Explicit Lagrangian Formulation of the Dynamic Regressors for Serial Manipulators

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In this paper an explicit formulation of the matrices used in the linear-regressor form of the dynamics of general n-dof serial manipulators is presented. First, a closed formula for the classical regressor $Y(q, \dot{q}, \ddot{q})$ is derived directly from the Lagrange equations, in terms of the Denavit-Hartenberg Jacobians, the configuration vector q and its derivatives $\dot{q}$, $\ddot{q}$. Then, a specialized version $Y_r(q, \dot{q}, \dot{q}_r, \ddot{q}_r)$ is obtained, which is suitable for application of the Slotine-Li adaptive control scheme.

The advantage of the proposed formulation is that an explicit analytic form of the regressor of a general manipulator can be produced systematically (a dedicated *Mathematica*™ package is made available). Furthermore, the analytic form of the regressor enables direct inspection of the physical properties of the manipulator, which is concealed by the recursive formulations available in the literature. To illustrate the proposed approach, explicit calculations are carried out for a simple two-dof manipulator.

1 INTRODUCTION

The manipulator regressor, usually denoted as $Y(q, \dot{q}, \ddot{q})$, is a matrix function originated by Atkeson [1], Khosla [2], and Kawasaki [3], employed to write the manipulator dynamics *linearly* in the inertial parameters

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Y(q, \dot{q}, \ddot{q})\pi = \tau, \quad (1)$$

where q, $\dot{q}$ and $\ddot{q} \in \mathbb{R}^n$ denote the joint position, velocity and acceleration, respectively; $B(q)$, $C(q, \dot{q}) \in \mathbb{R}^{n\times n}$ are the inertia and Coriolis matrices; $G(q) \in \mathbb{R}^n$ is the gravitational force vector. The left side of (1) represents the *inertial* and the *gravitational* forces in the dynamics of a n-link rigid-body manipulator while the right side is the input torque/force vector $\tau \in \mathbb{R}^n$. As evident from (1), the robot dynamic equations are linear with respect to a properly defined inertia parameter vector $\pi \in \mathbb{R}^r$ via the regressor matrix $Y(q, \dot{q}, \ddot{q}) \in \mathbb{R}^{n\times r}$.

The manipulator regressor $Y(q, \dot{q}, \ddot{q})$ is a key quantity in derivation as well as implementation of many established adaptive motion and force control algorithms [4].

A different definition of the manipulator regressor stems from the Slotine-Li adaptive algorithm [5], where a *novel* $Y_r(q, \dot{q}, \dot{q}_r, \ddot{q}_r)$ is introduced to fit

$$B(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + G(q) = Y_r(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\pi, \quad (2)$$

where $\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$ is the reference velocity; $\Lambda \in \mathbb{R}^{n\times n}$ is an arbitrary p.d. matrix and q_d is the desired joint trajectory. It is worth remarking here that in (1) multiple forms of $C(q, \dot{q})$ can be employed, as far as $C(q,\dot{q})\dot{q}$ is the same. On the other hand, for the Slotine-Li adaptive control scheme to work properly, it is required for the regressor to be written in a form $\hat{C}(q,\dot{q})$, obtained e.g. via the Christoffel symbols, which ensures the skew-symmetry of matrix $\dot{B}-2\hat{C}$.

In principle, the regressor can be obtained by following a two-step *indirect* approach as: (i) one first formulates the manipulator dynamics in the standard form[1]

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{3}$$

and (ii) having defined a vector of parameters π, one works on every entry on the left-hand side of (3) to obtain the desired version $Y\pi = \tau$. Even if the first step is straightforward, the parametric extraction is hardly systematic and computationally demanding as the entries of π are, in general, spread all over the entries of $B(q)$, $C(q,\dot{q})$ and $G(q)$. Moreover, for manipulators with "many" degrees of freedom, the computational burden can be hindering even for most advanced computer algebra systems (e.g., *Mathematica*™ [6]).

Motivated by the necessity of finding a direct way to obtain the regressor dynamics, several solutions have been proposed in the literature. Atkeson *et al.* [1] proposed a recursive Newton-Euler approach, but did not address the Slotine-Li version of the regressor. Gautier and Khalil [7] as well as Mayeda *et al.* [8] focused mainly on the identification of the minimum set of parameters actually appearing in the equations of motion.

The work of Lu and Meng [9] uses the Lagrangian approach to derive a closed formula for the classical regressor. Their analytical treatment does not generalize to the computation of the Slotine-Li regressor, and is quite different from the one developed in this paper.

Following the success of the Slotine-Li adaptive control scheme [5], several researchers have studied a regressor formulation that is compatible with the specific requirements of the method, i.e. the derivation of those forms $\hat{C}(q,\dot{q})$ of the Coriolis matrix that ensures skew-symmetry of $\dot{B}-2\hat{C}$. An approximate algorithm, based on a Newton-Euler recursive formulation, is provided already in [5]. Yuan and Yuan [10] employ a modified version of the Newton-Euler equations proposed in [11] to inject the reference velocities and accelerations in the expressions. The recursive nature of the algorithm entails that a formula for the regressor matrix $Y_r(q,\dot{q},\dot{q}_r,\ddot{q}_r)$ is not obtained, rather values are provided for the matrix product used in Slotine-Li parameter update equation (i.e., $Y^T s$).

In this work a different approach is pursued. By employing the Lagrange equations of motion, a closed formula for the classical regressor $Y(q,\dot{q},\ddot{q})$ is obtained by properly factorizing the expressions in the inertial parameters throughout the computations. Moreover, by directly imposing skew-symmetry of the terms corresponding to $N = \dot{B} - 2C$ and by introducing the reference velocity, the closed formula for the Slotine-Li regressor $Y_r(q,\dot{q},\dot{q}_r,\ddot{q}_r)$ alone is derived.

The advantage of our formulation is that an explicit analytic form of the regressor of a general manipulator can be produced in a thoroughly systematic way. Furthermore, the exact form of the regressor we provide enables analytic inspection of the physical properties of the manipulator, such as e.g. the possible irrelevance of some of the inertial parameters (corresponding to a zero column in $Y(\cdot)$), which is concealed by the recursive formulations available in the literature. The algorithm is implemented in a *Mathematica*™ package which is made available to the public. To illustrate the proposed approach, explicit calculations are carried out for the simple two-dof planar elbow manipulator.

[1] The indirect route to the definition of the Slotine-Li regressor $Y_r(q,\dot{q},\dot{q}_r,\ddot{q}_r)$ would entail the use of (2) and is here omitted for brevity.

2 LAGRANGIAN EQUATIONS OF ROBOT DYNAMICS

To establish the background and notation for the rest of the paper, we briefly review the Lagrangian formulation of the dynamics of a general *n*-dof serial manipulator.

Let $U^{(i)}$ and $T^{(i)}$ be, respectively, the potential and kinetic energy associated to the link *i* and q the $n \times 1$ vector of joint displacements; the Lagrangian of the serial chain is defined as

$$L\left(q, \dot{q}\right) = T(q, \dot{q}) - U(q) = \sum_{i=1}^{n} \left(T^{(i)}(q, \dot{q}) - U^{(i)}(q) \right) = \sum_{i=1}^{n} L^{(i)}(q, \dot{q}), \tag{4}$$

because $L\left(q, \dot{q}\right)$ is obviously linkwise additive.
Thus, adopting the Lagrange equations, the dynamics of the manipulator is

$$\left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right]^T = \sum_{i=1}^{n} \left[\frac{d}{dt} \frac{\partial L^{(i)}}{\partial \dot{q}} - \frac{\partial L^{(i)}}{\partial q} \right]^T = \tau, \tag{5}$$

where τ is the $n \times 1$ vector of applied joint torques or forces.

The key feature of robot dynamics exploited to derive adaptive control laws ([4], [5]) is its *linearity* with respect to some properly defined parameters π

$$Y(q, \dot{q}, \ddot{q})\, \pi = \tau, \tag{6}$$

where $Y(q, \dot{q}, \ddot{q})$ is the manipulator regressor.

Referring to (5) and denoting the block of the regressor associated to the link *i* with $Y^{(i)}$, we have

$$\sum_{i=1}^{n} \left[\frac{d}{dt} \frac{\partial L^{(i)}}{\partial \dot{q}} - \frac{\partial L^{(i)}}{\partial q} \right]^T = \sum_{i=1}^{n} Y^{(i)}\, \pi^{(i)}, \tag{7}$$

where $\pi^{(i)} \in \mathbb{R}^{r_i}$ is the vector of the parameters associated to link *i* and $Y^{(i)} \in \mathbb{R}^{n \times r_i}$.

Thus, for a *n*-link manipulator, we can write the complete regressor as $Y = \left[Y^{(1)} \ldots Y^{(n)} \right] \in \mathbb{R}^{n \times r}$ and the dynamic parameters as $\pi = \left[\pi^{(1)^T} \ldots \pi^{(n)^T} \right]^T \in \mathbb{R}^r$, with $r = \sum_{i=i}^{n} r_i$.

Considering (4) and (7), it follows that

$$\left[\frac{d}{dt} \frac{\partial T^{(i)}}{\partial \dot{q}} - \frac{\partial T^{(i)}}{\partial q} + \frac{\partial U^{(i)}}{\partial q} \right]^T = Y^{(i)}\, \pi^{(i)}. \tag{8}$$

3 DIRECT FORMULATION OF THE MANIPULATOR REGRESSOR

With reference to Fig. 1, we assume that each link *i* has a local coordinate system $\{i\}$ with the origin O_i fixed at joint $i + 1$, according to the classical Denavit-Hartenberg (DH) conventions. A coordinate system is fixed at the center of mass G_i, located with respect to O_i by the vector p_{iG_i}, and with respect to the global origin *O* by ${}^0p_{iG_i}$.

3.1 Kinetic Energy Terms

Applying the König theorem with respect to the global coordinate system O, the kinetic energy $T^{(i)}$ of the i-th link can be written as

$$T^{(i)} = \frac{1}{2} m_i\, {}^0v_{G_i}^T\, {}^0v_{G_i} + \frac{1}{2}\, {}^i\omega_i^T\, {}^iI_{G_i}\, {}^i\omega_i, \tag{9}$$

where ${}^0v_{G_i}$ is the G_i linear velocity, ${}^i\omega_i$ is the angular velocity and ${}^iI_{G_i}$ is the moment-of-inertia tensor about the center of mass G_i.

Furthermore, using the rotation matrix ${}^0R_i$ from the global frame $\{0\}$ to the D.-H. frame $\{i\}$, we obtain

$$ {}^i\omega_i = {}^iR_0\,{}^0\omega_i = {}^0R_i{}^T\,{}^0\omega_i. \tag{10} $$

Next, adopting the position and orientation DH Jacobians J_{v_i} and J_{ω_i}, we have

$$ {}^0v_{G_i} = {}^0v_i + {}^0\omega_i \times {}^0p_{iG_i} = J_{v_i}\dot{q} + J_{\omega_i}\dot{q} \times {}^0R_i\,p_{iG_i}; \qquad {}^0\omega_i = J_{\omega_i}\,\dot{q}. \tag{11} $$

Thus, substituting (10) and (11) in (9), we obtain

$$ \begin{aligned} T^{(i)} &= \frac{1}{2}\,m_i(J_{v_i}\dot{q} + J_{\omega_i}\dot{q} \times {}^0R_i\,p_{iG_i})^T(J_{v_i}\dot{q} + J_{\omega_i}\dot{q} \times {}^0R_i\,p_{iG_i}) \\ &+ \frac{1}{2}\,\dot{q}^T(J_{\omega_i}^T\,{}^0R_i\,{}^iI_{G_i}\,{}^0R_i{}^T\,J_{\omega_i})\dot{q}. \end{aligned} $$

Let $S(x) \in \mathbb{R}^{3\times3}$ be a skew-symmetric matrix such that $S(x)\,y = S^T(y)\,x = x \times y$ for any $x,\, y \in \mathbb{R}^3$; we have $S(R\,x) = R\,S(x)\,R^T$, where $R \in \mathbb{R}^{3\times3}$ represents a rotation matrix. Considering this latter property, we can write

$$ \begin{aligned} T^{(i)} &= \frac{1}{2}\,m_i\,\dot{q}^T\,(J_{v_i}^T\,J_{v_i})\,\dot{q} - \frac{1}{2}\,m_i\,\dot{q}^T\left\{J_{v_i}^T\,S({}^0R_i\,p_{iG_i})\,J_{\omega_i}\right\}\dot{q} \\ &+ \frac{1}{2}\,m_i\,\dot{q}^T\left\{J_{\omega_i}^T\,S({}^0R_i\,p_{iG_i})\,J_{v_i}\right\}\dot{q} \\ &+ \frac{1}{2}\,\dot{q}^T\left\{J_{\omega_i}^T\,{}^0R_i\left[{}^iI_{G_i} + m_i\,S^T(p_{iG_i})\,S(p_{iG_i})\right]{}^0R_i{}^T\,J_{\omega_i}\right\}\dot{q}. \end{aligned} \tag{12} $$

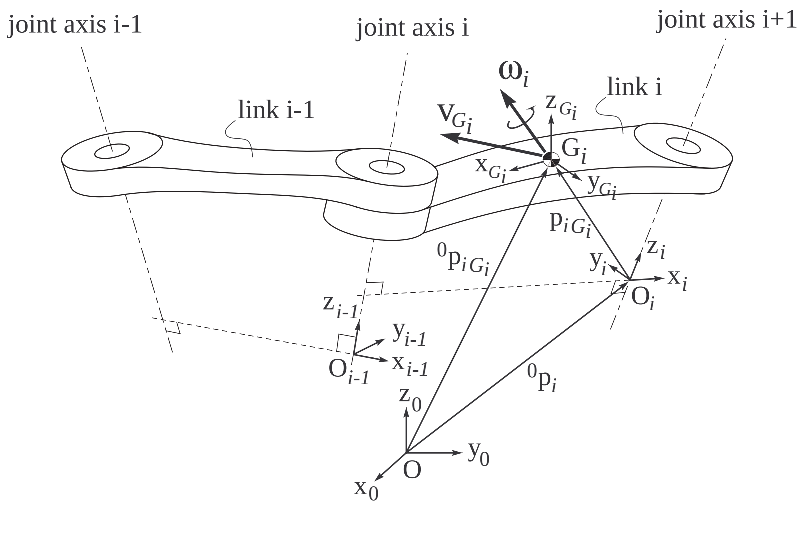

Figure 1: Local coordinate systems and vectors for the i-th link.

Taking the derivative of (12) according to (8) we get

$$\begin{aligned}\frac{\partial T^{(i)}}{\partial \dot{q}} &= m_i\,\dot{q}^T\,(J_{v_i}^T\,J_{v_i}) - m_i\,\dot{q}^T\left\{J_{v_i}^T\,S({}^0R_i\,p_{iG_i})\,J_{\omega_i}\right\} + m_i\,\dot{q}^T\left\{J_{\omega_i}^T\,S({}^0R_i\,p_{iG_i})\,J_{v_i}\right\}\\ &+ \dot{q}^T\left\{J_{\omega_i}^T\,{}^0R_i\,{}^iI_i\,{}^0R_i^T\,J_{\omega_i}\right\},\end{aligned} \tag{13}$$

where ${}^iI_i = {}^iI_{G_i} + m_i\,S^T(p_{iG_i})\,S(p_{iG_i})$ is the inertia tensor with respect to origin O_i (Steiner's theorem). Since ${}^iI_i$ is a symmetric matrix and thanks to the properties of skew-symmetric matrices, we can rearrange (13) as

$$\begin{aligned}\left[\frac{\partial T^{(i)}}{\partial \dot{q}}\right]^T &= (J_{v_i}^T\,J_{v_i})\,\dot{q}\,m_i + \left\{J_{v_i}^T\,S(J_{\omega_i}\,\dot{q})\,{}^0R_i - J_{\omega_i}^T\,S(J_{v_i}\,\dot{q})\,{}^0R_i\right\}m_i\,p_{iG_i}\\ &+ J_{\omega_i}^T\,{}^0R_i\,{}^iI_i\,{}^0R_i^T\,J_{\omega_i}\,\dot{q}.\end{aligned} \tag{14}$$

Eq. (14) is made of three terms: the first and the second ones are obtained by extracting respectively the mass m_i and the first order moment-of-inertia $m_i\,p_{iG_i}$; the elements of the matrix ${}^iI_i$ have to be extracted from the third term. For such purpose we can consider a third-order tensor $E \in \mathbb{R}^{3\times3\times6}$ and the vector of parameters $\bar{J}_i = \left[\bar{J}_{ixx}\;\bar{J}_{ixy}\;\bar{J}_{ixz}\;\bar{J}_{iyy}\;\bar{J}_{iyz}\;\bar{J}_{izz}\right]^T$, where

$$E = \left[\begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6\end{array}\right], \qquad {}^iI_i = \begin{bmatrix}\bar{J}_{ixx} & \bar{J}_{ixy} & \bar{J}_{ixz}\\ \bar{J}_{ixy} & \bar{J}_{iyy} & \bar{J}_{iyz}\\ \bar{J}_{ixz} & \bar{J}_{iyz} & \bar{J}_{izz}\end{bmatrix}, \tag{15}$$

with

$$\begin{aligned}E_1 &= \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \quad E_2 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \quad E_3 = \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\\ E_4 &= \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \quad E_5 = \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \quad E_6 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.\end{aligned}$$

In this way we can write the tensor ${}^iI_i$ as the inner product between E and $\bar{J}_i$. Hence

$${}^iI_i = E\,\bar{J}_i. \tag{16}$$

With this procedure, the third term of (14) becomes

$$\begin{aligned}J_{\omega_i}^T\,{}^0R_i\,{}^iI_i\,{}^0R_i^T\,J_{\omega_i}\dot{q} &= \left[J_{\omega_i}^T\,{}^0R_iE\,{}^0R_i^T\,J_{\omega_i}\dot{q}\right]\,\bar{J}_i\\ &= \left[J_{\omega_i}^T\,{}^0R_iE_1\,{}^0R_i^T\,J_{\omega_i}\dot{q}|\ldots|J_{\omega_i}^T\,{}^0R_iE_6\,{}^0R_i^T\,J_{\omega_i}\dot{q}\right]\,\bar{J}_i.\end{aligned} \tag{17}$$

Referring to (8), we are now able to extract the mass m_i and both $m_i\,p_{iG_i}$ and $\bar{J}_{ilm}$ $(l, m = x, y, z)$ from the first term of the Lagrange equations. Thus we get

$$\frac{d}{dt}\left[\frac{\partial T^{(i)}}{\partial \dot{q}}\right]^T = \dot{X}_0^{(i)}\,\pi_0^{(i)} + \dot{X}_1^{(i)}\,\pi_1^{(i)} + \dot{X}_2^{(i)}\,\pi_2^{(i)}, \tag{18}$$

where

$$
\begin{aligned}
X_0^{(i)} &= (J_{v_i}^T\, J_{v_i})\, \dot{q} \;\in \mathbb{R}^{n\times 1} \\
X_1^{(i)} &= \left\{ J_{v_i}^T\, S(J_{\omega_i}\, \dot{q}) - J_{\omega_i}^T\, S(J_{v_i}\, \dot{q}) \right\} {}^0R_i \;\in \mathbb{R}^{n\times 3} \\
X_2^{(i)} &= J_{\omega_i}^T\, {}^0R_i \left[E_1 \,|\, E_2 \,|\, \ldots \,|\, E_6 \right] {}^0R_i^{\,T}\, J_{\omega_i}\, \dot{q} \;\in \mathbb{R}^{n\times 6} \\
\pi_0^{(i)} &= m_i \;\in \mathbb{R} && (19)\\
\pi_1^{(i)} &= \begin{bmatrix} m_i p_{iG_{ix}} & m_i p_{iG_{iy}} & m_i p_{iG_{iz}} \end{bmatrix}^T \;\in \mathbb{R}^3 && (20)\\
\pi_2^{(i)} &= \bar{J}_i \;\in \mathbb{R}^6. && (21)
\end{aligned}
$$

The second term of the Lagrange equations (8) is a function of the kinetic energy as well. With reference to (12), we differentiate with respect to q. Hence

$$
\begin{aligned}
\left[\frac{\partial T^{(i)}}{\partial q}\right]^T = &\frac{1}{2}\left\{ \dot{q}^T \left[\frac{\partial}{\partial q}(J_{v_i}^T\, J_{v_i})\right] \dot{q} \right\}^T m_i \\
&+ \frac{1}{2}\left\{ \dot{q}^T \left[\frac{\partial}{\partial q}\left(J_{v_i}^T\, S(J_{\omega_i}\, \dot{q})^0R_i - J_{\omega_i}^T\, S(J_{v_i}\, \dot{q})^0R_i\right)\right] \right\}^T m_i p_{iG_i} \qquad (22)\\
&+ \frac{1}{2}\left\{ \dot{q}^T \left[\frac{\partial}{\partial q}\left(J_{\omega_i}^T\, {}^0R_i\, {}^iI_i\, {}^0R_i^{\,T}\, J_{\omega_i}\right)\right] \dot{q} \right\}^T .
\end{aligned}
$$

Let us consider the third term of the right part of the previous equation; again, by employing (16), we can extract the second order moment-of-inertia $\bar{J}_i$. It follows that

$$
\left\{ \frac{\partial}{\partial q}\left[J_{\omega_i}^T\, {}^0R_i\, {}^iI_i\, {}^0R_i^{\,T}\, J_{\omega_i} \right] \right\}^T = \begin{bmatrix} \frac{\partial}{\partial q}\left(J_{\omega_i}^T\, {}^0R_i\, E_1\, {}^0R_i^{\,T}\, J_{\omega_i}\right) \\ \vdots \\ \frac{\partial}{\partial q}\left(J_{\omega_i}^T\, {}^0R_i\, E_6\, {}^0R_i^{\,T}\, J_{\omega_i}\right) \end{bmatrix} \bar{J}_i.
$$

Thus, we can now write more compactly

$$
\left[\frac{\partial T^{(i)}}{\partial q}\right]^T = W_0^{(i)}\, \pi_0^{(i)} + W_1^{(i)}\, \pi_1^{(i)} + W_2^{(i)}\, \pi_2^{(i)}, \qquad (23)
$$

where

$$
\begin{aligned}
W_0^{(i)} &= \frac{1}{2}\,\dot{q}^T \begin{bmatrix} \frac{\partial}{\partial q_1}\left(J_{v_i}^T\, J_{v_i}\right) \\ \vdots \\ \frac{\partial}{\partial q_n}\left(J_{v_i}^T\, J_{v_i}\right) \end{bmatrix} \dot{q}, \\
W_1^{(i)} &= \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial q_1}\left[{}^0R_i^{\,T}\, S^T(J_{\omega_i}\, \dot{q})\, J_{v_i}\, \dot{q} - {}^0R_i^{\,T}\, S^T(J_{v_i}\, \dot{q})\, J_{\omega_i}\, \dot{q}\right] \\ \vdots \\ \frac{\partial}{\partial q_n}\left[{}^0R_i^{\,T}\, S^T(J_{\omega_i}\, \dot{q})\, J_{v_i}\, \dot{q} - {}^0R_i^{\,T}\, S^T(J_{v_i}\, \dot{q})\, J_{\omega_i}\, \dot{q}\right] \end{bmatrix}, \qquad (24)\\
W_2^{(i)} &= \frac{1}{2}\,\dot{q}^T \begin{bmatrix} \frac{\partial}{\partial q_1}\left(J_{\omega_i}^T\, {}^0R_i\, E\, {}^0R_i^{\,T}\, J_{\omega_i}\right) \\ \vdots \\ \frac{\partial}{\partial q_n}\left(J_{\omega_i}^T\, {}^0R_i\, E\, {}^0R_i^{\,T}\, J_{\omega_i}\right) \end{bmatrix} \dot{q}.
\end{aligned}
$$

3.2 Potential Energy Term

The last term of the Lagrange equations to be managed comes from the potential energy. The potential energy $U^{(i)}$ associated to the link *i* can be written as

$$U^{(i)} = -m_i\, g^T\, \left({}^0p_i \,+\, {}^0R_i\, p_{iG_i}\right), \tag{25}$$

where g is the gravitational acceleration vector with respect to the global frame and ${}^0p_i$ is the position vector from the origin O of the global coordinate system to the DH frame $\{i\}$ fixed at O_i.
We can derive the expression used in the Lagrange equations differentiating $U^{(i)}$ with respect to q. Hence

$$\left[\frac{\partial U^{(i)}}{\partial q}\right]^T = -\, m_i\, \left\{g^T\, \frac{\partial\, {}^0p_i}{\partial q} \,+ g^T\, \frac{\partial\, {}^0R_i}{\partial q}\, p_{iG_i}\right\}^T = -\, J_{v_i}^T\, g\, m_i - \left[\frac{\partial\, (g^T\, {}^0R_i)}{\partial q}\, m_i\, p_{iG_i}\right]^T. \tag{26}$$

Considering the last term of the right part of the previous equation, we get

$$\left[\frac{\partial\, (g^T\, {}^0R_i\, m_i\, p_{iG_i})}{\partial q}\right]^T = \left[\frac{\partial}{\partial q}\left\{(m_i\, p_{iG_i})^T\, {}^0R_i^T\, g\right\}\right]^T = \begin{bmatrix} \left(\frac{\partial({}^0R_i{}^T g)}{\partial q_1}\right)^T \\ \vdots \\ \left(\frac{\partial({}^0R_i{}^T g)}{\partial q_n}\right)^T \end{bmatrix} m_i\, p_{iG_i},$$

In this way we can express the term associated with potential energy in the Lagrange equations as a sum of two linear functions with respect to m_i and $m_i\, p_{iG_i}$. Equation (26) can also be written more compactly as

$$\left[\frac{\partial U^{(i)}}{\partial q}\right]^T = Z_0^{(i)}\, m_i \,+\, Z_1^{(i)}\, m_i\, p_{iG_i}, \tag{27}$$

where

$$Z_0^{(i)} = -\, J_{v_i}^T\, g, \qquad Z_1^{(i)} = -\begin{bmatrix} \left(\frac{\partial({}^0R_i{}^T g)}{\partial q_1}\right)^T \\ \vdots \\ \left(\frac{\partial({}^0R_i{}^T g)}{\partial q_n}\right)^T \end{bmatrix}. \tag{28}$$

3.3 Regressor and Inertial Parameters

The Lagrange equations related to the link *i* can now be written from (8), (18), (19), (20), (21), (23) and (27) as

$$\left[\frac{d}{dt}\frac{\partial T^{(i)}}{\partial \dot{q}} - \frac{\partial T^{(i)}}{\partial q} + \frac{\partial U^{(i)}}{\partial q}\right]^T = Y^{(i)}\pi^{(i)}, \tag{29}$$

where

$$Y^{(i)}\pi^{(i)} = \begin{bmatrix} Y_0^{(i)} & Y_1^{(i)} & Y_2^{(i)} \end{bmatrix} \begin{bmatrix} \pi_0^{(i)} \\ \pi_1^{(i)} \\ \pi_2^{(i)} \end{bmatrix}, \tag{30}$$

with

$$Y_0^{(i)} = \dot{X}_0^{(i)} - W_0^{(i)} + Z_0^{(i)} \in \mathbb{R}^{n\times 1} \quad (31)$$
$$Y_1^{(i)} = \dot{X}_1^{(i)} - W_1^{(i)} + Z_1^{(i)} \in \mathbb{R}^{n\times 3} \quad (32)$$
$$Y_2^{(i)} = \dot{X}_2^{(i)} - W_2^{(i)} \in \mathbb{R}^{n\times 6}. \quad (33)$$

Therefore, by simply juxtaposing the regressor blocks $Y^{(i)}$ associated to the link i, for $i = 1, \ldots, n$, the manipulator regressor can be written as

$$Y(q, \dot{q}, \ddot{q}) = \left[\begin{array}{ccc} Y^{(1)} & \cdots & Y^{(n)} \end{array} \right]. \quad (34)$$

Similarly, we build the inertial parameters vector by stacking the $\pi^{(i)}$ for each link

$$\pi = \left[\pi^{(1)T} \ \cdots \ \pi^{(n)T} \right]^T. \quad (35)$$

4 DIRECT FORMULATION OF THE SLOTINE-LI REGRESSOR

The result of the previous section, i.e. the direct formulation of the classical dynamic regressor, lends itself to generalization to a form that is compatible with the Slotine-Li version of the regressor.

Indeed, the Slotine-Li (SL) control algorithm and adaptation law are obtained introducing a robot regressor defined as [5]

$$B(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + G(q) = Y_r(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\,\pi, \quad (36)$$

where $\dot{q}_r$, was defined in (2), and $Y_r(q, \dot{q}, \dot{q}_r, \ddot{q}_r)$ is denoted as the SL regressor. It is worth noting the fundamental role played in (36) by that particular choice of $\hat{C}(q, \dot{q})$, since Slotine and Li [5] proved the stability of their adaptive controller using the skew-symmetric identity

$$s^T[\dot{B}(q, \dot{q}) - 2\hat{C}(q, \dot{q})]s = 0 \quad \forall s \neq 0 \in R^n. \quad (37)$$

In order to satisfy (37) matrix $\hat{C}(q, \dot{q})$ can be built via the Christoffel symbols of the first kind [11]. With this choice, the hj-th element of $\hat{C}(q, \dot{q})$ can be written as

$$\hat{c}_{hj} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial b_{hj}}{\partial q_k} + \frac{\partial b_{hk}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_h}\right)\dot{q}_k = \frac{1}{2}\sum_{k=1}^{n}\hat{c}_{hjk}\dot{q}_k, \quad (38)$$

where b_{hj} is the hj-th element of $B(q)$.

Let us now present an alternative form of (5) which is useful for further developments

$$\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q}\right]^T = B\,\ddot{q} + \left[\dot{B} - \frac{1}{2}\,\dot{q}^T\,\frac{\partial B}{\partial q}\right]\dot{q}, \quad (39)$$

where $\frac{\partial B}{\partial q} \in \mathbb{R}^{n\times n\times n}$ is a third-order tensor with elements $\left[\frac{\partial B}{\partial q}\right]_{jkh} = \frac{\partial b_{jk}}{\partial q_h}$. Here, $C(q, \dot{q}) = \dot{B}(q) - \frac{1}{2}\,\dot{q}^T\,\frac{\partial B(q)}{\partial q}$ is the *classical* Coriolis matrix.

Recalling the identity $C(q, \dot{q})\,\dot{q} = \hat{C}(q, \dot{q})\,\dot{q}$, (note the same velocity $\dot{q}$ is inside and outside parentheses) eq.(39) can be written in component-wise fashion as

$$\left[B(q)\ddot{q} + \hat{C}(q, \dot{q})\dot{q}\right]_h = \sum_{j=1}^{n} b_{hj}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}\left[\left(\frac{\partial b_{hj}}{\partial q_k} + \frac{\partial b_{hk}}{\partial q_j}\right)\dot{q}_k - \frac{\partial b_{jk}}{\partial q_h}\dot{q}_k\right]\dot{q}_j. \quad (40)$$

In this form, the injection of the reference velocity and acceleration $\dot{q}_r$ and $\ddot{q}_r$, as required from (36), leads to

$$\Big[B(q)\,\ddot{q}_r \;+\; \hat{C}(q,\dot{q})\,\dot{q}_r\Big]_h \;=\; \sum_{j=1}^{n} b_{hj}\,\ddot{q}_{r_j} + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}\left[\left(\frac{\partial b_{hj}}{\partial q_k} \;+\; \frac{\partial b_{hk}}{\partial q_j}\right)\dot{q}_k \;-\; \frac{\partial b_{jk}}{\partial q_h}\dot{q}_k\right]\dot{q}_{r_j}\,. \tag{41}$$

A further useful step to take (which will be clearer later) is to operate a different aggregation of the summation terms in (41) by defining $\mathfrak{X}$ and $\mathfrak{W}$, whose h-th components $[\mathfrak{X}]_h$ and $[\mathfrak{W}]_h$ are explicitly given by

$$[\mathfrak{X}]_h \;=\; \sum_{j=1}^{n} b_{hj}\,\ddot{q}_{r_j} + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}\left(\frac{\partial b_{hj}}{\partial q_k} + \frac{\partial b_{hk}}{\partial q_j}\right)\dot{q}_k\dot{q}_{r_j}\,, \quad [\mathfrak{W}]_h \;=\; \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{jk}}{\partial q_h}\dot{q}_k\dot{q}_{r_j}\,. \tag{42}$$

By handling with care the tensor dimensions, the terms in (42) can be cast in matrix notation as

$$\mathfrak{X} = B(q)\,\ddot{q}_r + \frac{1}{2}\big\{\big[\frac{\partial B}{\partial q} + \big(\frac{\partial B}{\partial q}\big)^{T_{132}}\big]\,\dot{q}\big\}\dot{q}_r, \quad \mathfrak{W} = \frac{1}{2}\big\{\big[\frac{\partial B}{\partial q}\big]^{T_{231}}\dot{q}\big\}\,\dot{q}_r. \tag{43}$$

Here, the generalized transpose operator was employed which is defined for tensors of arbitrary order. If we let $(A)_{i_1\,\ldots\,i_j\,\ldots i_k} = a_{i_1\,\ldots i_j\ldots i_k}$ be a tensor of order k, the operator $(\cdot)^{T_{n_1\,\cdots n_j\cdots n_k}}$ allows to specify an arbitrary reordering to apply to the indices of a tensor, such that if $\tilde{A} = A^{T_{n_1\,\cdots n_j\cdots n_k}}$ the j-th level in $\tilde{A}$ corresponds to the n_j-th level in A, as follows

$$\big(\tilde{A}\,\big)_{i_1\,\ldots i_j\ldots i_k} = a_{i_{n_1}\,\ldots i_{n_j}\ldots i_{n_k}}\,. \tag{44}$$

It is also worth pointing out that $\mathfrak{X}$ and $\mathfrak{W}$ are linkwise addiditive, that is

$$\mathfrak{X} \;=\; \sum_{i=1}^{n}\mathfrak{X}^{(i)}, \qquad \mathfrak{W} \;=\; \sum_{i=1}^{n}\mathfrak{W}^{(i)}, \tag{45}$$

since $B(q)$ itself enjoys the same property. In fact, with reference to (12) and (13), this can be written as

$$\begin{aligned} B(q) &= \sum_{i=1}^{n} B(q)^{(i)} \\ B(q)^{(i)} &= m_i\,(J_{v_i}^T\,J_{v_i}) \\ &\quad + m_i\big\{J_{\omega_i}^T\,S(\,^0R_i\,p_{iG_i})\,J_{v_i} - J_{v_i}^T\,S(\,^0R_i\,p_{iG_i})\,J_{\omega_i}\big\} \\ &\quad + \big\{J_{\omega_i}^T\,^0R_i\,\big[^iI_{G_i} + m_i\,S^T(p_{iG_i})\,S(p_{iG_i})\big]\,^0R_i{}^T\,J_{\omega_i}\big\} \end{aligned} \tag{46}$$

Now, meaningful expressions of (42) and/or (43) consistent with definition (46) and *explicitly* linear in the dynamic parameters can be sought for. The first term in (46) is already linear in the i-th link mass m_i; in order to extract the first order moment-of-inertia, the second term in the right-hand side of (46) can be rearranged as

$$m_i\big\{J_{\omega_i}^T\,^0R_iS(p_{iG_i})^0R_i{}^T J_{v_i} - J_{v_i}^T\,^0R_iS(p_{iG_i})^0R_i{}^T J_{\omega_i}\big\}. \tag{47}$$

Then, as done for (16), $S(p_{iG_i})$ is reshaped as an inner product of the third order tensor $Q \in \mathbb{R}^{3\times3\times3}$ with vector $p_{iG_i} = [p_{iG_{i_x}}\; p_{iG_{i_y}}\; p_{iG_{i_z}}]^T$, thus getting $S(p_{iG_i}) = Q\, p_{iG_i}$, where

$$\begin{aligned} Q &= \begin{bmatrix} Q_1 & Q_2 & Q_3 \end{bmatrix}, \\ Q_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{48}$$

Furthermore, by following the same steps as in (17), (46) is rewritten as

$$\begin{aligned} B(q)^{(i)} = m_i\,(J_{v_i}^T\, J_{v_i}) + \left\{ J_{\omega_i}^T\, {}^0R_i\, Q\, {}^0R_i{}^T\, J_{v_i} - J_{v_i}^T\, {}^0R_i\, Q\, {}^0R_i{}^T\, J_{\omega_i} \right\} m_i p_{iG_i} \\ + \left\{ J_{\omega_i}^T\, {}^0R_i\, E\, {}^0R_i{}^T\, J_{\omega_i} \right\} \bar{J}_i. \end{aligned} \tag{49}$$

By substituting (49) in each term of (45) and eliciting the structure of each $\mathfrak{X}^{(i)}$ and $\mathfrak{W}^{(i)}$ from the global versions (43), we are in a position to write (41) *linearly* in the parameters π.

Tedious but systematic calculations, lead us to the explicit form

$$\mathfrak{X}^{(i)} = \dot{X}_{0_r}^{(i)}\, \pi_0^{(i)} + \dot{X}_{1_r}^{(i)}\, \pi_1^{(i)} + \dot{X}_{2_r}^{(i)}\, \pi_2^{(i)}, \tag{50}$$

where

$$\begin{aligned} \dot{X}_{0_r}^{(i)} &= (J_{v_i}^T\, J_{v_i})\, \ddot{q}_r + \frac{1}{2}\left\{ \left[\frac{\partial (J_{v_i}^T\, J_{v_i})}{\partial q} + \left(\frac{\partial (J_{v_i}^T\, J_{v_i})}{\partial q} \right)^{T_{132}} \right] \dot{q} \right\} \dot{q}_r, \\ \dot{X}_{1_r}^{(i)} &= X_1^{(i)} \ddot{q}_r + \frac{1}{2}\left\{ \left[\frac{\partial X_1^{(i)}}{\partial q} + \left(\frac{\partial X_1^{(i)}}{\partial q} \right)^{T_{1324}} \right] \dot{q} \right\} \dot{q}_r, \\ \dot{X}_{2_r}^{(i)} &= X_2^{(i)}\, \ddot{q}_r + \frac{1}{2}\left\{ \left[\frac{\partial X_2^{(i)}}{\partial q} + \left(\frac{\partial X_2^{(i)}}{\partial q} \right)^{T_{1324}} \right] \dot{q} \right\} \dot{q}_r, \end{aligned} \tag{51}$$

with

$$\begin{aligned} X_1^{(i)} &= J_{\omega_i}^T\, {}^0R_i \left[Q_1 \,|\, Q_2 \,|\, Q_3 \right] {}^0R_i{}^T\, J_{v_i} - J_{v_i}^T\, {}^0R_i \left[Q_1 \,|\, Q_2 \,|\, Q_3 \right] {}^0R_i{}^T\, J_{\omega_i} \in \mathbb{R}^{n\times n\times 3}, \\ X_2^{(i)} &= J_{\omega_i}^T\, {}^0R_i \left[E_1 \,|\, E_2 \,|\, \dots \,|\, E_6 \right] {}^0R_i{}^T\, J_{\omega_i} \in \mathbb{R}^{n\times n\times 6}, \end{aligned} \tag{52}$$

In (51) the generalized transpose operator was applied to both 3-rd and 4-th order tensors. Representatives of these classes appearing in (51) are, respectively,

$$\frac{\partial (J_{v_i}^T\, J_{v_i})}{\partial q} \in \mathbb{R}^{n\times n\times n}, \quad \frac{\partial X_2^{(i)}}{\partial q} \in \mathbb{R}^{n\times n\times 6\times n} \tag{53}$$

The terms originating from $\mathfrak{W}$ in (42) can actually be computed as for $Y(q, \dot{q}, \ddot{q})$ with minor differences. These result in

$$\mathfrak{W}^{(i)} = W_{0_r}^{(i)}\, \pi_0^{(i)} + W_{1_r}^{(i)}\, \pi_1^{(i)} + W_{2_r}^{(i)}\, \pi_2^{(i)}, \tag{54}$$

where

$$W_{0_r}^{(i)} = \frac{1}{2}\,\dot{q}_r^T \begin{bmatrix} \frac{\partial}{\partial q_1}\left(J_{v_i}^T\, J_{v_i}\right) \\ \vdots \\ \frac{\partial}{\partial q_n}\left(J_{v_i}^T\, J_{v_i}\right) \end{bmatrix} \dot{q}$$

$$W_{1_r}^{(i)} = \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial q_1}\left[{}^0R_i{}^T\, S^T(J_{\omega_i}\,\dot{q})\, J_{v_i}\,\dot{q}_r \;-\; {}^0R_i{}^T\, S^T(J_{v_i}\,\dot{q})\, J_{\omega_i}\,\dot{q}_r\right] \\ \vdots \\ \frac{\partial}{\partial q_n}\left[{}^0R_i{}^T\, S^T(J_{\omega_i}\,\dot{q})\, J_{v_i}\,\dot{q}_r \;-\; {}^0R_i{}^T\, S^T(J_{v_i}\,\dot{q})\, J_{\omega_i}\,\dot{q}_r\right] \end{bmatrix}$$

$$W_{2_r}^{(i)} = \frac{1}{2}\,\dot{q}_r^T \begin{bmatrix} \frac{\partial}{\partial q_1}\left(J_{\omega_i}^T\,{}^0R_i\,E\,{}^0R_i{}^T\,J_{\omega_i}\right) \\ \vdots \\ \frac{\partial}{\partial q_n}\left(J_{\omega_i}^T\,{}^0R_i\,E\,{}^0R_i{}^T\,J_{\omega_i}\right) \end{bmatrix} \dot{q}.$$

Finally, the gravitational terms can be written as shown in (27), because $G(q)$ is not a function of the reference velocity $\dot{q}_r$.
The SL regressor block related to the i-th link becomes

$$Y_r^{(i)} = \begin{bmatrix} Y_{0_r}^{(i)} & Y_{1_r}^{(i)} & Y_{2_r}^{(i)} \end{bmatrix}, \tag{55}$$

where

$$Y_{0_r}^{(i)} = \dot{X}_{0_r}^{(i)} - W_{0_r}^{(i)} + Z_0^{(i)} \in \mathbb{R}^{n\times 1} \tag{56}$$

$$Y_{1_r}^{(i)} = \dot{X}_{1_r}^{(i)} - W_{1_r}^{(i)} + Z_1^{(i)} \in \mathbb{R}^{n\times 3} \tag{57}$$

$$Y_{2_r}^{(i)} = \dot{X}_{2_r}^{(i)} - W_{2_r}^{(i)} \in \mathbb{R}^{n\times 6}. \tag{58}$$

As a result, the Slotine-Li manipulator regressor can again be written juxtaposing the regressor blocks $Y_r^{(i)}$ as

$$Y_r(q,\dot{q},\dot{q}_r,\ddot{q}_r) = \begin{bmatrix} Y_r^{(1)} & \dots & Y_r^{(n)} \end{bmatrix}. \tag{59}$$

5 EXAMPLE

5.1 Explicit calculation of the Slotine-Li regressor for the planar elbow manipulator

A simple two-dof planar elbow manipulator, as shown in Fig. 2, is considered to test the proposed procedure for the calculation of the SL regressor. The manipulator is modeled as two rigid links of lengths a_1, a_2 and masses m_1, m_2. Let $p_{G_1} = [p_{G_{1x}}\, p_{G_{1y}}\, p_{G_{1z}}]^T$ and $p_{G_2} = [p_{G_{2x}}\, p_{G_{2y}}\, p_{G_{2z}}]^T$ be the position vector of the centroids G_i with respect to the origin of DH frame O_i, for $i = 1, 2$. Moreover, let I_1 and I_2 be the inertia tensors relative to the centroids of the two links, respectively. Finally, let $q = [q_1\; q_2]^T$ be the joint coordinates and let $\dot{q}_r = [\dot{q}_{r_1}\; \dot{q}_{r_2}]^T$ be the joint reference velocities.

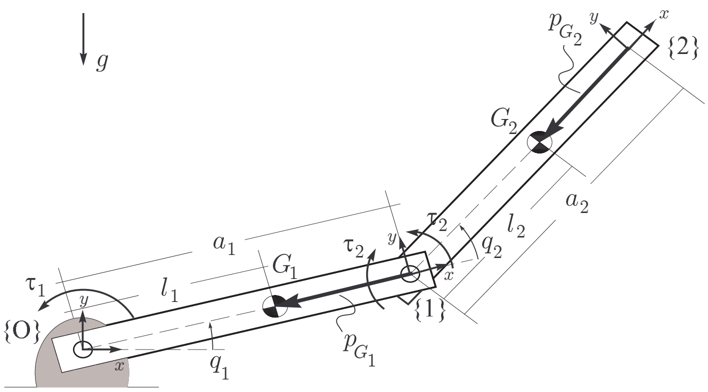

Figure 2: Planar elbow manipulator.

In the chosen coordinate frames, the computation of the D.-H. Jacobians yields

$$J_{v_1} = \begin{bmatrix} -a_1\,s_1 & 0 \\ a_1\,c_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_{v_2} = \begin{bmatrix} -a_1\,s_1 \,-\, a_2\,s_{12} & -a_2\,s_{12} \\ a_1\,c_1 \,+\, a_2\,c_{12} & a_2\,c_{12} \\ 0 & 0 \end{bmatrix},$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},$$

where $c_i = \cos(q_i)$, $s_i = \sin(q_i)$, for $i = 1, 2$, and $c_{12} = \cos(q_1 + q_2)$, $s_{12} = \sin(q_1 + q_2)$. Considering the definitions of $\dot{X}_r^{(i)}$, $W_r^{(i)}$ and $Z^{(i)}$, the terms contributing to the SL regressor block $Y^{(1)}$ for the first link are

$$\dot{X}_{0_r}^{(1)} = \begin{bmatrix} a_1^2\,\ddot{q}_{r_1} \\ 0 \end{bmatrix}, \qquad \dot{X}_{1_r}^{(1)} = \begin{bmatrix} 2\,a_1\,\ddot{q}_{r_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \dot{X}_{2_r}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \ddot{q}_{r_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W_{0_r}^{(1)} = \mathbf{0}_{2,\,1}, \qquad W_{1_r}^{(1)} = \mathbf{0}_{2,\,3}, \qquad W_{2_r}^{(1)} = \mathbf{0}_{2,\,6},$$

$$Z_0^{(1)} = \begin{bmatrix} a_1\,c_1\,g_y \\ 0 \end{bmatrix}, \qquad Z_1^{(1)} = \begin{bmatrix} c_1\,g_y & -g_y\,s_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the gravitational acceleration vector with respect to the global frame is $g = [0 \;\, -g_y \;\; 0]^T$ and

$\mathbf{0}_{n,\,m}$ is a $n \times m$ zero matrix. Analogously, for the second link we get

$$
\begin{aligned}
\dot{X}_{0_r}^{(2)} &= \begin{bmatrix} -b_6\,b_7 - b_6\,\dot{q}_{r_2}\,\dot{q}_2 + a_2^2\,b_4 + a_1^2\,\ddot{q}_{r_1} + b_4\,b_8 \\ -\frac{1}{2}\,b_6\,b_7 + b_8\,\ddot{q}_{r_1} + a_2^2\,b_4 \end{bmatrix}, \\
\dot{X}_{1_r}^{(2)} &= \begin{bmatrix} 2\,a_2\,b_4 + b_9\,b_3 - b_{10}\,b_2 & -b_9\,b_2 - b_{10}\,b_3 & 0 \\ b_9\,\ddot{q}_{r_1} - \frac{1}{2}\,b_{10}\,b_1 + 2\,a_2\,b_4 & -\frac{1}{2}\,b_9\,b_1 - b_{10}\,\ddot{q}_{r_1} & 0 \end{bmatrix}, \\
\dot{X}_{2_r}^{(2)} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 0 & b_4 \end{bmatrix}, \\
W_{0_r}^{(2)} &= \begin{bmatrix} 0 \\ -\frac{1}{2}\,a_1\,a_2\,s_2\,b_5 \end{bmatrix}, \qquad W_{1_r}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2}\,a_1\,s_2\,b_5 & -\frac{1}{2}\,a_1\,c_2\,b_5 & 0 \end{bmatrix}, \quad W_{2_r}^{(2)} = \mathbf{0}_{2,\,6}, \\
Z_0^{(2)} &= \begin{bmatrix} (a_1\,c_1 + a_2\,c_{12})\,g_y \\ a_2\,c_{12}\,g_y \end{bmatrix}, \quad Z_1^{(2)} = \begin{bmatrix} c_{12}\,g_y & -g_y\,s_{12} & 0 \\ c_{12}\,g_y & -g_y\,s_{12} & 0 \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 &= \dot{q}_{r_2}\,\dot{q}_1 + \dot{q}_{r_1}\,\dot{q}_2, & b_2 &= \dot{q}_{r_2}\,\dot{q}_1 + (\dot{q}_{r_1} + \dot{q}_{r_2})\,\dot{q}_2, \\
b_3 &= 2\,\ddot{q}_{r_1} + \ddot{q}_{r_2}, & b_4 &= \ddot{q}_{r_1} + \ddot{q}_{r_2}, \\
b_5 &= b_3\,\dot{q}_1 + \ddot{q}_{r_1}\,\dot{q}_2, & b_6 &= a_1\,a_2\,s_2, \\
b_7 &= \dot{q}_{r_2}\,\dot{q}_1 + \dot{q}_{r_1}\,\dot{q}_2, & b_8 &= a_1\,a_2\,c_2, \\
b_9 &= a_1\,c_2, & b_{10} &= a_1\,s_2.
\end{aligned}
$$

Thus, referring to (55), (56), (57) and (58) we can write the SL regressor blocks $Y_r^{(1)}$ and $Y_r^{(2)}$ associated to link 1 and 2. The manipulator regressor $Y_{r_{RR}}$ is obtained by juxtaposing these blocks as in (59).

Finally, considering (19), (20), (21) and (35) , we can build the inertial parameters vector as

$$
\begin{aligned}
\pi_0^{(1)} &= m_1\,, \quad \pi_1^{(1)} = \begin{bmatrix} m_1\,p_{G_{1x}} & m_1\,p_{G_{1y}} & m_1\,p_{G_{1z}} \end{bmatrix}^T\,, \\
\pi_2^{(1)} &= \begin{bmatrix} \bar{J}_{1xx} & \bar{J}_{1xy} & \bar{J}_{1xz} & \bar{J}_{1yy} & \bar{J}_{1yz} & \bar{J}_{1zz} \end{bmatrix}^T\,, \\
\pi_0^{(2)} &= m_2\,, \quad \pi_1^{(2)} = \begin{bmatrix} m_2\,p_{G_{2x}} & m_2\,p_{G_{2y}} & m_2\,p_{G_{2z}} \end{bmatrix}^T\,, \\
\pi_2^{(2)} &= \begin{bmatrix} \bar{J}_{2xx} & \bar{J}_{2xy} & \bar{J}_{2xz} & \bar{J}_{2yy} & \bar{J}_{2yz} & \bar{J}_{2zz} \end{bmatrix}^T\,, \\
\pi^{(1)} &= \begin{bmatrix} \pi_0^{(1)^T} & \pi_1^{(1)^T} & \pi_2^{(1)^T} \end{bmatrix}^T, \quad \pi^{(2)} = \begin{bmatrix} \pi_0^{(2)^T} & \pi_1^{(2)^T} & \pi_2^{(2)^T} \end{bmatrix}^T, \quad \pi = \begin{bmatrix} \pi^{(1)^T} & \pi^{(2)^T} \end{bmatrix}^T.
\end{aligned}
$$

5.2 Derivation of the classical regressor as a particular case

Further, we can derive the classical regressor (34) for the planar elbow (RR) manipulator $Y_{RR}(q, \dot{q}, \ddot{q})$ simply by substituting $\dot{q}_r = \dot{q}$ and $\ddot{q}_r = \ddot{q}$ in the obtained SL regressor $Y_{r_{RR}}(q, \dot{q}, \dot{q}_r, \ddot{q}_r)$. The classical regressor $Y_{RR}(q,\, \dot{q},\, \ddot{q})$ can be also computed by following the algorithm proposed in Section 3 with identical results. Adopting the proposed algorithms for the direct formulation of both the classical and Slotine-Li regressor, we get a vector π as function of ten inertial parameters per link. However, some of these parameters have no effect on the dynamics of the manipulator. We can determine the parameters not affecting the dynamic model simply inspecting the expressions of regressor blocks $Y^{(1)}$ and $Y^{(2)}$ and neglecting the parameters that correspond to the zero columns of the complete regressor Y_{RR}. Thus, for the considered RR manipulator we have

$$\begin{aligned} Y^{(1)} &= \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & y_{1,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ Y^{(2)} &= \begin{bmatrix} y_{1,11} & y_{1,12} & y_{1,13} & 0 & 0 & 0 & 0 & 0 & 0 & y_{1,20} \\ y_{2,11} & y_{2,12} & y_{2,13} & 0 & 0 & 0 & 0 & 0 & 0 & y_{2,20} \end{bmatrix} \end{aligned}$$

and the parameters that affect the dynamics are m_1, $m_1 p_{G_{1x}}$, $m_1 p_{G_{1y}}$, $\bar{J}_{1zz}$, m_2, $m_2 p_{G_{2x}}$, $m_2 p_{G_{2y}}$, $\bar{J}_{2zz}$. Hence the reduced regressor becomes a 2×8 matrix without zero columns and it is analogous to the one presented in [12] (Par. 4.3.2).

5.3 Implementation of the procedure in a software package

The results described in this section have been obtained by employing the *Wolfram Mathematica™* package *ScrewCalculus* developed by M. Gabiccini [13]. In this package the main function `Regressor` returns the expression of the complete manipulator regressor. A sample code that produces the classical and Slotine-Li regressors for the planar elbow manipulator with this *Mathematica™* function is shown below

```
<<ScrewCalculus`ScrewCalculus`
tab[q1_, q2_] = {{a1, 0, 0, q1, "R"},
                 {a2, 0, 0, q2, "R"}};
q[t_]   = {q1[t], q2[t]};
qd[t_]  = D[q[t], t];
qdd[t_] = D[qd[t], t];
v[t_]   = {v1[t], v2[t]};
vd[t_]  = D[v[t], t];
g0 = {0, -g, 0};
Y =Regressor[tab@@q[t],q[t],qd[t],qd[t],qdd[t],t,g0];
Yr=Regressor[tab@@q[t],q[t],qd[t],v[t],vd[t],t,g0];
```

The function `Regressor` returns the classical regressor `Y` and the SL regressor `Yr` evaluating the DH table of the manipulator `tab`, the vector of joint coordinates `q[t]`, its first and second derivative `qd[t]` and `qdd[t]` with respect to time `t`, the vector of reference velocity `v[t]`, its time derivative `vd[t]`, time `t` and the components of the gravitational acceleration vector with respect to the global frame, `g0`. For further details and examples regarding the package please refer to [13].

6 CONCLUSIONS

A direct formulation of the regressor for a general n-dof manipulator has been derived using the Lagrangian approach in a fresh way. In addition, a modified algorithm for the special regressor required in the Slotine-Li adaptive control algorithm has been presented. The developed algorithms has been implemented in a *Mathematica™* package which has been successfully tested on different manipulators.

The availability of closed-form expressions for arbitrary serial manipulators could be employed as a good starting point for further investigations on significant topics such as parameters identifiability and automatic definition of minimal sets of parameters.

References

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