A variational formulation in damaging plasticity for modelling Strong Discontinuities

Loredana Contrafatto[1], Massimo Cuomo[1]
[1] *Department of Civil and Environmental Engineering, University of Catania, Italy*
E-mail: loredana.contrafatto@dica.unict.it, massimo.cuomo@dica.unict.it

Keywords: Interfaces, Strong Discontinuities, Variational Formulation, Elements with Embedded Discontinuities.

SUMMARY. The model falls in the context of the Strong Discontinuities Approach (SDA). All the relevant equations of the model are obtained from a Hu-Washizu variational principle formulated in a general context, thus allowing also for nonlinear continua. The numerical implementation in the Finite Element Method is based on Elements with Embedded Discontinuities concept. An enhanced enrichment function is introduced for modelling the discontinuity in the displacement field. This leads to a symmetric formulation of the tangent operator based on the weak satisfaction of the internal continuity condition on the stress. The model is discussed with reference to a single Finite Element and compared with the classical Strong Discontinuities Approach and with the smeared crack model.

1 INTRODUCTION

One of the most important causes that can produce structural failure is material cracking evolving into collapse mechanisms. The simulation of the behaviour of structures and components with discontinuities has become an important research topic. The number of experimental and analytical studies has led to the conclusion that the cracking process in continuum media is preceded by a strain-localization phenomenon, characterized by the formation of strain localization zones in which damage and other inelastic effects accumulate, gradually turning into macroscopically observable discontinuities or cracks.

This phenomena can be effectively described by means of models that incorporate the kinematics of strong discontinuities obtained by an enrichment of the displacement field with a discontinuous term [1, 2]. Elements with Embedded Discontinuities [3] and the eXtended Finite Element Method [4] are the main tools for the discrete description of the problem. The method of Embedded Discontinuities, however, appears to present some advantage in the computational implementation, since the evaluation of the discontinuity can be made at the element level, eventually together with the evaluation of additional irreversible variables.

The paper presents the derivation from a generalized multi-field Hu-Washizu variational principle of the equations ruling the problem of the enriched continuum. The equilibrium, compatibility and constitutive equations constitute the Euler-Lagrange stationarity conditions of the functional. Because of the topology of the problem, an additional equilibrium condition at the interface is obtained. It guarantees for the continuity of the stress across the interface. This equation is known as the orthogonality condition between the stress and the enhanced deformation field. The possible choices of the approximations introduced in the discretized principle give raise to the different implementations of the method. Usually, in order to satisfy the interface equilibrium condition, a Petrov-Galerkin approximation is introduced. However the resulting stiffness matrix is non symmetric. In the paper it is proposed an enhancement of the Strong Discontinuous kinematics that allows the fulfilment of BC's. It is shown that in this way it is possible to obtain a symmetric formulation.

2 THE MODEL

The paper presents a variational formulation of the equilibrium problem for a continuum characterized by an elastic-plastic damaging behavior, in which the growth of interfaces S takes places. The presence of pre-assigned physical interfaces is also considered. The growth or the activation of an interface is ruled by a specific activation function, based on a cohesive fracture like criterion. In the general formulation the medium and the interface are ruled by different constitutive equations, defined by distinct free energy and dissipation functionals. The strong form of the equilibrium and compatibility conditions is presented, with special attention to the equilibrium conditions at the interfaces and to the satisfaction of the Dirichelet boundary conditions. Similarities and differences with respect to other formulations in the literature are highlighted.

2.1 *Enhanced kinematics of strong discontinuities*

Let S be an interface embedded within a continuous body occupying the domain $\Omega \subset \Re^3$. We will limit the present discussion to the case of a single interface. The unit normal vector **n** is defined on the surface S.

Let Ω_φ be a subdomain of Ω containing the discontinuity and such that S divides Ω_φ in two subdomains, $\Omega_\varphi^+, \Omega_\varphi^-$ (Figure 1(a)). The normal **n** is oriented toward the interior of Ω_φ^+. The boundary of Ω_φ is divided by the surface S in two parts, $\partial\Omega_\varphi^+$, $\partial\Omega_\varphi^-$. According to the position of the interface, part of the boundary of Ω_φ can belong to $\partial\Omega = \partial\Omega_u \bigcup \partial\Omega_q$. Across the interface S the displacement field is discontinuous and the jump is denoted by $[\![\mathbf{u}]\!]_S$. The displacement field in the continuum is described according to the format

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(\mathbf{x},t) + \tilde{\mathbf{u}}(\mathbf{x},t) \tag{1}$$

where $\hat{\mathbf{u}}$ is defined in Ω, and $\tilde{\mathbf{u}}$ is a function having as support Ω_φ, continuous and differentiable everywhere except on the interface S and such that

$$\begin{array}{ll} \tilde{\mathbf{u}}^+(\mathbf{x},t) - \tilde{\mathbf{u}}^-(\mathbf{x},t) = [\![\mathbf{u}]\!]_S(\mathbf{x},t) & \forall \mathbf{x} \in S \\ \tilde{\mathbf{u}}(\mathbf{x},t) = 0 & on \quad \partial\Omega_\varphi \end{array} \tag{2}$$

where $[\![\bullet]\!]_S$ indicates the discontinuity through the interface S.

Function $\tilde{\mathbf{u}}$ can be given in the general form

$$\tilde{\mathbf{u}}(\mathbf{x},t) = \bar{M}_S(\mathbf{x})\mathbf{a}(\mathbf{x},t) \tag{3}$$

where **a** is defined on S and $\bar{M}_S(\mathbf{x}) = M_S(\mathbf{x})N_S(\mathbf{x})$ is the enhanced enrichment function.

In classical Strong Discontinuities Approaches the enhanced displacement field $\tilde{\mathbf{u}}$ and function M_S are defined as follows:

$$\tilde{\mathbf{u}}(\mathbf{x},t) = M_S(\mathbf{x})\mathbf{a}(\mathbf{x},t) \quad M_S(\mathbf{x}) = (H_S(\mathbf{x}) - \varphi(\mathbf{x})) \tag{4}$$

H_S being the Heaviside function related to the surface S and defined on the domain Ω_φ and $\varphi(\mathbf{x})$ is a continuous and differentiable function on Ω_φ.

In Finite Element approximations the region Ω_φ usually coincides with a band having one element width [5]. This element contains the embedded discontinuity and function M_S is generated from the standard shape functions of the element N_i. In this way the essential boundary conditions can be satisfied only at the boundary nodes which belong to the element defining the domain Ω_φ, where the function $M_S(\mathbf{x})$ vanishes. In all the internal points of the element side laying on the border the essential conditions are not met if **a** $\neq 0$, as it is shown in fig. 1 in the case of a triangular Finite Elements.

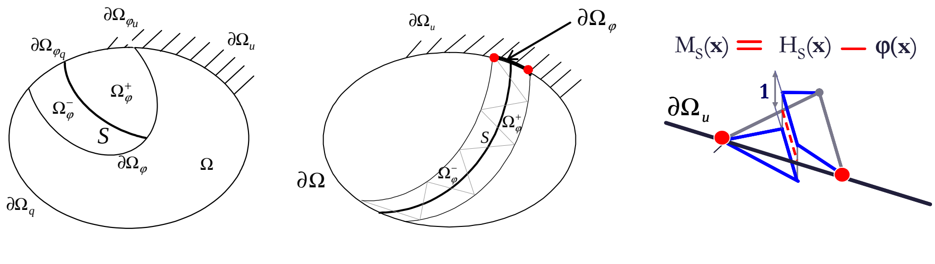


Figure 1: (a) Domain Ω and enhanced region Ω_φ. (b,c) FEM approximation of the enrichment displacement field on the constrained boundary $\partial\Omega_u$.

Let $\partial\Omega_{\varphi_u}$ be the constrained region of Ω_φ and $\partial\Omega_u/\partial\Omega_{\varphi_u}$ the remaining constrained part of $\partial\Omega_u$. Let $\mathbf{u}^\star(\mathbf{x})$ be the prescribed displacements on $\partial\Omega_u$.In order to enforce condition

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^\star \quad \forall \mathbf{x} \in \partial\Omega_{\varphi_u} \tag{5}$$

the enhanced enrichment function $\bar{M}_S$ is introduced, such that $\bar{M}_S(\mathbf{x})$ vanishes on the internal boundary of Ω_φ and on the restrained boundary $\partial\Omega_{\varphi_u}$ and presents an unit jump across S. It is given by

$$\bar{M}_S(\mathbf{x}) = N_S(\mathbf{x})(H_S - \varphi(\mathbf{x})) \tag{6}$$

where function $N_S(\mathbf{x})$ takes the role of annihilating the enhanced displacement field along the restrained portion of the boundary $\partial\Omega_{\varphi_u}$, so that $N_S(\mathbf{x}) = 0, \forall \mathbf{x} \in \partial\Omega_{\varphi_u}$. It can be obtained from the shape functions of the element, as it will be shown in section 3. The total displacement field (1) is C^0 everywhere except on S, while only $\hat{\mathbf{u}}$ has to satisfy the constraint on the boundary. The deformations, instead, are generally not continuous across the boundary of Ω_φ.

The properties of the enhancement can be summarized as follows:

$$\begin{array}{l}
\tilde{\mathbf{u}} = \bar{M}_S \mathbf{a} \quad in \quad \Omega_\varphi \\
\bar{M}_S = N_S M_S \quad in \quad \Omega_\varphi \\
N_S = 0 \quad on \quad \partial\Omega_{\varphi_u} \\
\bar{M}_S = 0 \quad on \quad \partial\Omega_\varphi/\partial\Omega_{\varphi_q} \\
[\![M_S]\!]_S = 1 \quad on \quad S \\
[\![N_S]\!]_S = \emptyset \quad on \quad S \\
[\![\bar{M}_S]\!]_S = \bar{N}_S \quad on \quad S \\
\Delta\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^+ - \tilde{\mathbf{u}}^- = [\![\mathbf{u}]\!]_S = \bar{N}_S \mathbf{a} \quad on \quad S
\end{array} \tag{7}$$

where $\bar{N}_S$ is the restriction of function $N_S(\mathbf{x})$ on S.

The present approach is based on two assumptions: the domain Ω_φ coincides with the band of elements that are cut by the discontinuity and the interpolation of function **a** is made element-wise. In this way, the nodal degrees of freedom coincide with the nodal displacements, and the jump function can be treated as an internal variable.

2.2 *The interface behaviour*

A cohesive model is used for the interface. The general case would include a reversible as well a dissipative regime, including damage in order to properly model the unloading path. Parallel attention has to be devoted to the idealization of the constitutive behaviour of the continuum, since damage and other irreversible phenomena can usually occur prior to the formation of a discontinuity. Given the limits of the present work, it is considered the case of an elastic continuum, and of a rigid-softening interface. The latter hypothesis leads to unrealistic predictions, since after unloading the interface remains open, and it should thus be necessary either to modify the form of the limit function for the interface, or to introduce damage in the model. However, in the paper only monotonic loading histories will be considered. This implies that the field **a** has no reversible component on S, so that $[\![\mathbf{u}]\!]_S = [\![\mathbf{u}]\!]_{S_p}$.

The elastic deformation of the continuum is ruled by a potential internal energy functional $\psi(\varepsilon_e)$, while for the interface it is used a rate independent associated cohesive model with softening. A failure condition is thus introduced for the traction acting on the discontinuity surface S, of the form

$$g(\mathbf{t}_S, \chi_S) \leq 0 \tag{8}$$

where $\mathbf{t}_S$ is the stress acting on the interface. We assume the failure criterion in the form

$$g(\mathbf{t}_S, \chi_S) = \tilde{g}(\mathbf{t}_S) - (f_y - \chi_S) \tag{9}$$

The scalar variable χ_S is conjugated to the internal softening variable α_{S_e} and is obtained from the gradient of an internal hardening potential $\psi_S(\alpha_{S_e})$, whose form determines the shape of the softening branch of the cohesive law. For linear softening it is

$$\psi_S(\alpha_{S_e}) = \int_S \frac{1}{2} H_S \alpha_{S_e}^2 \, dS \tag{10}$$

H_S being the finite softening modulus of the interface.

In the context of the standard generalised material model [6], assuming convexity of g and associativity, the displacement jump is given by the subdifferential of (8), $[\![\mathbf{u}]\!]_S = \lambda \partial_{\mathbf{t}_S} g(\mathbf{t}_S, \chi_S)$, and it can be introduced a dissipation potential for the interface that turns out to be the support function of the admissibility domain for the traction acting on the discontinuity surface:

$$\begin{aligned} &K = \{(\mathbf{t}_S, \chi_s) : g(\mathbf{t}_S, \chi_S) \leq 0\} \\ &d_S([\![\dot{\mathbf{u}}]\!]_{S_p}, \dot{\alpha}_{S_p}) = \mathrm{supp}_K = \underset{(\mathbf{t}_S, \chi_s) \in K}{sup} \left(\mathbf{t}_S \cdot [\![\dot{\mathbf{u}}]\!]_{S_p} + \chi_s \cdot \dot{\alpha}_{S_p} \right) \end{aligned} \tag{11}$$

3 VARIATIONAL FORMULATION OF THE BVP PROBLEM

The kinematics defined in section 2.1 is used to develop a structural model for the simulation of growth and propagation of interfaces inside a continuum medium. The basic equations are derived following a variational approach. Specifically a generalized Hu Washizu principle is considered.

The constitutive hypothesis of section 2.2 is assumed. The deformation energy $\Pi(\varepsilon_e, \alpha_{S_e}) = \psi(\varepsilon_e) + \psi_S(\alpha_{S_e})$ is given by the sum of the standard elastic energy and the hardening energy, different from zero only in the region Ω_φ, for which the quadratic form (10) is assumed.

The solution of the structural problem is characterized by means of the multi-fields functional

$$
\begin{aligned}
\Pi^{HW} = & \int_{\Omega/S} \sigma \cdot (\nabla^S \hat{\mathbf{u}} + \nabla^S \tilde{\mathbf{u}} - \varepsilon) d\Omega + \int_S \mathbf{t}_S \cdot (\Delta\tilde{\mathbf{u}} - [\![\mathbf{u}]\!]_S) dS - \int_S \chi_S (\alpha_{Se} + \alpha_{Sp}) dS \\
& + \int_{\Omega/S} \psi(\varepsilon) d\Omega + \int_S \psi_S(\alpha_{S_e}) dS + \int_S d_S([\![\dot{\mathbf{u}}]\!]_{S_p}, \dot{\alpha}_{S_p}) \Delta t dS \\
& - \int_{\Omega/S} \mathbf{b} \cdot (\hat{\mathbf{u}} + \tilde{\mathbf{u}}) d\Omega - \int_{\partial\Omega_q} \mathbf{q} \cdot (\hat{\mathbf{u}} + \tilde{\mathbf{u}}) d\Gamma \\
& - \int_{\partial\Omega_u} \mathbf{r} \cdot (\hat{\mathbf{u}} + \tilde{\mathbf{u}} - \mathbf{u}^\star) d\Gamma - \int_{\partial\Omega_\varphi / \partial\Omega_{\varphi q}} \rho \cdot \tilde{\mathbf{u}}
\end{aligned}
\tag{12}
$$

where the additive decomposition for the internal variables $[\![\mathbf{u}]\!]_S$ and α_S has been assumed:

$$
\begin{aligned}
[\![\mathbf{u}]\!]_S = & \quad [\![\mathbf{u}]\!]_{S_p} = [\![\mathbf{u}]\!]_{S_{p0}} + [\![\dot{\mathbf{u}}]\!]_{S_p} \Delta t \\
\alpha_S = & \quad \alpha_{S_e} + \alpha_{S_p} = \alpha_{S_e} + \alpha_{S_{p0}} + \dot{\alpha}_{S_p} \Delta t
\end{aligned}
\tag{13}
$$

and $\varepsilon = \varepsilon_e \quad in \quad \Omega/S$.

Functional Π^{HW} includes the compatibility conditions for both the continuum and the interface, the constitutive potentials of the continuum and the interface, and the duality pairings of the relevant state variables. Please note that only elastic deformations have been considered in the continuum, while the internal energy of the interface is only associated to the internal variable (that is, no elastic opening has been considered). In (12) $\mathbf{r}$ denotes the reactions on the constrained boundary $\partial\Omega_u$. A further Lagrangian multiplier, ρ, enforces variationally the boundary conditions for the enhanced displacement field on the boundary of Ω_φ.

In order to obtain a formulation similar to the familiar one proposed by Mosler [7], from (12) a generalised form of the Hellinger-Reissner functional is derived, by means of the introduction of the complementary energy functionals ψ', ψ'_S,d'_S. The compatibility conditions on the interface are strongly enforced assuming $\tilde{\mathbf{u}}$ in the form (3). By eliminating variables $\varepsilon_e, \alpha_{S_e}, [\![\dot{\mathbf{u}}]\!]_{S_p}, \dot{\alpha}_{S_p}$ the generalized Hellinger-Reissner functional $\Pi^{HR}(\hat{\mathbf{u}}, \mathbf{a}, \sigma, \mathbf{t}_S, \chi_S, \mathbf{r})$ is obtained:

$$
\begin{aligned}
\Pi^{HR} = & \int_{\Omega/S} \sigma \cdot \left[\nabla^S \hat{\mathbf{u}} + \nabla^S (\bar{M}_S \mathbf{a})\right] d\Omega - \int_{\Omega/S} \psi'(\sigma) d\Omega - \int_S \psi'_S(\mathbf{t}_S, \chi_S) dS \\
& - \int_S d'_S(\mathbf{t}_S, \chi_S) dS + \int_S \mathbf{t}_S \cdot [\![\mathbf{u}]\!]_S dS - \int_S (\chi_S \cdot \alpha_{S_{p0}} + \mathbf{t}_S \cdot [\![\mathbf{u}]\!]_{S_{p0}}) dS \\
& - \int_{\Omega/S} \mathbf{b} \cdot (\hat{\mathbf{u}} + \bar{M}_S \mathbf{a}) d\Omega - \int_{\partial\Omega_q} \mathbf{q} \cdot (\hat{\mathbf{u}} + \bar{M}_S \mathbf{a}) d\Gamma - \int_{\partial\Omega_u} \mathbf{r} \cdot (\hat{\mathbf{u}} + \bar{M}_S \mathbf{a} - \mathbf{u}^\star) d\Gamma
\end{aligned}
\tag{14}
$$

The optimization problem is stated as:

$$
\inf_{(\hat{\mathbf{u}}, \mathbf{a})} \sup_{(\sigma, \mathbf{t}_S, \chi_S)} \Pi^{HR}
\tag{15}
$$

The stationarity conditions of functional Π^{HR} give the relevant equations of the model:

$$
\begin{array}{lll}
\delta_{\hat{\mathbf{u}}}\Pi^{HR} & \Rightarrow & \left\{\begin{array}{lll} div\sigma + \mathbf{b} = 0 & in & \Omega/S \\ \sigma\mathbf{n} = \mathbf{q} & on & \partial\Omega_q \\ \sigma\mathbf{n} = \mathbf{r} & on & \partial\Omega_u \end{array}\right. \\
\delta_{\mathbf{a}}\Pi^{HR} & \Rightarrow & \mathbf{t}_S = \sigma\mathbf{n} \qquad on \quad S \\
\delta_{\sigma}\Pi^{HR} & \Rightarrow & \nabla^S(\hat{\mathbf{u}} + \bar{M}_S\mathbf{a}) = \nabla_\sigma\phi'(\sigma) \qquad in \quad \Omega/S \\
\delta_{\mathbf{t}_S}\Pi^{HR} & \Rightarrow & [\![\mathbf{u}]\!]_S + \nabla_{\mathbf{t}_S}\phi'_S(\mathbf{t}_S,\chi_S) + \nabla_{\mathbf{t}_S}d'_S(\mathbf{t}_S,\chi_S) - \alpha_{S_{p_0}} \qquad on \quad S \\
\delta_{\chi_S}\Pi^{HR} & \Rightarrow & -\nabla_{\chi_S}\phi'_S(\mathbf{t}_S,\chi_S) - \nabla_{\chi_S}d'_S(\mathbf{t}_S,\chi_S) - \alpha_{S_{p_0}} = 0 \qquad on \quad S \\
\delta_{\mathbf{r}}\Pi^{HR} & \Rightarrow & \hat{\mathbf{u}} + \bar{M}_S\mathbf{a} = \mathbf{u}^\star \qquad on \quad \partial\Omega_u
\end{array}
\tag{16}
$$

It is significant to dedicate a closer examination to the equilibrium equation obtained from the variation w.r.t. the enhanced displacement field **a**. Its weak form, from (14), is

$$
\begin{aligned}
&\int_{\Omega_\varphi/S} \sigma\cdot\delta\nabla^S(\bar{M}_S\mathbf{a})d\Omega + \int_S \mathbf{t}_S\cdot\bar{N}_S\delta\mathbf{a}dS - \int_{\Omega_\varphi/S}\mathbf{b}\cdot\bar{M}_S\delta\mathbf{a}d\Omega - \\
&\int_{\partial\Omega_{\varphi_q}}\mathbf{q}\cdot\bar{M}_S\delta\mathbf{a}d\Gamma - \int_{\partial\Omega_{\varphi_u}}\mathbf{r}\cdot\bar{M}_S\delta\mathbf{a}d\Gamma = 0
\end{aligned}
\tag{17}
$$

where we have assumed for the sake of simplicity strong satisfaction of the boundary conditions on $\partial\Omega_{\varphi_u}$ and the properties (7) have been used. Assuming absence of body and surface forces and in the case of a constant field **a**, condition (17) yields

$$
\int_{\Omega_\varphi/S}\sigma\nabla^S\bar{M}_S\cdot\delta\mathbf{a}d\Omega + \int_S \bar{N}_S\mathbf{t}_S\cdot\delta\mathbf{a}dS = 0 \tag{18}
$$

It has to be underlined that condition (18) leads to a wrong result in the classical approach for which the enrichment function M_S (4) is used, so that it has been suggested to replace the virtual enhanced displacement field with $-A_S/V_\varphi\mathbf{n}$. If instead the enriched enhancement function (6) is used, it can be shown that applying the Gauss' theorem equation (18) is always satisfied. Therefore, at least in principle, a standard Galerkin discretisation can be used, as opposed to the Petrov Galerkin approach usually adopted. Condition (18), or more generally (17), states that the interface traction can be obtained from a weighted average of the stress field; however, the condition is valid only globally on region Ω_φ and not element-wise.

4 ANALYSIS OF A FOUR-NODES ELEMENT

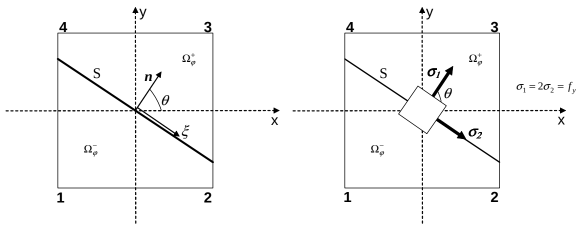
Consider the case of a discontinuity crossing a single plane quadrilateral element, as shown in figure 2(a), totally constrained on its border. Let 2 be the dimension of the element sides. Let N_1, N_2, N_3, N_4 be the linear shape functions. The stress field is defined as a function of the angle θ between the normal vector **n** and the x axis:

$$
\sigma = \left(\begin{array}{cc} \sigma_1\sin^2\theta + \sigma_2\cos^2\theta & \sin\theta\cos\theta(\sigma_2-\sigma_1) \\ \sin\theta\cos\theta(\sigma_2-\sigma_1) & \sigma_1\cos^2\theta + \sigma_2\sin^2\theta \end{array}\right) \tag{19}
$$

A Rankine failure criterion is considered:

$$
g = \mathbf{t}_S\cdot\mathbf{n} - f_y + \chi_s \leq 0 \quad \forall\mathbf{n} \tag{20}
$$

Under this condition, if $\sigma_1 > \sigma_2 > 0$, the maximum and the minimum of the stress are σ_1 and σ_2 along the n and ξ directions respectively and the direction of activation of the interface is **n**, as represented in figure 2(b). According to (20) the growth of the interface S occurs when $\sigma_1 = f_y$. The value $f_y = 1$ has been assumed in what follows.



(a) (b)

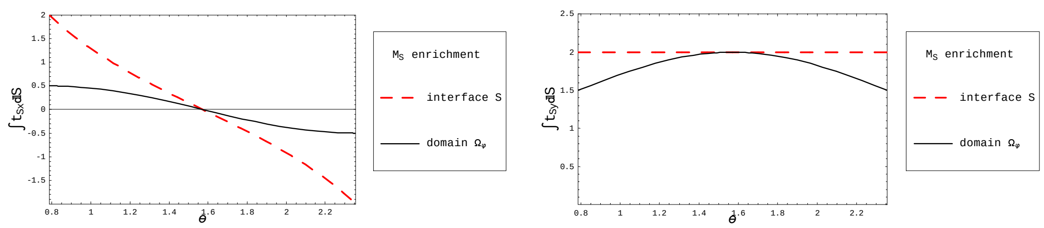
Figure 2: Four-nodes Finite Element. (a) Geometry. (b) Interface activation stress state.

If θ is in the range $\theta \in [\pi/4, 3\pi/4]$ the generic interface S cuts sides 1-4 and 2-3.

The standard enrichment function $M_S(\mathbf{x})$ is given by $H_S(\mathbf{x}) - (N_3(\mathbf{x}) + N_4(\mathbf{x}))$, whose gradient is $(0, -\frac{1}{2})$. The standard continuity equation leads to

$$\int_{\Omega_\varphi/S} \sigma \cdot \nabla^S M_S d\Omega = \begin{bmatrix} (\sigma_1 - \sigma_2)\sin 2\theta \\ -2(\sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta \end{bmatrix} \qquad \int_S \mathbf{t}_S dS = \begin{bmatrix} 2\sigma_2 \cot\theta \\ 2\sigma_2 \end{bmatrix} \tag{21}$$

In figure 3 the values of the two terms in (21) are reported, in the case $\sigma_1 = 2\sigma_2 = \sigma_0 = 1$. It can be observed that the continuity is satisfied only when $\theta = \pi/2$.



(a) (b)

Figure 3: Standard enrichment. (a) x component (b) y component.

The enhanced enrichment function $\bar{M}_S(\mathbf{x}) = M_S(\mathbf{x}) N_S(\mathbf{x})$ can be defined using for $N_S(\mathbf{x})$ the expression

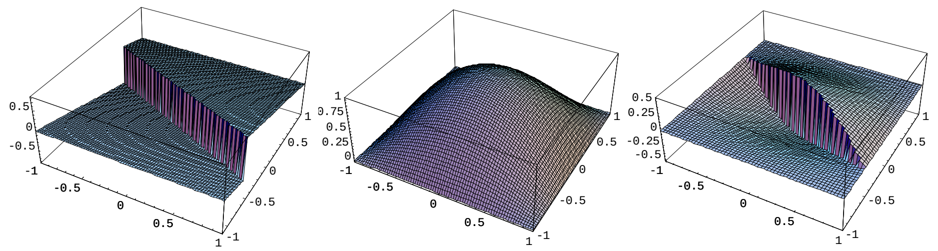
$$N_S(\mathbf{x}) = 8\left[N_2(\mathbf{x})N_4(\mathbf{x}) + N_1(\mathbf{x})N_3(\mathbf{x})\right] \tag{22}$$

satisfying the requirement of null displacement on the border of the element.

In this case function $\varphi(\mathbf{x})$ is given by

$$\varphi(\mathbf{x}) = N_3(\mathbf{x}) + N_4(\mathbf{x}) \tag{23}$$

Figure 4 shows the plots of functions $M_S(\mathbf{x})$, $N_S(\mathbf{x})$ and $\bar{M}_S(\mathbf{x})$. Along the interface the discontinuity is equal to 1 at the center, 0 on the edges.



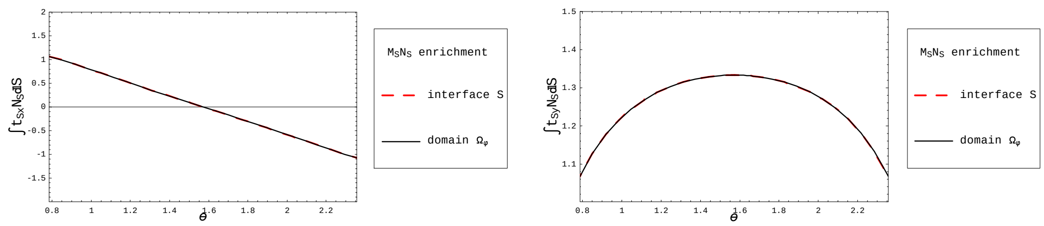
(a) (b) (c)

Figure 4: 4-nodes element. Discontinuity for $\theta = \pi/3$. (a) Enrichment function $M_S(\mathbf{x})$. (b) Modulation function $\bar{N}_S(\mathbf{x})$. (c) Enhanced enrichment function $\bar{M}_S(\mathbf{x})$.

Simple calculations allow to evaluate the two integrals in expression (18) for the considered stress field. It is obtained

$$\int_{\Omega_\varphi/S} \sigma \nabla^S \bar{M}_S d\Omega = \begin{bmatrix} \frac{4}{15}\sigma_2 \cot\theta(\cot^2\theta - 5) \\ \frac{4}{15}\sigma_2(\cot^2\theta - 5) \end{bmatrix} \quad \int_S \bar{N}_S \mathbf{t}_S dS = \begin{bmatrix} -\frac{4}{15}\sigma_2 \cot\theta(\cot^2\theta - 5) \\ -\frac{4}{15}\sigma_2(\cot^2\theta - 5) \end{bmatrix} \tag{24}$$

so that (18) is satisfied $\forall\theta \in [\pi/4, 3\pi/4]$. The variability of the two integrals in (24) is shown in figure 5, as opposed to the case in figure 3.



(a) (b)

Figure 5: Enhanced enrichment. Integral on Ω_φ and on S of (18). (a) x component (b) y component.

The traction $\mathbf{t}_S$ on the discontinuity is obtained as

$$\mathbf{t}_S = \frac{1}{\int_S \bar{M}_S(\xi)d\xi} \int_{\Omega_\phi} \sigma \nabla^S \bar{M}_S d\Omega = \frac{1}{\int_S \bar{N}_S(\xi)d\xi} \int_{\Omega_\phi} \sigma \nabla^S \bar{M}_S d\Omega \tag{25}$$

being for properties (7) $[\![M_S]\!]_S = 1$.

5 INCREMENTAL DISPLACEMENT

In this section the local incremental inelastic step is examined. The relevant equations are obtained from the variation of the principle (14) with respect to $\mathbf{t}_S$ and σ. The first yields the flow rule for the discontinuity:$[\![\dot{\mathbf{u}}]\!]_{S_p} = \mu \partial_{\mathbf{t}_S} g(\mathbf{t}_S, \chi_S)$.Note that in this context the multiplier μ has the dimensions of a displacement. The variation with respect to σ yields the strong form of the elastic constitutive equation in the continuum,$\nabla^S \hat{\mathbf{u}} + (\nabla \bar{M}_S \otimes \mathbf{a})^S = \mathbf{E}^{-1}\sigma$.

In order to analyze the structure of the incremental relationships, the common hypothesis that the jump is constant in the element is made. Assuming the Rankine failure criterion (20) and linear softening $\chi_S = H\alpha_{Se}$, H being the tangent softening modulus, it is obtained that the jump $[\![\mathbf{u}]\!]_S$ is directed normally to the interface so that it is useful to introduce the notation $\mu = \lambda \mathbf{n}$, with λ a scalar. Enforcing the conditions $\dot{g} = 0$ after substituting $\mathbf{t}_S$ from (25), the plastic-like multiplier, that coincides with the module of the displacement jump, can be evaluated as

$$\lambda = -\frac{\int \mathbf{E}\nabla^S \hat{\mathbf{u}} \cdot (\nabla M_S \otimes \mathbf{n})^S d\Omega}{\int \mathbf{E}(\nabla M_S \otimes \mathbf{n})^S \cdot (\nabla M_S \otimes \mathbf{n})^S d\Omega + H_S \int_S M_S(\xi) d\xi} \tag{26}$$

The last integral at the denominator is performed along the interface, and represents an internal length that yields the continuous equivalent softening modulus. Since either $\bar{M}_S(\mathbf{x}) = N_S(\mathbf{x})M_S(\mathbf{x})$ and the length of the interface depend on the angle θ, the internal length is not constant, accordingly to what happens with the smeared crack model while for the standard SDA the internal length is constant. Figure 5 compares the results for the present model with either $N_S = 1$ or N_S given by (22) with the smeared crack results.

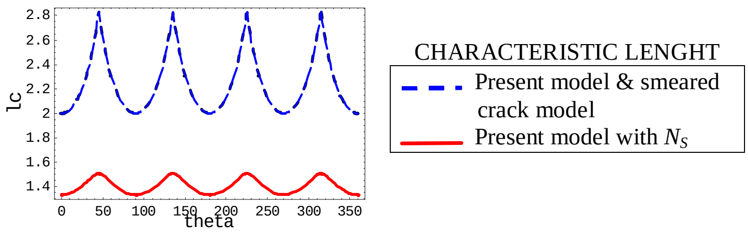


Figure 6: Internal length predicted by the models

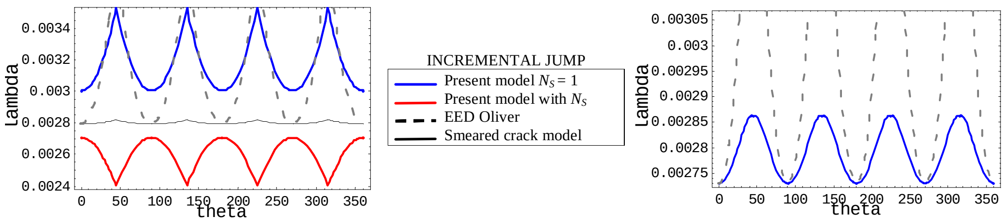
For the standard SDA model equation (26) has to be replaced by

$$\lambda = \frac{\int \mathbf{E}\nabla^S \hat{\mathbf{u}} \cdot (\mathbf{n} \otimes \mathbf{n})^S d\Omega}{-\int \mathbf{E}(\nabla M_S \otimes \mathbf{n})^S \cdot (\mathbf{n} \otimes \mathbf{n})^S d\Omega + H_S} \tag{27}$$

Figure 7(a) compares expressions (26) and (27). The results for the smeared crack model are also shown. The present model differs from both the other two examined. It can be noted that in the case H_S is very small the present model coincides with SDA at multiples of $\pi/2$, that is when the crack is parallel to the sides of the element (see fig. 7(b)).

6 CONCLUSIONS

In the paper it has been given a consistent kinematic characterization of the SDA for interface problems. In order to comply with boundary conditions the enhancement function has to assume



(a) (b)

Figure 7: Incremental displacement. (a) $H_S = -50$. (b) $H_S = -0.1$.

special forms near the boundary of the enriched region Ω_φ. This has consequences also on the internal continuity condition, that is obtained from the variational formulation. It has been shown that the weak form of the equations can be obtained employing a Hu-Washizu mixed functional: the equilibrium relations are directly obtained using the discontinuous kinematics and the internal energy given by the deformation energy of the continuum medium plus the energy dissipated on the interface. The orthogonality condition is fulfilled in a global sense, but in general not locally for each element. Further investigation are needed for evaluating the performance of the model proposed in section 2.1 in a more general context.

References

[1] Simo, J., Oliver, J., and Armero, F. “An analysis of strong discontinuities induced by strain softening in rate independent inelastic solids”. *Computat. Mech.* **12**, 277–296 (1993).

[2] Oliver, J., Cervera, M., and Manzoli, O. “Strong discontinuities and continuum plasticity models: The strong discontinuity approach”. *Int.J. Plasticity* **15**, 319–351 (1999).

[3] Jirásek, M. and Zimmermann, T. “Embedded crack model: Part I: Basic formulation”. *Int. J. Num. Meth. Eng.* **50**, 1269–1290 (2001).

[4] Moës, N., Dolbow, I., and Belytschko, T. “A finite element method for crack growth without remeshing”. *Int. J. Num. Meth. Eng.* **46**, 131–150 (1999).

[5] Simo, J. e Oliver, J. *Fracture and Damage in Quasi-brittle Structures*, chapter A new approach to the analysis and simulation of strong discontinuities, 25–39. E and FN Spon (1994).

[6] Halphen, B. and Nguyen, Q. “Sur les matriaux standards gnraliss”. *Journal de Mecanique* **14**, 39–63 (1975).

[7] Mosler, J. “On the modelling of highly localized deformations induced by material failure: the strong discontinuity approach”. *Archives of Computational Methods in Engineering* **11**, 389–446 (2004).