcal{L}}{\partial\boldsymbol{z}^2}\right|_{\boldsymbol{z}=\boldsymbol{z}_j}$$

The quantities θ_j usually unitary in the case of a full Newton step, are selected in order to maintain positivity of the slack variables $\boldsymbol{s}_{j+1}$ and $\boldsymbol{\mu}_{j+1}$. The sequence so generated, for $\omega_j \longrightarrow 0$ gives the first order condition of the original problem exactly (19). In this fashion it can be seen that, the optimal solution is reached faster than by following the other paths. Actually the implementation needs some other manipulation to stay in the neighborhood of the central path while the efficiency of the method is heavily conditioned by the nature of the inequality constraints equations. In particular the better efficiency is obtained with linear or conic constraints.

In the present case eq.(22) can be written as:

$$\boldsymbol{r}_j = \begin{bmatrix} r_{\lambda j} \\ \boldsymbol{r}_{\sigma j} \\ \boldsymbol{r}_{eqj} \\ \boldsymbol{r}_{\mu j} \\ \boldsymbol{r}_{sj} \end{bmatrix} = \begin{bmatrix} 1 - \Delta\boldsymbol{u}_j^T\boldsymbol{p} \\ \boldsymbol{Q}\Delta\boldsymbol{u}_j - \mathbf{F}\Delta\mathbf{t}_j - \boldsymbol{A}_{\sigma j}\boldsymbol{\mu}_j \\ \boldsymbol{Q}^T\mathbf{t}_j - \lambda_j\boldsymbol{p} \\ -\boldsymbol{f}[\mathbf{t}_j] - \boldsymbol{s}_j \\ \boldsymbol{S}_j\boldsymbol{\mu}_j - \omega_j\mathrm{e} \end{bmatrix} = \mathbf{0}$$

where the last equation, to avoid numerical singularities, has been multiplied for the diagonal matrix $\boldsymbol{S}$

$$\boldsymbol{S} = \mathrm{diag}\begin{bmatrix} s_1 & \cdots & s_{N_\sigma} \end{bmatrix}$$

while $\boldsymbol{e}$ denotes a vector of 1 of the appropriate dimension and:

$$\Delta \boldsymbol{u}_j = \boldsymbol{u}_j - \boldsymbol{u}_0 \quad , \quad \Delta \mathbf{t}_j = \mathbf{t}_j - \mathbf{t}_0$$

The j Newton iteration then becomes

$$\begin{bmatrix} \cdot & \cdot & -\boldsymbol{p}^T & \cdot & \cdot \\ \cdot & -(\mathbf{F}+\boldsymbol{H}_j) & \boldsymbol{Q} & -\boldsymbol{A}_{\sigma j} & \cdot \\ -\boldsymbol{p} & \boldsymbol{Q}^T & \cdot & \cdot & \cdot \\ \cdot & -\boldsymbol{A}_{\sigma j}^T & \cdot & \cdot & -\boldsymbol{I} \\ \cdot & \cdot & \cdot & \boldsymbol{S}_j & \boldsymbol{\Upsilon}_j \end{bmatrix} \begin{bmatrix} \dot{\lambda} \\ \dot{\mathbf{t}} \\ \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\mu}} \\ \dot{\boldsymbol{s}} \end{bmatrix} = - \begin{bmatrix} r_{\lambda j} \\ \boldsymbol{r}_{\sigma j} \\ \boldsymbol{r}_{eqj} \\ \boldsymbol{r}_{\mu j} \\ \boldsymbol{r}_{sj} \end{bmatrix} \tag{23}$$

where $\cdot$ stands for a zero vector or matrix of the appropriate dimension and

$$\boldsymbol{\Upsilon} = \mathrm{diag}\begin{bmatrix}\mu_1 & \mu_2 & \dots & \mu_{N_\sigma}\end{bmatrix} \quad , \quad \boldsymbol{H} = \sum_g \mu_g \frac{\partial^2 f}{\partial \boldsymbol{\sigma}^2}$$

The last two equations can be solved at the stress node level:

$$\begin{aligned} \dot{\boldsymbol{s}} &= \boldsymbol{r}_{\mu j} - \boldsymbol{A}_{\sigma j}^T \dot{\mathbf{t}} \\ \dot{\boldsymbol{\mu}} &= -\boldsymbol{S}_j^{-1}\left(\boldsymbol{r}_{sj} + \boldsymbol{\Upsilon}_j \boldsymbol{r}_{\mu j} - \boldsymbol{\Upsilon}_j \boldsymbol{A}_{\sigma j}^T \dot{\mathbf{t}}\right) \end{aligned}$$

and substituted in the stress equation to give the following condensed system:

$$\begin{bmatrix} \cdot & \cdot & -\boldsymbol{p}^T \\ \cdot & -\mathbf{F}_{tj} & \boldsymbol{Q} \\ -\boldsymbol{p} & \boldsymbol{Q}^T & \cdot \end{bmatrix} \begin{bmatrix} \dot{\lambda} \\ \dot{\mathbf{t}} \\ \dot{\boldsymbol{u}} \end{bmatrix} = - \begin{bmatrix} r_{\lambda} \\ \boldsymbol{g}_j \\ \boldsymbol{r}_{eqj} \end{bmatrix}$$

where

$$\mathbf{F}_{tj} \equiv (\mathbf{F} + \boldsymbol{H}_j + \boldsymbol{A}_{\sigma j}\boldsymbol{\Omega}_j \boldsymbol{A}_{\sigma j}^T) \quad , \quad \boldsymbol{\Omega}_j = \boldsymbol{S}_j^{-1}\boldsymbol{\Upsilon}_j$$

and

$$\boldsymbol{g}_j \equiv \boldsymbol{r}_{\sigma j} + \boldsymbol{A}_{\sigma j}\boldsymbol{S}_j^{-1}(\boldsymbol{r}_{sj} + \boldsymbol{\Upsilon}_j \boldsymbol{r}_{\mu j})$$

Also when $\mathbf{F}_t$ is not singular it is possible to perform the further stress condensation

$$\dot{\mathbf{t}} = \mathbf{E}_j \boldsymbol{Q}\dot{\boldsymbol{u}} - \boldsymbol{q}_j \quad , \quad \boldsymbol{q}_j = -\mathbf{E}_j \boldsymbol{g}_j$$

where, using the Sherman-Morrison equation, we have:

$$\mathbf{E}_j \equiv \mathbf{F}_j^{-1} = \boldsymbol{C}_j - \boldsymbol{C}_j \boldsymbol{A}_{\sigma j}\boldsymbol{D}_j \boldsymbol{A}_{\sigma j}^T \boldsymbol{C}_j \quad , \quad \boldsymbol{D}_j = \boldsymbol{\Upsilon}_j(\boldsymbol{S}_j + \boldsymbol{A}_{\sigma j}^T \boldsymbol{C}_j \boldsymbol{A}_{\sigma j}\boldsymbol{\Upsilon}_j)^{-1}$$

Note that $\mathbf{F}_j^{-1}$, $\mathbf{E}_j$ and $\boldsymbol{D}_j$ are block–diagonal matrices that can be assembled and inverted at the stress point level. That is at the global level, in terms of displacement variables and of the multiplier parameters λ, we obtain the following condensed system of equations

$$\begin{bmatrix} \cdot & -\boldsymbol{p}^T \\ -\boldsymbol{p} & \boldsymbol{K}_j \end{bmatrix} \begin{bmatrix} \dot{\lambda} \\ \dot{\boldsymbol{u}} \end{bmatrix} = - \begin{bmatrix} r_{\lambda j} \\ \boldsymbol{r}_{ecj} \end{bmatrix} \quad \text{with} \quad \begin{cases} \boldsymbol{r}_{ecj} = \boldsymbol{r}_{eqj} + \boldsymbol{Q}^T \mathbf{E}_j \boldsymbol{g}_j \\ \qquad = \boldsymbol{Q}^T(\mathbf{t}_j - \boldsymbol{q}_j) - \lambda_j \boldsymbol{p} \\ \boldsymbol{K}_j = \boldsymbol{Q}^T \mathbf{E}_j \boldsymbol{Q} \end{cases}$$

that is the search direction is:

$$\begin{cases} \dot{\boldsymbol{u}} = \dot{\lambda}\hat{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \\ \dot{\lambda} = \dfrac{r_{\lambda j} - \boldsymbol{p}^T\tilde{\boldsymbol{u}}}{\boldsymbol{p}^T\hat{\boldsymbol{u}}} \end{cases} \quad \text{being} \quad \begin{cases} \hat{\boldsymbol{u}} = \boldsymbol{K}_j^{-1}\boldsymbol{p} \\ \tilde{\boldsymbol{u}} = -\boldsymbol{K}_j^{-1}\boldsymbol{r}_{ec} \end{cases}$$

Having evaluated $\dot{\boldsymbol{u}}$ from back–substitution to the single Gauss point

$$\begin{bmatrix} -(\mathbf{F}+\boldsymbol{H}_j) & -\boldsymbol{A}_{\sigma j} & 0 \\ -\boldsymbol{A}_{\sigma j}^T & 0 & -\boldsymbol{I} \\ 0 & \boldsymbol{S}_j & \boldsymbol{\Upsilon}_j \end{bmatrix} \begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\boldsymbol{\mu}} \\ \dot{\boldsymbol{s}} \end{bmatrix} = - \begin{bmatrix} \boldsymbol{r}_{\sigma j} + \boldsymbol{Q}\dot{\boldsymbol{u}} \\ \boldsymbol{r}_{\mu j} \\ \boldsymbol{r}_{sj} \end{bmatrix}$$

we obtain $\dot{\mathbf{t}}$, $\dot{\boldsymbol{s}}$, $\dot{\boldsymbol{\mu}}$. In particular note that

$$\mathbf{t}_{j+1} = \mathbf{t}_j + \mathbf{E}_j\boldsymbol{Q}\dot{\boldsymbol{u}} - \boldsymbol{q}_j$$

3.5 Strain driven analysis

The strain–driven algorithm pursues the limit load (or the shakedown multiplier λ_a [2, 4]) by a step-by-step sequence of safe states $\boldsymbol{z}^{(k)} := \{\lambda^{(k)}, \mathbf{t}^{(k)}, \boldsymbol{u}^{(k)}\}$ in the sense of static theorem by using a Riks scheme that with a non decreasing multiplier converges to the desired solution. A new point on the equilibrium path is obtained from that previously evaluated by means of the following (statically admissible) scheme

$$\begin{cases} \boldsymbol{u}_{j+1} := \boldsymbol{u}_j + \dot{\boldsymbol{u}} \\ \lambda_{j+1} := \lambda_j + \dot{\lambda} \end{cases} \quad \text{where} \quad \begin{cases} \dot{\boldsymbol{u}} = \dot{\lambda}\hat{\boldsymbol{u}}_j + \tilde{\boldsymbol{u}}_j \\ \dot{\lambda} = \dfrac{r_{\lambda,j} - \boldsymbol{p}^T\tilde{\boldsymbol{u}}_j}{\boldsymbol{p}^T\hat{\boldsymbol{u}}_j} \end{cases} \quad \text{and} \quad \begin{cases} \hat{\boldsymbol{u}}_j = \boldsymbol{K}_j^{-1}\boldsymbol{p} \\ \tilde{\boldsymbol{u}}_j = -\boldsymbol{K}_j^{-1}\boldsymbol{r}_{eqj} \end{cases} \tag{24}$$

with $\boldsymbol{r}_{eqj} := \lambda_j\boldsymbol{p} - \boldsymbol{Q}^T\mathbf{t}_j$ being the equilibrium equations with plastically admissible stresses obtained by a return mapping procedure $\boldsymbol{r}_{\lambda,j}$ which has the same meaning as in the previous section, while $\boldsymbol{K}_j$ is the algorithmic tangent matrix which are the derivatives with respect to the residual providing the initial tangent on $(\boldsymbol{u}_j, \lambda_j)$. In order to improve the performance and reduce the computational effort, it is possible to use as iteration matrix the elastic one $\mathbf{K}_e$ following a modified Newton scheme.

It is worth noting that if the admissibility condition is fulfilled exactly $\boldsymbol{r}_\mu = \boldsymbol{r}_s = \mathbf{0}$ from eq.(23) we obtain the strain–driven scheme.

The plastically admissible stress vector $\mathbf{t}_j$ is obtained starting from an elastic predictor $\mathbf{t}_j^{tr} = \mathbf{t}_0 + \mathbf{F}^{-1}\boldsymbol{Q}\Delta\boldsymbol{u}_j$ solving the *return mapping* nonlinear convex problem

$$\mathbf{t}_j : \begin{cases} \text{minimize} & \dfrac{1}{2}(\mathbf{t}_j - \mathbf{t}_j^{tr})^T\mathbf{F}(\mathbf{t}_j - \mathbf{t}_j^{tr}) \\ & \boldsymbol{f}[\mathbf{t}_j] + \boldsymbol{s}_j = \mathbf{0} \\ & \boldsymbol{s}_j \geq \mathbf{0} \end{cases} \tag{25}$$

that due to the block diagonal properties of $\mathbf{F}$ can be solved at the single stress control point level.

Although generally simple, the solution of eq. (25) can became expensive when applied to large structures with hundreds of thousands of Gauss points especially for multisurface plasticity or shakedown. The use of ad hoc tools (highly specialized) to perform this optimization task can improve the efficiency and robustness of the strain driven strategy.

3.6 Strain driven versus Interior Point strategies

When applied to the single finite step of elasto-plastic analysis the Newton systems solved by the two methods are similar. The differences are only related to the exact solution, at each iteration (23), of the plastic admissibility and slackness conditions performed by the return mapping process. In fact if in the interior point equations (23) we assume fulfilling exactly, at each iteration, the following residual equations with $\omega_j = 0$ we obtain:

$$\begin{bmatrix} \boldsymbol{r}_{\sigma j} \\ \boldsymbol{r}_{\mu j} \\ \boldsymbol{r}_{sj} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{F}\Delta\mathbf{t}_j + \boldsymbol{A}_{\sigma j}\boldsymbol{\mu}_j - \boldsymbol{Q}\Delta\boldsymbol{u}_j \\ \boldsymbol{f}[\mathbf{t}_j] + \boldsymbol{s}_j \\ \boldsymbol{S}_j\boldsymbol{\mu}_j \end{bmatrix} = \mathbf{0}$$

that are the first order condition of the convex minimization problem (25).

The difference in the two strategies is then that the IP method follows a central path (characterized by $\omega_j \neq 0$) and attempts to solve eq. (19) without exactly fulfilling any of them if not in the final step, while strain driven methods solve exactly at each iteration, plastic–admissibility and consistency and so delete the corresponding residual equations from the global system (19). This last aspect appears to be dictated more by the simplicity of solving return mapping in a single step, at least for standard elasto-plastic problems, than from real numerical convenience. Finally the nonlinearity of the path obtained following the central path or the complementary slackness can be easily tested by adding, in the (25) the barrier term:

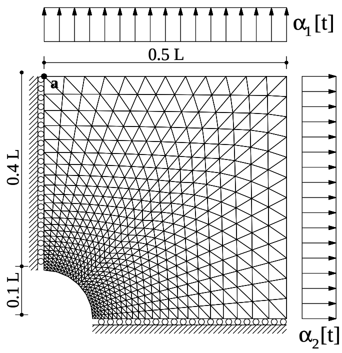
$$\mathbf{t}_j : \begin{cases} \text{minimize} \quad \dfrac{1}{2}(\mathbf{t}_j - \mathbf{t}_j^{tr})^T \mathbf{F}(\mathbf{t}_j - \mathbf{t}_j^{tr}) + \omega_j \displaystyle\sum_{g=1}^{N_\sigma} \log(s_g) \\ \boldsymbol{f}[\mathbf{t}_j] + \boldsymbol{s}_j = \mathbf{0} \end{cases} \tag{26}$$

From the point of view of the direct evaluation of the safety factor, that is starting directly from limit and shakedown static and kinematic theorems, the strain driven methods represent a primal optimization method that keeps improving a primal feasible solution, maintains the zero-duality gap (complementarity slackness condition) and moves toward dual feasibility. Note that the first primal solution, that is the elastic limit, is known or easily evaluated. On the contrary, when applied to evaluate the load multiplier, primal dual IP methods use the logarithmic barrier to move in the neighborhood of the primal dual central path (the sets of feasible solutions for which the complementary condition is set equal to ω_j), always maintain primal and dual feasibility and move to the optimal solution characterized by zero complementary slackness ($\omega = 0$). Also note that using a single iteration for each new point, that is evaluating only a search direction, the new point in general will not be on the central path and may also not be feasible at all. For this reason the method works well even when the constraints are linear, in this case the only nonlinearity being in the consistency equation. Furthermore note as for conic constraints, including practically all the technical relevant yield conditions, the performance of the methods are practically the same as for linear constraints.

4 NUMERICAL RESULTS

Numerical results where obtined with both the proposed strain-driven implementation and using the conic optimization program `MOSEK` a useful tool in limit and shakedown analysis used also for comparing the efficiency with a widely used and carefully tuned optimization software.

We refer to a square plate with a circular hole. The analysis was carried out using the standard strain-driven algorithm based on a modified Newton iteration scheme (MN) and the interior point algorithm (IP) in different fashions. In particular a direct formulation of the static theorem, implementing the problem (15) and using a MN version of it have been used. Similarly the elastoplastic formulation of the shakedown problem is also made by using a full Newton scheme and its MN version.



λ_c =***0.894***		
Algorithm	N. Steps	N. loops
Strain-driven.	18	224
Direct Interior Point(IP)	21	-
MN Direct IP	x	-
Elastic Plastic IP	22	107
MN Elastic Plastic IP	17	79

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