Structural modification of vibrating systems: an approach based on a constrained inverse eigenvalue problem

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SUMMARY. The inverse structural modification problem consists in determining the mass and stiffness parameters ensuring the desired eigenstructure in a vibrating undamped linear system. Such a problem is usually formulated as an unconstrained inverse eigenvalue problem, which simplifies the computation of a solution but does not ensure its feasibility.

In this work, in order to assure technically feasible modifications, the structural modification problem is formulated as a constrained inverse eigenvalue problem and is solved within the frame of convex constrained quadratic optimization. Such an approach ensures the existence of an unique, and hence global, optimal and feasible solution. Moreover, powerful numerical tools allow this solution to be efficiently computed.

A numerical proof of the method effectiveness is provided by applying it to a distributed parameter multi-body system.

1 INTRODUCTION

The design and the optimization of a vibrating system are generally aimed at obtaining the desired dynamic response. The modification of the system dynamic response is usually carried out by modifying its physical parameters. However, establishing the relationship between the changes in the physical properties of a system and the corresponding alterations in the dynamic response (i.e. the system eigenstructure) is not trivial and is currently matter of investigation. Structural modification techniques, in particular, aim at identifying the mass and stiffness modifications which allow achieving the desired eigenstructure

There are basically two opposite classes of structural modifications problems. The first one, referred to as "direct structural modification problem", is the forward problem for the prediction of the new eigenstructure of a dynamic system subjected to structural modification. The second one, referred to as "inverse structural modification problem", aims at precisely defining the system inertial and stiffness parameters on the basis of the desired system dynamic behaviour, often expressed in terms of its spectral data.

The inverse structural modification problem is the most interesting from a practical point of view since it can be applied both to improve the preliminary design of a system and to optimize the performances of existing systems. In the last decades the interest towards the study of inverse problems has therefore arisen, and several solutions have been developed and proposed in literature. Such problems have also inherent mathematical and algebraic interest because of their nonlinear characteristics, their frequent ill-conditioning and the topology imposition on the involved system matrices. Heuristic or iterative solutions are hence usually adopted.

The early inverse structural modification techniques are based on the Rayleigh matrix perturbation approach [1] and on the sensitivity analysis of the eigenstructure, developed by several researchers in the following decades [2]. These studies take advantage of an approximate solution of the derivatives of the eigenvalues and the eigenvectors of a dynamic system with respect to system perturbation, such as modification of materials or geometrical parameters. A thorough historical review is provided in [3] and [4]. The main problem associated with the perturbation approach is that it provides reliable and effective results as long as modifications are "small". If the modifications are not "small", accurate solutions might be achieved through iterations. These approaches, however, usually require small computational effort.

In the last two decades, the structural modification problem has been more frequently formulated as an inverse eigenvalue problem. The chief aspects of the inverse problem in vibration, including inverse eigenvalue problems, are covered by Gladwell's book [5]. The developed approaches employ both the system eigenstructure and the Frequency Response Function (FRF).

In their pioneering work, Ram and Braun [6] formulated the modification problem as an optimised inverse eigenvalue problem, minimising the Frobenious norm of a residual matrix. The resulting solutions are optimal in the sense of the Rayleigh-Ritz idea. The optimality in the sense of Rayleigh-Ritz is exploited also in [7], where a method for determining mass and stiffness modifications is presented for overcoming the difficulties arising from the availability of truncated modal data. This paper is one of the few works which explicitly allows dealing with interrelated mass and stiffness. A more intuitive approach, based on the solution of linear equations, is instead proposed in [8]. Nevertheless, as the Authors admit, it can usually be applied when either just masses or stiffnesses are to be modified, and the number of parameters to be modified is equal to the number of degrees-of-freedom (dofs) of the system model adopted. In addition, no condition assures the feasibility of the solutions.

Examples of FRF-based approaches are instead the works presented in [9] and [10]. The former tackles the problem of assigning natural frequencies to a multi-dof undamped system by adding a mass supported through one or more springs. In [10] the exact solutions of the inverse eigenvalue problem are studied. These solutions are provided for the specific case of an equal number of mode shape constraints and structural modification parameters, which implies that the system modification becomes the unique solution of a linear problem. When the problem admits infinite solutions, this paper suggests an approximate procedure for examining the feasibility of the available set of modifications. In the selection of the best solution, however, no optimization problem is explicitly defined, which implies that such a selection approach is not straightforward.

A wide number of structural modification problems cope with the location of anti-resonances in order to attain local vibration absorption. The state of art of the mathematical theory of vibration absorption is presented and illustrated in the thorough review proposed by Mottershead and Ram [11]. In [12] the general theory of multi-degree-of-freedom dynamic absorbers is investigated, by providing interesting results for the modification of vibrating system.

Some recent developments on structural modification have been devoted to the optimal design and placement of structural supports (see e.g. [13]). In particular, the placement problem is not trivial because of its strongly nonlinear characteristics.

Differently from what has been proposed in literature so far, this work introduces a formulation of the structural modification problem ensuring an unique feasible solution. Basically a constrained inverse eigenvalue problem is formulated and solved within the frame of convex quadratic optimization. Starting from the dynamic model of the original system, the proposed formulation allows finding a solution for any number of alterable parameters in a multi-degree-of-freedom linear system. The system model needs to be expressed in physical coordinates, and the topology of the modification matrices must be coherent with the feasible modifications. The method developed is particularly suitable for the optimization of multi-body systems under single and constant frequency harmonic excitation and where a single major modal contribution arises. Such a working condition is very popular in industrial applications: examples of industrial devices operating under a single harmonic forced motion are sieves, conveyors and linear feeders. In these cases structural optimization should be performed in a narrow frequency range, by assigning the desired eigenstructure of the mode with the most significant modal participation factor at the excitation frequency.

A peculiar feature of the proposed approach is the convexity of the optimization problem. The optimization of convex functions on convex domains ensures that a globally optimal solution is found and that efficient and reliable algorithms are available for solving the optimization problem [14]. Furthermore, the presence of bounds on all the modification parameters, the a-priori assumed structure of the modification matrices, and the addition of a penalty term for large modifications ensure attaining a solution which is technically feasible and makes sense from a physical point of view. In the proposed formulation of the optimization problem, in particular, the aforementioned penalty term also plays a crucial role since it increases the numerical reliability of the solution, and its robustness with respect to system model uncertainty.

The paper is organized as follows: in Section 2 the optimization problem is defined starting from the assignment of the desired eigenstructure. The problem analytical solution holding for the unconstrained case is also presented. The addition of a regularizing term to the problem formulation is argued for in Section 3, where both an analytical solution is attained for the unconstrained optimization problem and a numerical solution is discussed for the constrained problem. A numerical test case is presented in Section 4 for validating the method. It is a distributed parameter multi-body system, modelled through beam finite elements and recalling a simplified model of linear feeder. Finally, concluding remarks and future directions are given in Section 5.

2 INVERSE EIGENVALUE PROBLEM

Let consider $\mathbf{M} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ respectively the mass and stiffness matrices of an undamped N-degree-of-freedom linear system. Let also consider the symmetric matrices $\mathbf{\Delta M} \in \mathbb{R}^{N\times N}$ and $\mathbf{\Delta K} \in \mathbb{R}^{N\times N}$ which represent the feasible modifications of **M** and **K**. The structure of $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ is a priori defined reflecting technically feasible and realistic modifications, i.e. modifications which are coherent with the system design requirements. Admittedly, technical requirements and constraints usually impose system modifications to be performed only on a reduced number of masses and stiffness. There follows that $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ are, in general, sparse matrices. Additionally, it is assumed that the modifications do not increase the number of the system degrees of freedom. Hence, the eigenvalue problem of the modified system becomes

$$\omega^2 (\mathbf{M} + \Delta\mathbf{M})\mathbf{u} = (\mathbf{K} + \Delta\mathbf{K})\mathbf{u} \tag{1}$$

where ω is the generic eigenvalue and **u** the corresponding eigenvector of the modified system. In the inverse eigenvalue problem the matrices $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ are calculated after imposing the desired eigenpair (ω, **u**) (i.e. the system new dynamic requirements).

Let also define:

- $\Delta\mathbf{m} \in \mathbb{R}^{N_m}$ the vector collecting the N_m unknown terms of $\mathbf{\Delta M}$
- $\Delta\mathbf{k} \in \mathbb{R}^{N_k}$ the vector collecting the N_k unknown terms of $\mathbf{\Delta K}$.

The vector of the problem unknowns $\mathbf{x} \in \mathbb{R}^{N_x}$, $N_x = N_m + N_k$ can be defined

$$\mathbf{x} := \left\{ \Delta\mathbf{m}^{\mathrm{T}} \quad \Delta\mathbf{k}^{\mathrm{T}} \right\}^{\mathrm{T}} . \tag{2}$$

Two real matrices $\mathbf{U}_{\mathbf{M}} \in \mathbb{R}^{N \times N_m}$ and $\mathbf{U}_{\mathbf{K}} \in \mathbb{R}^{N \times N_k}$ do exist, so that

$$\Delta\mathbf{M}\mathbf{u} = \mathbf{U}_{\mathbf{M}} \Delta\mathbf{m} \tag{3}$$
$$\Delta\mathbf{K}\mathbf{u} = \mathbf{U}_{\mathbf{K}} \Delta\mathbf{k} . \tag{4}$$

Both $\mathbf{U}_{\mathbf{M}}$ and $\mathbf{U}_{\mathbf{K}}$ only depend on the desired eigenvector $\mathbf{u}$.
After defining

$$\mathbf{U} := \left[\omega^2 \mathbf{U}_{\mathbf{M}}, \mathbf{U}_{\mathbf{K}} \right] \tag{5}$$
$$\mathbf{b} := \mathbf{K}\mathbf{u} - \omega^2 \mathbf{M}\mathbf{u} \tag{6}$$

the inverse eigenvalue problem can be stated as

$$\mathbf{U}\mathbf{x} = \mathbf{b} \tag{7}$$

where $\mathbf{U} \in \mathbb{R}^{N \times N_x}$ and $\mathbf{b} \in \mathbb{R}^N$ only depend on the desired vibration mode and on the original system parameters.

Since no assumptions on N_x and on the rank of $\mathbf{U}$ are made, the linear equation $\mathbf{U}\mathbf{x} = \mathbf{b}$ does not admit an exact solution for the general case. Additionally, the solution must in general belong to a constrained region

$$\Gamma = \left\{ \mathbf{x} : \ \mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{x}_{\max} \right\} \tag{8}$$

ensuring that the system modifications are technically feasible. As a consequence, the inverse problem should be formulated as a constrained quadratic optimization problem

$$\min_{\mathbf{x}} \left\{ \left\| \mathbf{U}\mathbf{x} - \mathbf{b} \right\|_2^2 , \mathbf{x} \in \Gamma \right\} \tag{9}$$

where the goal is to minimize the norm of the residual of the linear system (7).

Since an analytical solution of problem (9) does not exists, a numerical method must be employed for computing the optimal solution. Nevertheless the proposed formulation of the inverse structural modification problem is a convex optimization problem, since it consists in minimizing a convex function on a convex domain [14]. This fact ensures that an unique global optimum does exist, and it can be computed numerically regardless of the initial solution guess.

In literature, several efficient solution algorithms have been proposed: for example, the Reflective Newton Method [15] is suitable for large scale problems and ensures quadratic convergence to the unique solution. An analytical solution of the problem (9) is available for the special case in which the solution set Γ is the whole $\mathbb{R}^{N_x}$ (unconstrained least-squares problem):

$$\mathbf{x} = \left(\mathbf{U}^{\mathrm{T}} \mathbf{U} \right)^{-1} \mathbf{U}^{\mathrm{T}} \mathbf{b} . \tag{10}$$

It must be observed that robustness issues arise in calculating the numerical value of the analytical solution (10). Generally speaking, the robust solutions of the inverse eigenvalue problem are those which exhibit small variation with respect to variations of the problem parameter (e.g. **U** and **b**). These variations should be taken into account because of both the inevitable system parameter uncertainty (influencing vector **b**) and the possible variations in the desired vibration mode (influencing both matrix **U** and vector **b**). The robustness issue is strictly related to the ill-posed nature of the problem (9), such that the numerical value of the analytical solution (10) of the unconstrained least-squares problem generally provides an overestimation of the actual solution [16] and therefore it is not a useful approximation of the desired structural modification. Both the issues can be overcome through the addition of a Tikhonov's regularization term to the problem formulation [14]. Clearly this concepts can be extended to the solution of the constrained problem (9).

3 REGULARIZED INVERSE EIGENVALUE PROBLEM

Beside increasing the problem robustness, the addition of the regularization term allows penalizing large parameter modifications. Large parameter modifications are in fact usually undesirable from a technical and economical point of view and can also cause spillover phenomena in the modes which have been neglected in the modification problem.

The inverse eigenvalue problem can then be stated as a Tikhonov regularized bi-criterion quadratic optimization problem of the convex function $f(\mathbf{x})$ on a convex domain Γ.

$$\min_{\mathbf{x}}\left\{f(\mathbf{x}):=\left\|\mathbf{Ux}-\mathbf{b}\right\|_2^2+\lambda\left\|\mathbf{\Omega_x x}\right\|_2^2,\mathbf{x}\in\Gamma\right\}. \tag{11}$$

The scalar positive value λ is the regularization parameter, trading between the cost of missing the target specifications, $\left\|\mathbf{Ux}-\mathbf{b}\right\|_2^2$, and the cost of using large values of the design variables $\left\|\mathbf{\Omega_x x}\right\|_2^2$. The definite positive diagonal matrix $\mathbf{\Omega_x}\in\mathbb{R}^{N_x\times N_x}$ is the regularization operator, and defines a scalar product which induces a norm on $\mathbb{R}^{N_x}$. It is employed in order to provide different weighs to the components of **x**, i.e. to penalize the modifications of some design variables, according to their technological and economical feasibility.

For any rank or dimension of matrix **U** and for any λ>0, the problem (11) has an analytical solution for the unconstrained case ($\Gamma=\mathbb{R}^{N_x}$):

$$\mathbf{x}=\left(\mathbf{U^TU}+\lambda\mathbf{\Omega_x}^{\mathbf{T}}\mathbf{\Omega_x}\right)^{-1}\mathbf{U^Tb}. \tag{12}$$

If **x** does not lie in the feasible set Γ, the constrained optimal solution of (11) must be computed numerically, as stated in Section 2.

The analytical solution (12) also allows better understanding the capability of the added regularization term to improve the numerical conditioning and the robustness of the solution. In fact, the condition number of the matrix $\mathbf{U^TU}+\lambda\mathbf{\Omega_x}^{\mathbf{T}}\mathbf{\Omega_x}$, defined as $\left\|\left(\mathbf{U^TU}+\lambda\mathbf{\Omega_x}^{\mathbf{T}}\mathbf{\Omega_x}\right)\right\|\left\|\left(\mathbf{U^TU}+\lambda\mathbf{\Omega_x}^{\mathbf{T}}\mathbf{\Omega_x}\right)^{-1}\right\|$, is a decreasing function of λ, and therefore the bigger λ is, the less reliable and sensitive the solution is.

The selection of an optimal value of λ clearly depends on the acceptability of missing the target specifications and of large parameter modification, which are often conflicting requirements.

A popular method for the choice of a suitable value is the curve:

$$L := \left\{ \left(\log \left\| \mathbf{\Omega}_{\mathbf{x}} \mathbf{x}_{\lambda} \right\|_2^2 , \log \left\| \mathbf{U} \mathbf{x}_{\lambda} - \mathbf{b} \right\|_2^2 \right) : \lambda > 0 \right\} \tag{13}$$

where $\mathbf{x}_{\lambda}$ is the solution of problem (11) for a given λ.

The resulting curve is usually called the *L*-curve, since its graph looks like a letter "L", when plotted with a log-log scale. In [16] it is proposed to choose the value of λ that corresponds to the point at the "vertex" of the "L", where the vertex is defined to be the point on the L-curve with the largest magnitude curvature. The selection of λ can be easily based on the L-curve of the unconstrained problem, since it has an analytical expression (see Eq. (12)).

4 NUMERICAL TEST CASE

In order to assess the effectiveness of the proposed method, a test case is discussed (Fig. (1)). It is a distributed parameter system consisting of a beam connected to the ground through two linear springs (k_3, k_4), and two concentrated masses (m_1, m_2) connected to the beam through two linear springs (k_1, k_2). The beam is modeled through four concatenated Euler-Bernoulli beam finite elements. The springs k_3, k_4 are placed at the beam tips, while the springs k_1, k_2 are placed at a distance d from the nearest beam tip. The test case proposed has therefore twelve dofs, and recalls a simplified model suitable for linear feeders. As a matter of fact, the four beam elements (with linear density ρ, bending stiffness EJ and total length l) may be adopted to model the tray of a feeder, while the two springs (k_3, k_4), connecting the frame and the beam may describe the system elastic supports. Finally, the two masses (m_1, m_2) can represent the feeder electromagnetic actuators. The nominal values of the aforementioned inertial and stiffness characteristics are listed in the second column of Table 1.

Let **u** be an eigenvector of the system, where *u(1)*, *u(3)*, *u(5)*, *u(7)*, *u(9)* are the eigenvector components related to the beam nodal displacements, *u(2)*, *u(4)*, *u(6)*, *u(8)*, *u(10)* are those related to the beam nodal rotations, and *u(11)*, *u(12)* are those related to the lumped mass displacements. In the presence of a 60 Hz single harmonic excitation, the mode with the highest participation factor is the one listed in the second column of Table 2 and depicted in Fig. (2).

In the proposed example it is assumed that modifications of the stiffness and of the concentrated masses are allowed, while the beam properties and the distance d are assumed constant. Additionally, three further springs (k_5, k_6, k_7) and three masses (m_3, m_4, m_5) may be added, as represented by the symbols depicted with dotted lines in Fig. (1). Figure (1) also depicts the position where such additional masses and stiffness are to be placed. The mass and stiffness modifications are constrained by the values listed in the third column of the Table 1, which allows defining the feasible domain Γ.

This leads to twelve design parameters, i.e. to twelve problem unknowns.

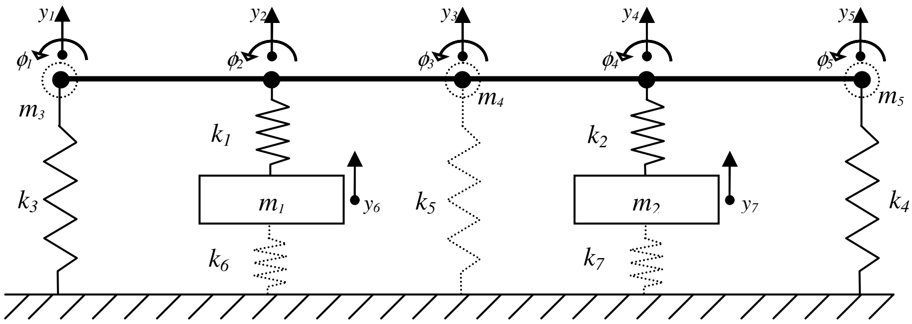

Figure 1: Dynamic model with distributed and lumped parameters.

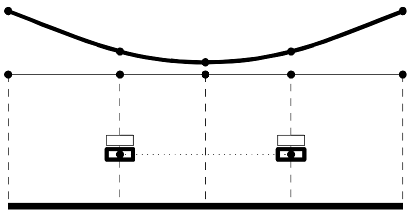

Figure 2: original mode shape

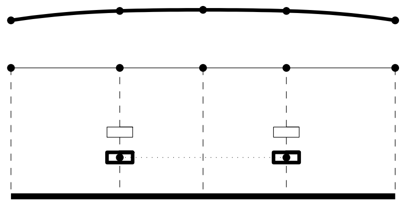

Figure 3: modified mode shape

The theory developed has been applied in order to get a system eigenpair consisting in a natural frequency at 60 Hz and a corresponding mode shape characterized by both a constant displacement along the tray and a desired equal relative displacement between the tray and the two masses. The imposed eigenvector **u** is explicitly defined in the third column of Table 2.

The regularization operator $\mathbf{\Omega_x} \in \mathbb{R}^{12\times 12}$ and parameter λ have been chosen as follows:

- $\mathbf{\Omega_x} = diag(k_1^{-1}, k_2^{-1}, 1.5k_3^{-1}, 1.5k_4^{-1}, 1.5k_5^{-1}, 1.5k_6^{-1}, 1.5k_7^{-1}, m_1^{-1}, m_2^{-1}, m_{3,\max}^{-1}, m_{4,\max}^{-1}, m_{5,\max}^{-1})$
- $\lambda = 1e\text{-}5$.

This selection of the entries of $\mathbf{\Omega_x}$ ensures that the relative modifications of the springs connected to the frame (k_3, k_4, k_5, k_6, k_7) are more penalized than the relative modifications of the other springs and masses, with respect to their initial value or to their largest allowed modification (when the initial value is equal to zero). The regularization parameter λ has been chosen according to the *L*-curve of the unconstrained problem, which is plotted in dotted line in Fig. (4). Figure (4) also depicts the *L*-curve for the constrained problem. It should be noticed that the constrained problem provides solution with higher $\mathbf{\Omega_x}$-norm than the unconstrained one. This is a consequence of the definition of the norm adopted and of the bound of the feasible set. Clearly, the unconstrained solution $\mathbf{\Omega_x}$-norm is smaller than the largest allowed (feasible) modification $\mathbf{\Omega_x}$-norm ($\mathbf{\Omega_x^{\max}} := \max\left\{\left\|\mathbf{\Omega_x x}\right\|_2^2, \mathbf{x}\in\Gamma\right\}$=117.11).

The unconstrained and constrained solutions of the structural modification are listed respectively in the fourth and in the fifth columns of Table 1. It can be noticed that the solution (12) of the unconstrained problem does not ensure achieving a feasible solution, since $\mathbf{x} \notin \Gamma$.

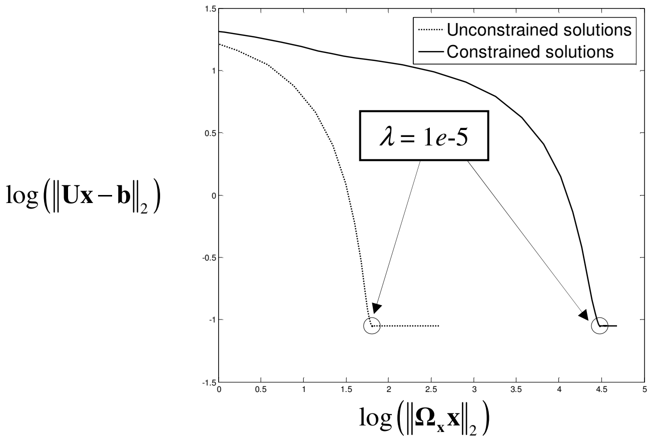

Figure 4: *L*-curves of the regularized problem

The solution computation has been performed by implementing both the analytical expression of Eq. (12) and the numerical solution of problem (11) on a 2.0 GHz PC running Microsoft Windows Xp. As far as the computational effort is concerned, it should be pointed out that a computation time equal to 0.44 seconds have been required for performing the numerical optimization.

Table 1: Distributed system properties and modifications

	Nominal Values	**Constrains**	**Unconstrained Solution**	**Constrained Solution**
m_1 [kg]	25	[- 10, 10]	- 11.67	- 10.00
m_2 [kg]	25	[- 10, 10]	- 11.67	- 10.00
m_3 [kg]	0	[0, 10]	- 3.06	0.00
m_4 [kg]	0	[0, 10]	- 5.76	0.00
m_5 [kg]	0	[0, 10]	- 3.06	0.00
k_1 [kN/m]	1.0*e*3	[- 0.8*e*3, 1.0*e*3]	- 3.69*e*2	- 3.67*e*2
k_2 [kN/m]	1.0*e*3	[- 0.8*e*3, 1.0*e*3]	- 3.69*e*2	- 3.67*e*2
k_3 [kN/m]	1.0*e*2	[- 0.8*e*2, 5.0*e*2]	0.96	4.33*e*2
k_4 [kN/m]	1.0*e*2	[- 0.8*e*2, 5.0*e*2]	0.96	4.33*e*2
k_5 [kN/m]	0	[0, 1.0*e*3]	1.80	8.15*e*2
k_6 [kN/m]	0	[0, 5.0*e*2]	0.58	2.25*e*2
k_7 [kN/m]	0	[0, 5.0*e*2]	0.58	2.25*e*2
EJ [Nm^2]	1.75*e*5	-	-	-
ρ [kg/m]	8.883	-	-	-
l [m]	3	-	-	-
d [m]	0.85	-	-	-

The effectiveness of the results attained is assessed by evaluating three parameters:

- $\|\mathbf{Ux}-\mathbf{b}\|_2^2$: the norm square of the non regularized problem residual;
- $f(\mathbf{x})$: the objective function (see Eq. (11));

- $\delta = \cos^{-1}\left(\left(\mathbf{u}^{\mathrm{T}} \bullet \tilde{\mathbf{u}}\right) / \left(\|\mathbf{u}\|\|\tilde{\mathbf{u}}\|\right)\right)$: the angle between the desired eigenvector $\mathbf{u}$ and the actual eigenvector $\tilde{\mathbf{u}}$.

Table 2 analytically allows comparing the desired eigenpair (ω, $\mathbf{u}$) (first column) with those of the original system (column two) and of the modified systems (third and fourth columns). In particular column three states the eigenpair attained by solving the unconstrained optimization problem (i.e. the unfeasible modification), while the eigenpair attained by solving the constrained optimization problem (i.e. the feasible modification) are stated in the fourth column. The effectiveness of the feasible modification is more clearly stressed by Figs. (2) and (3) which schematically compares the mode shapes of the original system and of the modified (constrained). The thin lines show the relative positions of the masses in the undeformed configuration while the thick lines show the dynamic displacements occurring at the attained natural frequency. The dotted lines connecting the centres of the two small masses further clarify the relative displacements, while the dashed lines show the positions of the springs.

Clearly, the unconstrained solution provides a good values on the three evaluation parameters ($\|\mathbf{Ux}-\mathbf{b}\|_2^2 = 0.1219$, $f(\mathbf{x})$=0.1223 and δ = 12.55°). In fact, the mode natural frequency (ω = 59.55 Hz) approaches the desired one and the angle δ is significantly reduced with respect to the one of the original system ($\|\mathbf{Ux}-\mathbf{b}\|_2^2$ = 0.2003, δ = 49.65°). As far as the solution of the constrained problem is concerned, it provides an effective solution to the modification problem, since both the eigenvector and the eigenvalue approach very closely the desired values (δ= 13.61° and ω = 58.98 Hz). These results lead to a negligible increase in the residual norm ($\|\mathbf{Ux}-\mathbf{b}\|_2^2$=0.1229) while the objective function increases ($f(\mathbf{x})$=0.2002), because of the higher $\mathbf{\Omega}_{\mathbf{x}}$-norm of the constrained problem solution (see Fig. 4).

Table 2: Eigenstructure comparison

	Original	**Desired**	**Unconstrained**	**Constrained**
u(1)	0.4835	0.4264	0.3723	0.3706
u(2)	- 0.3862	0	0.1416	0.1526
u(3)	0.1757	0.4264	0.4378	0.4442
u(4)	- 0.2654	0	0.0225	0.0279
u(5)	0.0936	0.4264	0.4451	0.4526
u(6)	0	0	0	0
u(7)	0.1757	0.4264	0.4378	0.4442
u(8)	0.2654	0	- 0.0225	- 0.0279
u(9)	0.4835	0.4264	0.3723	0.3706
u(10)	0.3862	0	- 0.1416	- 0.1526
u(11)	0.1065	- 0.2132	- 0.2238	- 0.1973
u(12)	0.1065	- 0.2132	- 0.2238	- 0.1973
ω [Hz]	51.82	60.00	59.55	58.98

5 CONCLUSIONS

In this work an original formulation based on convex constrained optimization has been proposed for the optimal inverse structural modification of multi-body vibrating systems. The approach developed aims at computing the optimal inertial and stiffness parameter modifications for attaining a desired eigenstructure, and is particularly suitable when a single mode shape at a specific natural frequency needs to be imposed in the system dynamics.

A strength of the proposed formulation is that it allows finding a global optimal solution for any number of dofs and design parameters. Besides that, the presence of constraints on all the parameters and the introduction of the regularization term penalizing large modifications, ensure attaining strictly technically feasible solutions.

The numerical results obtained studying a distributed parameter multi-body system including beam finite elements prove the effectiveness of the method and its small computational cost.

Future works will be devoted to the experimental validation of the method proposed and to the generalization of the approach to discrete modifications.

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