# Nonlinear Kinematic Hardening Modeling in Plasticity

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SUMMARY. In the present article a comparative analysis is made between the linear and the nonlinear kinematic hardening assumption for elastoplastic materials. A solution procedure which preserves a quadratic rate of convergence is used for the simulation of a typically adopted nonlinear kinematic hardening law. Numerical computations and results are reported. A comparison is made between the adoption of the assumption of linear versus nonlinear kinematic hardening rule by considering different types of material properties. The analysis allows to have a better insight on the conditions upon which such assumption may be considered as effective.

## 1 INTRODUCTION

Computational modeling of rate independent plasticity has nowadays achieved significant improvements, both in the proper formulation of the theoretical laws governing the inelastic behaviour and in the definition of the numerical procedures for the integration of the boundary value problem (see among others Simo and Hughes [1] and Zienkiewicz and Taylor [2]). In the numerical integration of rate independent plasticity problems the assumption of a linear kinematic hardening behaviour is often adopted. This assumption proves to be very advantageous from the numerical point of view since it often provides numerical algorithms characterized by computational efficiency and a symmetric tangential stiffness matrix. Nevertheless, in the last decades in the literature it has been outlined the necessity of adopting a nonlinear kinematic hardening rule in order to properly simulate experiments on real materials, see e.g. Armstrong and Frederick [3], Dafalias and Popov [4], McDowell [5], Chaboche [6], Lubliner et al. [7]. This is expecially true under cyclic loading conditions and when simulating experiments for describing ratchetting effects or more complex material behaviours, see for instance Chaboche [8], and Auricchio and Taylor [9]. For a review paper see e.g. Chaboche [10]. However, the adoption of nonlinear kinematic hardening rules for plasticity models is not a trivial task to be accomplished in the computational procedures. In fact it poses a number of challenges if a fast and robust computational solution is to be pursued. As a matter of fact the investigation for fast and robust integration methods for nonlinear kinematic hardening models and complex loading conditions currently represents an open topic of research. At this regard see for instance Auricchio and Taylor [9], Chaboche and Cailletaud [11], Hartmann et al. [12], Dettmer and Reese [13], Tsakmakis and Willuweit [14], Artioli et al. [15]. In the present article a comparative analysis between the linear and the nonlinear kinematic hardening assumption is made. Some considerations regarding the applicability of linear and nonlinear kinematic hardening rules in the computational simulations of material behaviour are presented. An integration scheme which preserves a quadratic rate of convergence is applied to the simulation of a typically adopted nonlinear kinematic hardening law. Numerical computations and results are reported in order to illustrate the effectiveness of the algorithmic procedure. A comparison is made between the adoption of the assumption of linear versus nonlinear kinematic hardening rule by selecting different types of material parameters. The performed analysis allows to have a better understanding on the conditions upon which such assumption may be considered efficacious.

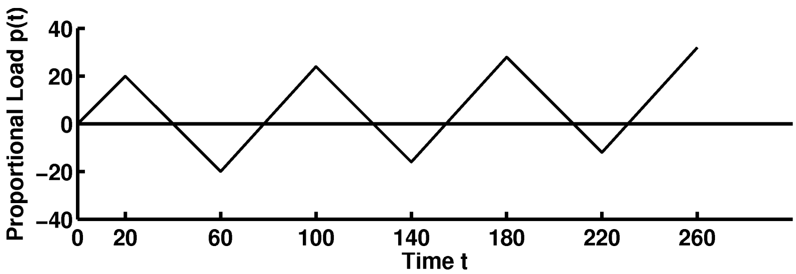


Figure 1: Cyclic loading program in tension-compression with increasing mean stress.

## 2 THE CONTINUUM EVOLUTIVE MODEL

Let us consider the body $\mathcal{B}$ in the reference configuration $\Omega \subset \Re^n$, with $1 \leq n \leq 3$. We indicate with $\mathcal{T} \subset \Re_+$ the time interval of interest and with $\boldsymbol{V}$ the space of displacements, $\boldsymbol{D}$ the strain space and $\boldsymbol{S}$ the dual stress space. We also denote by $\mathbf{u} : \Omega \times \mathcal{T} \to \boldsymbol{V}$ the displacement and by $\boldsymbol{\sigma} : \Omega \times \mathcal{T} \to \boldsymbol{S}$ the stress tensor. The compatible strain tensor is defined as $\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} : \Omega \times \mathcal{T} \to \boldsymbol{D}$, where $\nabla^s$ is the symmetric part of the gradient.

We assume the stress tensor to be additively decomposed into its deviatoric and spherical parts

$$\boldsymbol{\sigma} = \mathbf{s} + p\,\mathbf{1}, \tag{1}$$

where $\mathbf{s} \overset{\text{def}}{=} \text{dev}\boldsymbol{\sigma} = \boldsymbol{\sigma} - p\,\mathbf{1}$ is the stress deviator, $p = \frac{1}{3}tr(\boldsymbol{\sigma})$ is the pressure of the spherical part $p\,\mathbf{1}$ and $\mathbf{1}$ is the rank two identity tensor. Accordingly, the strain tensor can be decomposed into the deviatoric and volumetric parts

$$\boldsymbol{\varepsilon} = \mathbf{e} + \frac{1}{3}\theta\,\mathbf{1}, \tag{2}$$

where $\mathbf{e} \overset{\text{def}}{=} \text{dev}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - \frac{1}{3}\theta\,\mathbf{1}$ is the strain deviator and $\theta$ is the change in volume.

The elastic relation between the volumetric part of the stress and the volumetric part of the strain is expressed as

$$p = K\,\theta, \tag{3}$$

where $K$ is the bulk modulus.

The linear elastic relation between the stress deviator and the elastic deviatoric strain is

$$\mathbf{s} = 2G\,\mathbf{e}^e = 2G[\mathbf{e} - \mathbf{e}^p], \tag{4}$$

where $G$ is the shear modulus, and the deviatoric part of the total strain has been additively decomposed into an elastic and a plastic part $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$.

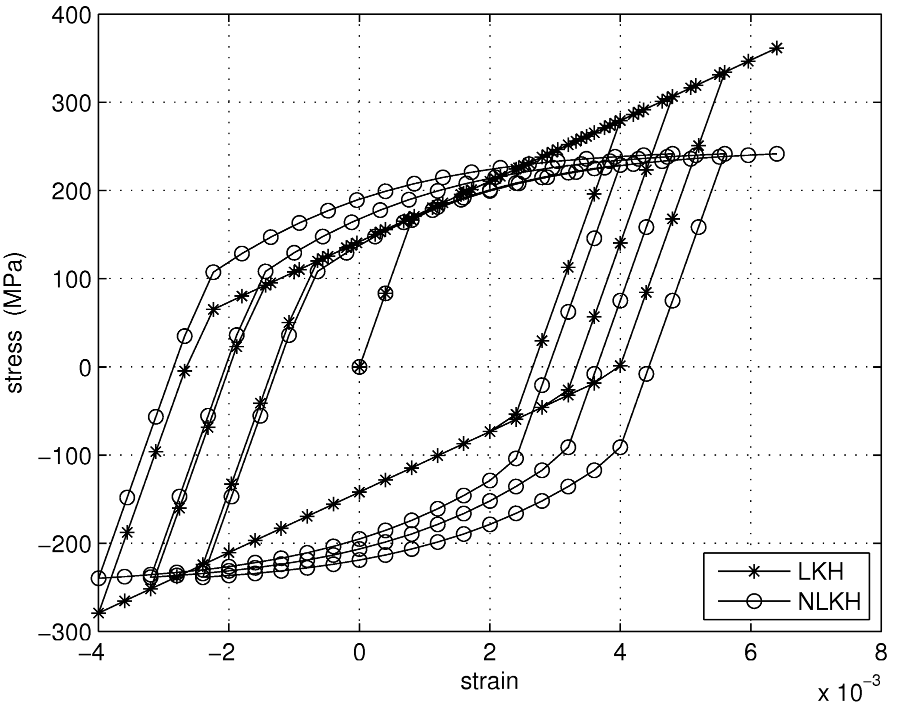


Figure 2: Cyclic loading conditions in tension-compression for the steel X5CrNi 18·9 (loading program in fig.1). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

The relative stress $\mathbf{\Sigma}$ is expressed as

$$\mathbf{\Sigma} = \mathbf{s} - \boldsymbol{\alpha}, \tag{5}$$

where $\boldsymbol{\alpha}$ represents the deviatoric back stress.

We consider herein the equations governing the behaviour of a $J_2$ material model. Consequently, a von Mises yield criterion is adopted in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \kappa) = ||\text{dev}\boldsymbol{\sigma} - \boldsymbol{\alpha}|| - \kappa(\chi_{iso}) = ||\mathbf{s} - \boldsymbol{\alpha}|| - \sqrt{\frac{2}{3}}(\sigma_{yo} + \chi_{iso}) \leq 0, \tag{6}$$

where $\kappa(\chi_{iso}) = \sqrt{\frac{2}{3}}\,\sigma_y = \sqrt{\frac{2}{3}}(\sigma_{yo} + \chi_{iso})$ represents the current radius of the yield surface in the deviatoric plane and $\sigma_{yo}$ denotes the uniaxial yield stress of the virgin material. For a linear isotropic hardening behaviour the static internal variable related to isotropic hardening is specified as $\chi_{iso} = H_{iso}\bar{e}^p$, where the dual kinematic internal variable $\bar{e}^p$ represents the equivalent (accumulated) plastic strain $\bar{e}^p \overset{\text{def}}{=} \int_0^t \sqrt{\frac{2}{3}}||\dot{\mathbf{e}}^p||\,\mathrm{d}\,t$.

In the framework of associative plasticity the evolutive flow law representing the constitutive equation for the deviatoric plastic strain rate is expressed as

$$\dot{\mathbf{e}}^p = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} = \dot{\gamma}\,\mathbf{n}, \tag{7}$$

where $\dot{\gamma}$ is the plastic multiplier and the second rank tensor $\mathbf{n}$ is defined as

$$\mathbf{n} \overset{\text{def}}{=} \frac{\boldsymbol{\Sigma}}{||\boldsymbol{\Sigma}||}, \tag{8}$$

and it has unit norm.

Accordingly, the equivalent (accumulated) plastic strain rate can be formulated as

$$\dot{\bar{e}}^p = \sqrt{\frac{2}{3}}\,\dot{\gamma}. \tag{9}$$

In the assumption of linear kinematic hardening behaviour the back stress rate is expressed as

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}\,H_{kin}\,\dot{\mathbf{e}}^p. \tag{10}$$

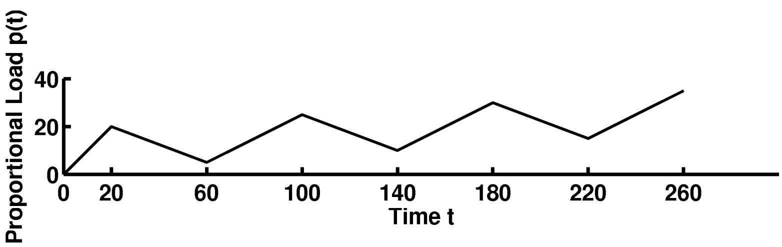


Figure 3: Cyclic loading program with increasing levels of loading.

For the nonlinear kinematic hardening behaviour in the literature it is often adopted the constitutive equation originally proposed by Armstrong and Frederick [3] (see e.g. Chaboche [6], and Hartmann et al. [12]), which can be written as

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}\,H_{kin}\,\dot{\mathbf{e}}^p - H_{nl}\,\dot{\bar{e}}^p\boldsymbol{\alpha}, \tag{11}$$

where $H_{nl}$ is a non-dimensional material dependent parameter characterizing nonlinear kinematic hardening behaviour and $H_{nl} = 0$ stands for linear kinematic hardening behaviour.

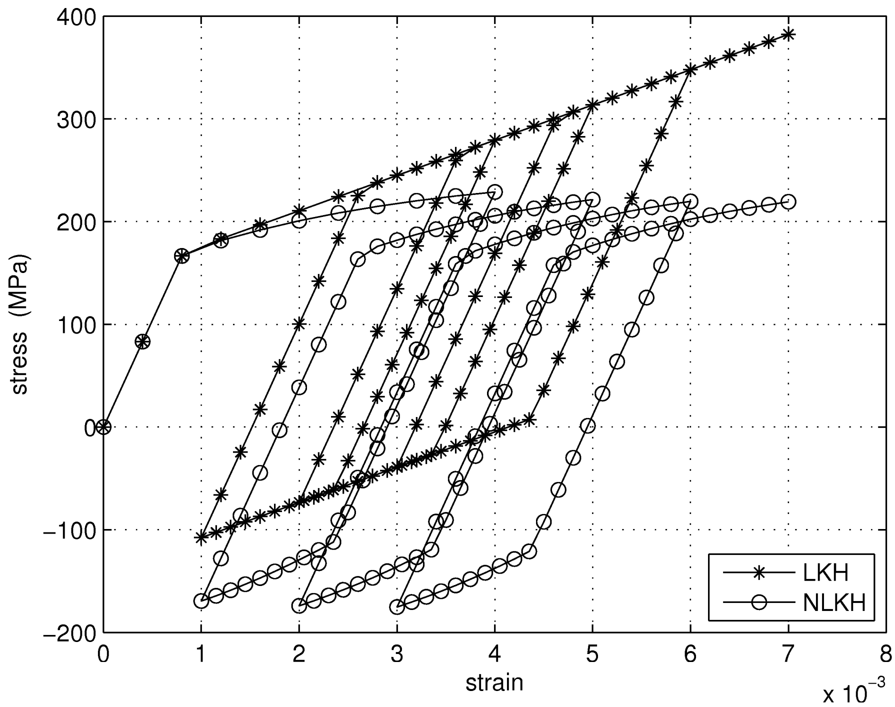


Figure 4: Cyclic loading conditions with increasing levels of loading for the steel X5CrNi 18·9 (loading program in fig.3). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

## 3 NUMERICAL RESULTS

The investigation for fast and robust integration methods relative to nonlinear kinematic hardening models is currently a topic of active research in the literature. As a matter of fact the adoption of the nonlinear kinematic hardening rule (11) for plasticity models produces an unsymmetric tangential stiffness matrix and it often results in a reduction of the rate of convergence typical of computationally efficient numerical schemes.

In the present paper a solution procedure which preserves the quadratic rate of convergence typical of Newton's schemes is used. The procedure is illustrated in detail in De Angelis and Taylor [16], where the development of a consistent tangent operator for nonlinear kinematic hardening plasticity which preserves a quadratic rate of convergence is also addressed.

In the sequel a comparative analysis between the linear and the nonlinear kinematic hardening assumption is illustrated and discussed. The linear kinematic hardening law is assumed in the form (10), whereas for the nonlinear kinematic hardening behaviour the Armstrong and Frederick's law (11) is adopted.

The numerical simulations are performed using a three-dimensional finite element, based on a mixed approach (Simo et al. [17]) and implemented into the Finite Element Analysis Program (FEAP) ( Zienkiewicz and Taylor [2], Taylor [18]).

In the tests a cubic specimen of side length equal to 5 is loaded by imposing a uniform displacement on the top boundary of the specimen and with the appropriate boundary conditions. The sample is modeled with only one element. Different types of material properties have been selected in the numerical examples in order to analyze the effectiveness of the adoption of linear versus nonlinear kinematic hardening rule for the simulation of the material constitutive behaviour.

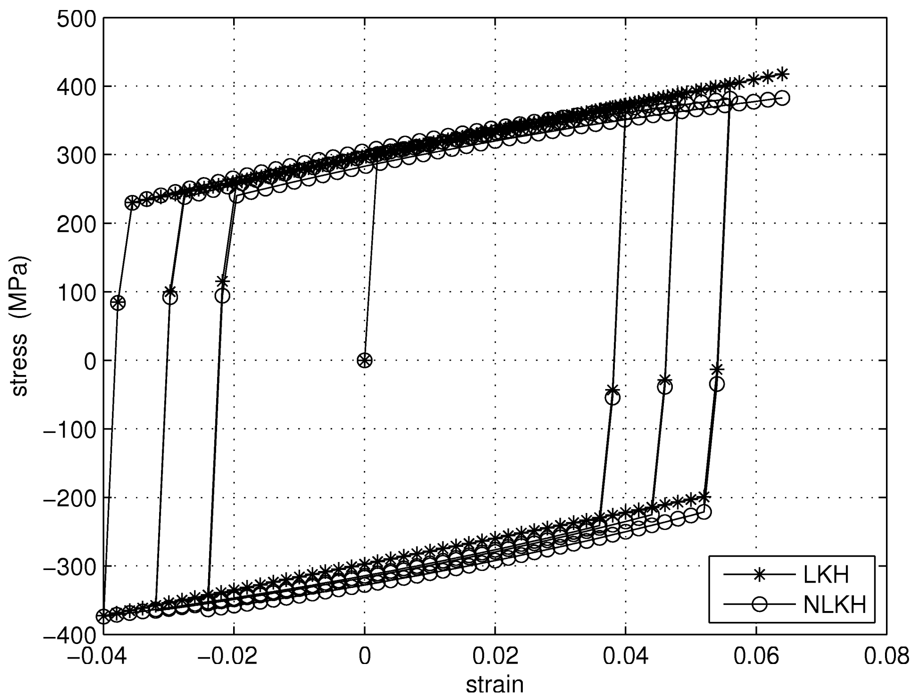


Figure 5: Cyclic loading conditions in tension-compression for the mild steel Ck 15 (loading program in fig.1). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

### 3.1 *Example 1*

As a first example we consider the material properties given in Hartmann et al. [12], which identify the steel X5CrNi 18·9. The material properties are: elastic modulus $E = 208000$ MPa, Poisson's ratio $\nu = 0.3$, yield limit $\sigma_{yo} = 170$ MPa, kinematic hardening modulus $H_{kin} = 41080$

MPa, nonlinear kinematic hardening parameter $H_{nl} = 525$, isotropic hardening modulus $H_{iso} = 0$ MPa. The imposed displacement of the top boundary has been assigned as $u_o = 0.001$. The evolution with time of the proportional load coefficient $p(t)$ amplifies the imposed displacement and describes the loading history according to relation $u(t) = p(t)u_o$.

A cyclic loading program in tension-compression with increasing mean stress has been performed in order to simulate the behaviour of the material model. The adopted loading history is illustrated in Figure 1. The related stress-strain curves comparing linear kinematic hardening (LKH) and nonlinear kinematic hardening behaviour (NLKH) are reported in Figure 2.

A cyclic loading history with increasing levels of loading is illustrated in Figure 3. This test has been performed in order to evaluate the ratchetting effects of the constitutive material behaviour. The stress-strain curves for this cyclic loading condition with ratchetting effects are illustrated in Figure 4 with the comparison of the linear kinematic hardening rule (LKH) and the nonlinear kinematic hardening rule (NLKH) in plasticity.

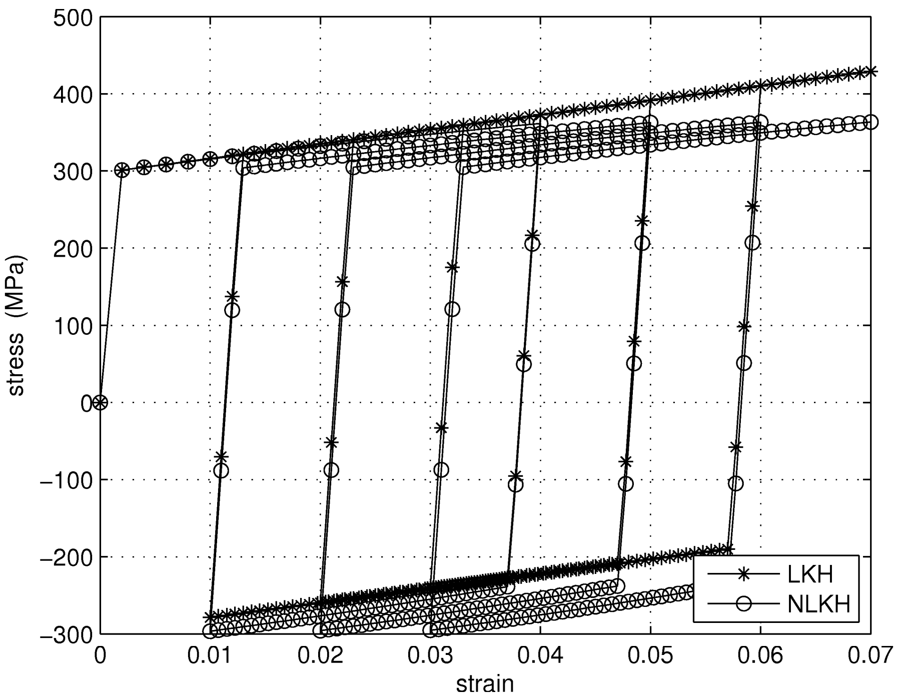


Figure 6: Cyclic loading conditions with increasing levels of loading for the mild steel Ck 15 (loading program in fig.3). Comparison between linear kinematic hardening (LKH) and nonlinear kinematic hardening (NLKH) in plasticity.

### 3.2 *Example 2*

As a second example we consider the material properties given in Dettmer and Reese [13] and Lührs et al. [19], which identify the mild steel Ck15. The material properties are: elastic modulus $E = 208000$ MPa, Poisson's ratio $\nu = 0.3$, yield limit $\sigma_{yo} = 300$ MPa, kinematic hardening modulus $H_{kin} = 1900$ MPa, nonlinear kinematic hardening parameter $H_{nl} = 8.5$, isotropic hardening modulus $H_{iso} = 0$ MPa. The imposed displacement of the top boundary has been assigned as $u_o = 0.01$. The evolution with time of the proportional load coefficient $p(t)$ describes the loading history $u(t) = p(t)u_o$.

The cyclic loading program in tension-compression illustrated in Figure 1 has been adopted for the simulation. The related stress-strain curves for the linear kinematic hardening (LKH) and the nonlinear kinematic hardening behaviour (NLKH) are reported in Figure 5.

Furtherly, the cyclic loading history with increasing levels of loading illustrated in Figure 3 has been adopted in order to analyze the behaviour of the material model with ratchetting effects. For this cyclic loading condition the stress-strain curves are reported in Figure 6. In Figure 6 it is illustrated the comparison of the linear kinematic hardening rule (LKH) and the nonlinear kinematic hardening rule (NLKH) in plasticity.

## 4 CONCLUSIONS

In the present article a comparative analysis between the linear and the nonlinear kinematic hardening behaviour has been performed. In the finite elements applications a linear kinematic hardening behaviour is usually adopted since this assumption ensures a symmetric tangent stiffness matrix and computationally efficient solution procedures. However, in the last years, in the literature it has been outlined the necessity of adopting a nonlinear kinematic hardening rule in order to properly simulate experiments on real materials. The computational implementation of nonlinear kinematic hardening rules is not a trivial task since it usually produces an unsymmetric tangential stiffness matrix and it often results in a reduction of the rate of convergence typical of computationally efficient numerical schemes.

In the present article a comparative analysis between the linear and the nonlinear kinematic hardening assumption has been performed by using a suitable solution procedure which preserves a quadratic rate of convergence. Numerical computations and results have been reported by selecting different types of material properties. A comparison between the adoption of the assumption of linear versus nonlinear kinematic hardening rule has been illustrated. The performed analysis allows to have a better insight on the conditions upon which the assumption of linear versus nonlinear kinematic hardening may be considered as effective.

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