A Simplified Finite Element-based Algorithm to Study Vibrations of Parallel Kinematic Machines

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SUMMARY. Most of commercial finite element softwares analyze models starting from assigned poses. The proposed method might help designers to integrate in a single package the elastodynamics analysis along with the inverse positioning and orienting problems of PKMs, with ensuing saved time and avoiding annoying manual procedures.

1 INTRODUCTION

Evaluating stiffness helps designers to predict positioning and orienting errors of the manipulators and to individuate resonance frequencies.

A common method to study the stiffness of robotic manipulators consists in creating FEA models [1, 2], but it implies very tedious routines as these models have to be remeshed over and over again when the robot changes its pose inside its workspace.

In the next sections, an algorithm to study the elastodynamics of parallel kinematic machines (PKMs), with arbitrary number of limbs and links, will be proposed. The algorithm is general and based on the matrix structural analysis [3–9].

In Section 2 the outlines of the method are presented: segmentation of a real PKM, nodal arrays, nodal matrices and joint arrays are introduced and explained in detail.

In Section 3 the algorithm is applied to a planar PKM, the 2$\underline{\text{P}}$RRR robot. Results are validated by means of Nastran2005.

Finally, conclusions summarize results of the proposed method and point out its feasible applications.

2 GENERAL DESCRIPTION OF THE ALGORITHM

The proposed algorithm is aimed to find the stiffness matrix and the natural frequencies of a PKM. Only linear analysis and small deformations will be considered. A generic PKM is thought composed of a rigid moving platform (MP) connected to a fixed base, i.e. the base platform (BP), by means of, at least, two or more limbs. Each limb, in turn, is made up of a number of links connected by means of joints. Here, for the sake of simplicity, all links are modeled as beam finite elements. Moreover, the actuated joints are considered clamped in their reference position.

Then, the algorithm proceeds with the following introductory steps:

1. each link with mass and inertia is split into two flexible bodies joined by a fixed connection *F* at its mass center; MP is modeled as a rigid body.
2. the bodies and joints are enumerated[1] from BP to MP: each body is included between two consecutive joints, whereas prismatic *P*, revolute *R*, universal *U*, spherical *S* and fixed *F* joints are considered.

[1] Hereafter, the letter *i* will be associated to bodies, while the letter *j* will be coupled with joints.

3. nodal arrays, thus nodal coordinates, are introduced for each joint. All joints, but the fixed, count for two nodal vectors; the fixed joint counts for one nodal vector.

The generic $6-$dimensional nodal array $\mathbf{u}_i^j = \left[\begin{array}{cccccc} (\mathbf{u}_i^j)_x & (\mathbf{u}_i^j)_y & (\mathbf{u}_i^j)_z & (\mathbf{u}_i^j)_\varphi & (\mathbf{u}_i^j)_\theta & (\mathbf{u}_i^j)_\psi \end{array} \right]^T$ includes six nodal coordinates: three translations and three rotations, of the section of the body *i* located at the joint *j*. Otherwise, $\mathbf{u}_{i+1}^j$ will indicate the nodal array of the body $i+1$ located at the same joint *j*. The bond between two nodal arrays located at the same joint *j* is

$$\mathbf{u}_i^j = \mathbf{u}_{i+1}^j + \mathbf{H}^j \boldsymbol{\theta}^j \tag{1}$$

where $\mathbf{H}^j$ is a $6 \times m(j)$ matrix depending on the dimension $m(j)$ of the joint array $\boldsymbol{\theta}^j$. The matrix $\mathbf{H}^j$ and the joint array $\boldsymbol{\theta}^j$ depend on the nature of the joint: the former containing unit vectors indicating geometric axes; the latter containing joint coordinates, either linear s^j, for translations, or angular ϑ^j, for rotations. The following expressions for principal classes of joints are derived:

Prismatic joint

$$\mathbf{H}^P = \left[\begin{array}{c} \mathbf{w}^j \\ \mathbf{0} \end{array} \right], \quad \theta^P = s^j \tag{2}$$

where $\mathbf{w}^j$ is the unit vector parallel to the direction of translation of the prismatic joint *P*, s^j is the scalar length of translation and **0** is the 3-dimensional zero vector.

Revolute joint

$$\mathbf{H}^R = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{w}^j \end{array} \right], \quad \theta^R = \vartheta^j \tag{3}$$

where $\mathbf{w}^j$ is the unit vector along the axis of the revolute joint *R* and ϑ^j is the rotation angle about the said axis.

In similar manner, other class of joints may be introduced.
For the case of the fixed joint *F* the following expression stands:

$$\mathbf{u}_i^j \equiv \mathbf{u}_{i+1}^j = \overline{\mathbf{u}}^j \tag{4}$$

in which the dependence from the body index *i* has been deleted.
Notice that each column of the matrix $\mathbf{H}$ is the Plücker array of the generic joint [10].

2.1 GENERALIZED STIFFNESS MATRIX

The stiffness of each body, modeled as a flexible beam, is generally expressed by a 12×12 stiffness matrix $\mathbf{K}_i$. Let *L* be the length of the generic body, *A* the area of the orthogonal cross section of the body, *J* the torsional constant, *I* the mass moment of inertia, *E* the Young modulus and *G* the shear modulus, depending on the material, here considered homogeneous and isotropic; then, the stiffness matrix $\mathbf{K}_i$ of the *i*th body, expressed in the local frame of the same body, involved between the two consecutive joints j and $j+1$, is

$$\mathbf{K}_i = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12I}{L^2} & 0 & 0 & 0 & \frac{6I}{L} & 0 & -\frac{12I}{L^2} & 0 & 0 & 0 & \frac{6I}{L} \\ 0 & 0 & \frac{12I}{L^2} & 0 & -\frac{6I}{L} & 0 & 0 & 0 & -\frac{12I}{L^2} & 0 & -\frac{6I}{L} & 0 \\ 0 & 0 & 0 & \frac{GJ}{E} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{E} & 0 & 0 \\ 0 & 0 & -\frac{6I}{L} & 0 & 4I & 0 & 0 & 0 & \frac{6I}{L} & 0 & 4I & 0 \\ 0 & \frac{6I}{L} & 0 & 0 & 0 & 4I & 0 & -\frac{6I}{L} & 0 & 0 & 0 & 4I \\ -A & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12I}{L^2} & 0 & 0 & 0 & -\frac{6I}{L} & 0 & \frac{12I}{L^2} & 0 & 0 & 0 & -\frac{6I}{L} \\ 0 & 0 & -\frac{12I}{L^2} & 0 & \frac{6I}{L} & 0 & 0 & 0 & \frac{12I}{L^2} & 0 & \frac{6I}{L} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{E} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{E} & 0 & 0 \\ 0 & 0 & -\frac{6I}{L} & 0 & 4I & 0 & 0 & 0 & \frac{6I}{L} & 0 & 4I & 0 \\ 0 & \frac{6I}{L} & 0 & 0 & 0 & 4I & 0 & -\frac{6I}{L} & 0 & 0 & 0 & 4I \end{bmatrix} \tag{5}$$

The total deformation energy V_{PKM} of the PKM is the sum of the contributes V_l of each limb; in turn, the latter are partitioned as sum of the contributes $V_{S\mathbf{j}(k-1),\mathbf{j}(k)}$ of each subchain. Thus,

$$V_{PKM} = V_1 + V_2 + \cdots + V_n \tag{6}$$

where n is the number of limbs. For the case of lth-limb the following expression for V_l stands:

$$\begin{aligned} V_l = & \frac{1}{2}\begin{bmatrix} \mathbf{u}_1^1 \\ \mathbf{u}_1^2 \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_1^{1,1} & \mathbf{K}_1^{1,2} \\ \mathbf{K}_1^{2,1} & \mathbf{K}_1^{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^1 \\ \mathbf{u}_1^2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{u}_2^2 \\ \mathbf{u}_2^3 \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_2^{2,2} & \mathbf{K}_2^{2,3} \\ \mathbf{K}_2^{3,2} & \mathbf{K}_2^{3,3} \end{bmatrix} \begin{bmatrix} \mathbf{u}_2^2 \\ \mathbf{u}_2^3 \end{bmatrix} + \cdots \\ & + \frac{1}{2}\begin{bmatrix} \mathbf{u}_i^{j-1} \\ \mathbf{u}_i^j \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_i^{j-1,j-1} & \mathbf{K}_i^{j-1,j} \\ \mathbf{K}_i^{j,j-1} & \mathbf{K}_i^{j,j} \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^{j-1} \\ \mathbf{u}_i^j \end{bmatrix} + \cdots \\ & + \frac{1}{2}\begin{bmatrix} \mathbf{u}_{n_i}^{n_j-1} \\ \mathbf{u}_{n_i}^{n_j} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{n_i}^{n_j-1,n_j-1} & \mathbf{K}_{n_i}^{n_j-1,n_j} \\ \mathbf{K}_{n_i}^{n_j,n_j-1} & \mathbf{K}_{n_i}^{n_j,n_j} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{n_i}^{n_j-1} \\ \mathbf{u}_{n_i}^{n_j} \end{bmatrix} \end{aligned} \tag{7}$$

in which n_i and n_j are the total numbers of bodies and joints inside the l-th limb, respectively.[2] The nodal array $\mathbf{q}_M$ of MP, along with the nodal arrays $\overline{\mathbf{u}}^{\mathbf{j}^l(k)}$, $l = 1, \ldots, n$, that is the nodal arrays of the sections which the lumped masses are attached to, will be the independent generalized coordinates of the system, hereafter denoted with overlined arrays. Hence, we define a global array $\tilde{\mathbf{q}}$ containing all the aforementioned nodal arrays:

$$\tilde{\mathbf{q}} = \begin{bmatrix} \overline{\mathbf{u}}_M{}^T & \overline{\mathbf{u}}^{\mathbf{j}^1(1)T} & \overline{\mathbf{u}}^{\mathbf{j}^1(2)T} & \ldots & \overline{\mathbf{u}}^{\mathbf{j}^1(n_k^1)T} & \ldots & \overline{\mathbf{u}}^{\mathbf{j}^n(n_k^n)T} \end{bmatrix}^T \tag{8}$$

where n_k^l stands for the number of independent joints with lumped masses of the l-th limb. Then, after this general overview, the steps of the algorithm are introduced with direct application to a case study.

[2]Notice that $n_j = n_i + 1$.

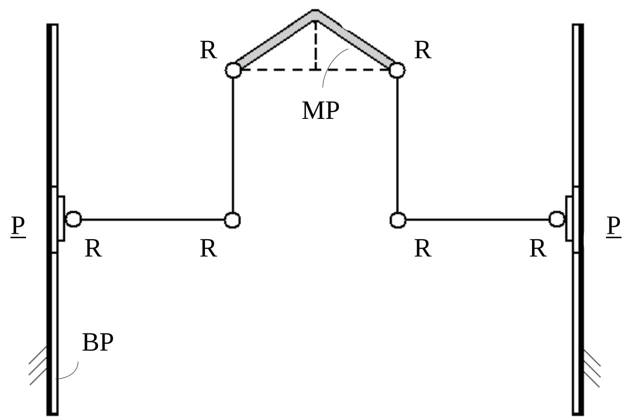

Figure 1: Schematic model of the 2-PRRR manipulator

Parameters of the 2-PRRR			
Notation	Description	Value	Unit
L_1	proximal link's length	1	[m]
L_2	distal link's length	1	[m]
d	$\overline{OM_l}$	$\sqrt{0.5}$	[m]
r	radius of cylindrical links	0.03	[m]
A	link's orthogonal cross section area	$2.827 \cdot 10^{-3}$	[m^2]
I	mass moment of inertia	$6.362 \cdot 10^{-7}$	[kg m^2]
J	torsional constant	$1.272 \cdot 10^{-6}$	[kg m^2]
E	Young modulus	$210 \cdot 10^{9}$	[Pa]
G	shear modulus	$79.545 \cdot 10^{9}$	[Pa]
ρ	density of the material	4222.71	[kg/m^3]
m_p	MP's mass	100	[kg]
$\mathbf{I}_p$	MP's inertia matrix	$0.01\mathbf{1}$	[kg m^2]

Table 1: The 2-PRRR geometric, inertial and structural parameters

3 CASE STUDY: 2PRRR PARALLEL ROBOT

The planar manipulator, investigated in this section, is shown in Fig.1. It consists of a mobile platform and two identical limbs: each limb is of PRRR type and it is composed of two links: the horizontal and the vertical link. All links are modeled as beam elements, while MP is a rigid body. In Tab.1 the geometrical, structural and inertial parameters are reported.

3.1 APPLICATION OF THE ALGORITHM

The algorithm proceeds with the following steps:

1. Considering the reference posture of the manipulator, body-reference frames $O(x_l, y_l, z_l)$ are defined for each link. Then, rotation matrices $\mathbf{R}_0^l$, $l = 1, 2$, are introduced in order to express

all vectors into the inertial frame $O(x,y,z)$:

$$\mathbf{R}_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_0^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Similarly, the body-frame of the vertical frame are expressed into the frames of the horizontal links by means of rotation matrices $\mathbf{R}_{1l}^{2l},\ l = 1,2$, i.e.

$$\mathbf{R}_{11}^{21} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}_{12}^{22} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Moreover, all revolute joints[3] have axes normal to the plane of the manipulator, hence

$$\mathbf{w}_1^1 \equiv \mathbf{w}_1^3 \equiv \mathbf{w}_1^5 \equiv \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \equiv \mathbf{w}_2^1 \equiv \mathbf{w}_2^3 \equiv \mathbf{w}_2^5 \tag{11}$$

where $\mathbf{w}_l^j$, with $l, j = 1,2$ are the unit vectors along the axes of the *j*-th revolute joint of the *l*-th limb. The matrices $\mathbf{H}_l^j$, $l, j = 1,2$, are the same column arrays, i.e.

$$\mathbf{H}_1^1 \equiv \mathbf{H}_1^3 \equiv \mathbf{H}_1^5 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \equiv \mathbf{H}_2^1 \equiv \mathbf{H}_2^3 \equiv \mathbf{H}_1^5 \tag{12}$$

Once the partition of the links is made, the i-th body, $(i = 1,2,3,4)$, enclosed between the joints *j* and *j*+1, has a (12×12) stiffness matrix $\mathbf{K}_i$, as expressed in eq.(5) into the local body-frame. Then, by expressing all stiffness matrices into the inertial frame, matrices $\widetilde{\mathbf{K}}_i$ are obtained as

$$\widetilde{\mathbf{K}}_i = \overline{\mathbf{T}}_i^T \mathbf{K}_i \overline{\mathbf{T}}_i \tag{13}$$

where the rotation matrices $\overline{\mathbf{T}}_i$ need to produce the said transformation:

$$\mathbf{T}_i = \begin{bmatrix} \mathbf{R}_0^{l\,T} \mathbf{R}_l^{1l} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_0^{l\,T} \mathbf{R}_l^{1l} \end{bmatrix} \quad \mathbf{T}_i = \begin{bmatrix} \mathbf{R}_0^{l\,T} \mathbf{R}_l^{1l} \mathbf{R}_{1l}^{2l} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_0^{l\,T} \mathbf{R}_l^{1l} \mathbf{R}_{1l}^{2l} \end{bmatrix} \tag{14}$$

Hereafter, $\mathbf{O}$ and $\mathbf{1}$ will be the 3×3 zero- and identity-matrix, respectively.

2. With reference to Fig.2, we define the array $\mathbf{j}^l$:

$$\mathbf{j}^l = \begin{bmatrix} 2 & 4 & M \end{bmatrix} \tag{15}$$

   containing the numbers of the independent nodal arrays inside the lth-limb. Besides, we settle the nodal arrays according to Fig.2.

3. The joint angles θ^j, $j = 1,3,5$, are calculated for each limb through the minimization of the deformation energy V_l with respect to θ^j, i.e.

$$\frac{dV_l}{d\theta^j} = 0 \tag{16}$$

   Thus, the following expressions are derived

$$\theta^1 = \mathbf{F}^{1,2}\overline{\mathbf{u}}^2; \quad \theta^3 = \mathbf{F}^{3,2}\overline{\mathbf{u}}^2 + \mathbf{F}^{3,3}\mathbf{u}_3^3; \quad \theta^5 = \mathbf{F}^{5,4}\overline{\mathbf{u}}^4 + \mathbf{F}^{5,5}\mathbf{u}_{M_l}; \tag{17}$$

[3] The enumeration of the joints is shown in Fig.2.

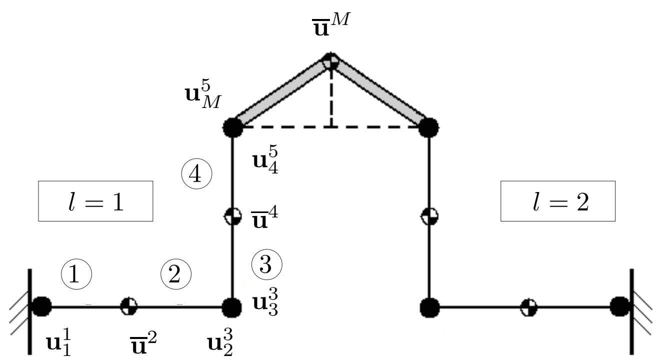

Figure 2: Nodal arrays enumeration

where the matrices $\mathbf{F}^{1,2}$, $\mathbf{F}^{j,j-1}$ and $\mathbf{F}^{j,j}$, $j \ni \mathbf{j}^1$, are

$$\mathbf{F}^{1,2} \equiv -(\mathbf{H}^{1^T}\mathbf{K}_1^{1,1}\mathbf{H}^1)^{-1}\mathbf{H}^{1^T}\mathbf{K}_1^{1,2} \equiv \left[\begin{array}{cccccc} 0 & 3.000 & 0 & 0 & 0 & -0.500 \end{array}\right] \tag{18a}$$

$$\mathbf{F}^{3,2} \equiv -(\mathbf{H}^{3^T}\mathbf{K}_2^{3,3}\mathbf{H}^3)^{-1}\mathbf{H}^{3^T}\mathbf{K}_2^{3,2} \equiv \left[\begin{array}{cccccc} 0 & -3.000 & 0 & 0 & 0 & -0.500 \end{array}\right] \tag{18b}$$

$$\mathbf{F}^{3,3} \equiv -(\mathbf{H}^{3^T}\mathbf{K}_2^{3,3}\mathbf{H}^3)^{-1}\mathbf{H}^{3^T}\mathbf{K}_2^{3,3} \equiv \left[\begin{array}{cccccc} 0 & 3 & 0 & 0 & 0 & -1 \end{array}\right] \tag{18c}$$

$$\mathbf{F}^{5,4} \equiv -(\mathbf{H}^{5^T}\mathbf{K}_4^{5,5}\mathbf{H}^5)^{-1}\mathbf{H}^{5^T}\mathbf{K}_4^{5,4} \equiv \left[\begin{array}{cccccc} 3.000 & 0 & 0 & 0 & 0 & -0.500 \end{array}\right] \tag{18d}$$

$$\mathbf{F}^{5,5} \equiv -(\mathbf{H}^{5^T}\mathbf{K}_4^{5,5}\mathbf{H}^5)^{-1}\mathbf{H}^{5^T}\mathbf{K}_4^{5,5} \equiv \left[\begin{array}{cccccc} -3 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \tag{18e}$$

It should be noted that, hereafter, only the first limb will be analyzed, omitting the same procedure for the second limb.

4. The dependent nodal arrays $\mathbf{u}_{i+1}^j$ are calculated with similar equations: thus, by minimizing the deformation energy V_l with respect to $\mathbf{u}_{i+1}^j$, we have

$$\frac{dV_l}{d\mathbf{u}_{i+1}^j} = \mathbf{0}^T \tag{19}$$

from which we obtain

$$\mathbf{u}_1^1 = \mathbf{G}^{1,2}\overline{\mathbf{u}}^2; \quad \mathbf{u}_3^3 = \mathbf{G}^{3,2}\overline{\mathbf{u}}^2 + \mathbf{G}^{3,4}\overline{\mathbf{u}}^4; \quad \mathbf{u}_5^5 = \mathbf{G}_1\overline{\mathbf{u}}^M; \tag{20}$$

where $\mathbf{G}^{1,2}$, $\mathbf{G}^{j,j-1}$ and $\mathbf{G}^{j,j+1}$ have the following expressions

$$\mathbf{G}^{1,2} = \mathbf{H}^1\mathbf{F}^{1,2} \equiv \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.0 & 0 & 0 & 0 & -0.5 \end{array}\right] \tag{21a}$$

$$\mathbf{G}^{3,2} = -(\mathbf{K}_2^{3,3} + \mathbf{K}_3^{3,3} + \mathbf{K}_2^{3,3}\mathbf{H}^3\mathbf{F}^{3,3})^{-1}(\mathbf{K}_2^{3,2} + \mathbf{K}_2^{3,3}\mathbf{H}^3\mathbf{F}^{2,3}) \equiv$$

$$
\equiv \begin{bmatrix} 0.997 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.003 & 0 & 0 & 0 & -0.001 \\ 0 & 0 & 0.500 & -0.054 & 0.196 & 0 \\ 0 & 0 & -1.261 & 0.295 & -0.495 & 0 \\ 0 & 0 & -1.261 & -0.136 & 0.074 & 0 \\ 2.992 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21b}
$$

$$
\mathbf{G}^{3,4} = -(\mathbf{K}_2^{3,3} + \mathbf{K}_3^{3,3} + \mathbf{K}_2^{3,3}\mathbf{H}^3\mathbf{F}^{3,3})^{-1}(\mathbf{K}_3^{3,4}) \equiv
$$

$$
\equiv \begin{bmatrix} 0.003 & 0 & 0 & 0 & 0 & 0.001 \\ 0 & 0.997 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.500 & -0.196 & 0.054 & 0 \\ 0 & 0 & 1.261 & 0.074 & -0.136 & 0 \\ 0 & 0 & 1.261 & -0.495 & 0.295 & 0 \\ -2.992 & 0 & 0 & 0 & 0 & -0.496 \end{bmatrix} \tag{21c}
$$

moreover, $\mathbf{G}_1$ is defined as

$$
\mathbf{G}_1 = \begin{bmatrix} \mathbf{1} & -\mathbf{Q}\,\mathrm{CPM}(\mathbf{d}_1) \\ \mathbf{O} & \mathbf{1} \end{bmatrix} \equiv \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 1.0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 1.0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix} \tag{22a}
$$

where $\mathbf{Q}$ and CPM are the rotation matrix and the *cross product matrix* of the position vector $\mathbf{d}_l$ [10].

Thus, replacing eq.(20) into (17), we obtain the arrays $\boldsymbol{\theta}^j$ in terms of the independents nodal arrays, i.e.

$$
\theta^1 = \mathbf{Y}^{1,2}\overline{\mathbf{u}}^2; \quad \theta^3 = \mathbf{Y}^{3,2}\overline{\mathbf{u}}^2 + \mathbf{Y}^{3,4}\overline{\mathbf{u}}^4; \quad \boldsymbol{\theta}^5 = \mathbf{Y}^{5,4}\overline{\mathbf{u}}^4 + \mathbf{Y}^{5,5}\overline{\mathbf{u}}_M; \tag{23}
$$

where

$$
\mathbf{Y}^{1,2} = \begin{bmatrix} 0 & 3.000 & 0 & 0 & 0 & -0.500 \end{bmatrix} \tag{24a}
$$

$$
\mathbf{Y}^{3,2} = \begin{bmatrix} -2.992 & -2.992 & 0 & 0 & 0 & -0.496 \end{bmatrix} \tag{24b}
$$

$$
\mathbf{Y}^{3,4} = \begin{bmatrix} 2.992 & 2.992 & 0 & 0 & 0 & 0.496 \end{bmatrix} \tag{24c}
$$

$$
\mathbf{Y}^{5,4} = \begin{bmatrix} 3.000 & 0 & 0 & 0 & 0 & -0.500 \end{bmatrix} \tag{24d}
$$

$$
\mathbf{Y}^{5,5} = \begin{bmatrix} -3.000 & 0 & 0 & 0 & 0 & -2.500 \end{bmatrix} \tag{24e}
$$

5. The dependents nodal arrays $\mathbf{u}_i^j$ are calculated by substituting eq.(20) and (23) into eq.(1), thus, obtaining

$$
\mathbf{u}_2^3 = \mathbf{X}^{3,2}\overline{\mathbf{u}}^2 + \mathbf{X}^{3,4}\overline{\mathbf{u}}^4, \quad \mathbf{u}_4^5 = \mathbf{X}^{5,4}\overline{\mathbf{u}}^4 + \mathbf{X}^{5,M}\overline{\mathbf{u}}^M \tag{25}
$$

where

$$
\mathbf{X}^{3,2} = \begin{bmatrix} 0.997 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.003 & 0 & 0 & 0 & 0.001 \\ 0 & 0 & 0.500 & -0.054 & -0.196 & 0 \\ 0 & 0 & -1.261 & 0.295 & 0.495 & 0 \\ 0 & 0 & 1.261 & 0.136 & 0.074 & 0 \\ 0 & -2.992 & 0 & 0 & 0 & -0.496 \end{bmatrix} \tag{26a}
$$

$$\mathbf{X}^{3,4} = \begin{bmatrix} 0.003 & 0 & 0 & 0 & 0 & 0.001 \\ 0 & 0.997 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.500 & -0.196 & -0.054 & 0 \\ 0 & 0 & 1.261 & 0.074 & 0.136 & 0 \\ 0 & 0 & -1.261 & 0.495 & 0.295 & 0 \\ 0 & 2.992 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26b}$$

$$\mathbf{X}^{5,4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3.0 & 0 & 0 & 0 & 0 & -0.5 \end{bmatrix} \tag{26c}$$

$$\mathbf{X}^{5,M} = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 1.0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 1.0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ -3.0 & 0 & 0 & 0 & 0 & -1.5 \end{bmatrix} \tag{26d}$$

6. The deformation energy V_1 is readily obtained in terms of the independent nodal arrays of the first limb $\mathbf{q}^1$, i.e.

$$V_1 = \frac{1}{2}\mathbf{q}^{1^T}\mathbf{K}_1\mathbf{q}^1 \tag{27}$$

where $\mathbf{K}_1$ is the generalized stiffness matrix of the first limb, i.e.

$$\mathbf{K}_1 = \mathbf{K}_{1,2} + \mathbf{K}_{2,4} + \mathbf{K}_{4,M} \tag{28}$$

Then, the final expression of the stiffness matrix $\mathbf{K}_{2RRR}$ is obtained upon assembling the two terms V_1 and V_2 coming from the two limbs:

$$V_{2RRR} = V_1 + V_2 \equiv \frac{1}{2}\tilde{\mathbf{q}}^T\mathbf{K}_{2RRR}\tilde{\mathbf{q}}$$

Where $\tilde{\mathbf{q}}$ is the global independent array, which is defined as $\tilde{\mathbf{q}} \equiv \begin{bmatrix} \overline{\mathbf{u}}^M & \overline{\mathbf{u}}^{2^1} & \overline{\mathbf{u}}^{4^1} & \overline{\mathbf{u}}^{2^2} & \overline{\mathbf{u}}^{4^2} \end{bmatrix}$ We only show a part of the (30×30) matrix so obtained:

$$\mathbf{K}_{2RRR} = 10^9 \begin{bmatrix} 0.006 & 0 & 0 & \dots & 0 & 0 & 0.002 \\ 0 & 2.375 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0.0257 & \dots & -0.003 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & -0.003 & \dots & 0.001 & 0.000 & 0 \\ 0 & 0 & 0 & \dots & 0.0001 & 0.000 & 0 \\ 0.002 & 0 & 0 & \dots & 0 & 0 & 0.002 \end{bmatrix} \tag{29}$$

7. Introducing the generalized inertia matrix $\mathbf{M}_{2RRR}$ of the PKM, not reported here for brevity, the natural frequencies of the $2-\underline{P}RRR$ can be calculated by the dynamics equations:

$$\mathbf{M}_{2RRR}\ddot{\tilde{\mathbf{q}}} + \mathbf{K}_{2RRR}\tilde{\mathbf{q}} = \mathbf{0} \tag{30}$$

where the latter have been linearized at an equilibrium configuration.

No.	A (Hz)	B (Hz)	rel.-err. [%]	No.	A (Hz)	B (Hz)	rel.-err. [%]
1	1,16E-05	8,75E-06	-	16	2,03E+02	2,02E+02	0,401
2	4,26E-05	3,72E-07	-	17	2,03E+02	2,03E+02	0,401
3	4,26E-05	7,04E-06	-	18	2,23E+02	2,22E+02	0,400
4	3,91E+00	3,91E+00	0,074	19	2,31E+02	2,28E+02	1,349
5	3,84E+01	3,84E+01	0,155	20	2,31E+02	2,28E+02	1,346
6	4,63E+01	4,62E+01	0,192	21	1,21E+03	1,21E+03	0,363
7	8,19E+01	8,18E+01	0,066	22	1,22E+03	1,22E+03	0,322
8	1,05E+02	1,05E+02	0,139	23	1,32E+03	1,31E+03	0,439
9	1,16E+02	1,16E+02	0,397	24	1,33E+03	1,32E+03	0,417
10	1,18E+02	1,18E+02	0,397	25	1,59E+03	1,59E+03	0,007
11	1,19E+02	1,19E+02	0,397	26	1,59E+03	1,59E+03	0,007
12	1,38E+02	1,37E+02	0,397	27	1,77E+03	1,77E+03	0,007
13	1,77E+02	1,76E+02	0,550	28	4,17E+03	4,11E+03	1,494
14	1,81E+02	1,81E+02	0,521	29	6,16E+03	6,07E+03	1,537
15	2,02E+02	2,01E+02	0,401	30	3,89E+04	3,89E+04	0,009

Table 2: Natural frequencies of the 2-PRRR in the *home-posture*; *A* indicates the outputs of the algorithm; *B* the outputs of Nastran2005

3.2 RESULTS

The segmentation of the $2-\underline{P}RRR$ manipulator has led to thirty independent nodal coordinates. Hence, thirty natural frequencies, shown in Tab.2, have been obtained at the reference posture and compared to output results of $Nastran$®. The $2-\underline{P}RRR$ is a planar mechanism that can undergo rigid motions of MP along the x- and y-axes, along with a rotation about the z-axis, normal to the plane of the mechanism. Thus, natural frequencies associated with the said degrees of freedom are zero. The relative error, shown in Fig.3, reveals good accuracy of the method. Future works will be aimed to: extend the method to hybrid robots with internal loops, generalize the method to any partition of links, study singularity loci in space.

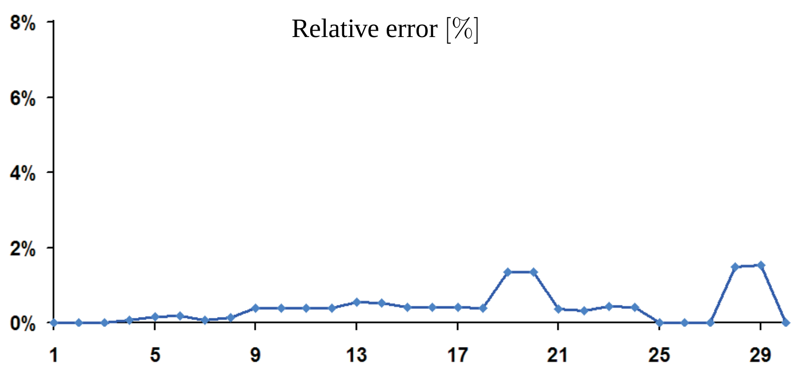

Figure 3: Relative error

4 CONCLUSIONS

The study of the elastodynamics of PKMs by means of a systematic algorithm based on the stiffness matrix theory was introduced. The proposed method can be easily adapted to a large class of planar and spatial PKMs and easily extended by introducing more complex finite elements. The method was applied to a planar parallel robot of 2PRRR type. The generalized stiffness matrix was first derived and natural frequencies were calculated and compared to the software Nastran2005 to validate the model. Results showed good accuracy, especially for lower frequencies.

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