An adaptive fast multipole boundary element method for the Helmholtz equation

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SUMMARY. The present paper intends to couple the Fast Multipole Method (FMM) with the Boundary Element Method (BEM) in 2D acoustic problems. The evaluation of the integrals involved in the governing Boundary Integral Equations (BIEs) is fasten by the FMM contribution. The multipole expansion and some suitable moment translations make the procedure much faster if compared to the conventional approach. The generalised minimal residual iterative solver (GMRES) is adopted to improve the overall computational efficiency. A simple numerical example is shown to demonstrate the reliability of the method.

1 INTRODUCTION

The BEM has been used to solve interior/exterior acoustic problems for many years because of its boundary only discretisation and automatically satisfaction of the radiation condition at infinity. The main drawback is related to the final system of equations which results to have dense, non-symmetrical and sometimes ill-conditioned coefficient matrices. Solving the system of equations becomes prohibitively expensive when applied at large-scale engineering problems. In fact, the computation of the coefficients of the matrices governing the discrete problem require $O(N^2)$ operations and another $O(N^3)$ operations are necessary to solve the system using any direct solver (let N be the number of equations).

In 1983 Rokhlin [1] proposed an algorithm for rapid solution of classical boundary value problems for the Laplace equation based on iteratively solving integral equations of potential theory. The CPU time requirement obtained was proportional to N. The starting point was the harmonic expansion of the kernel. The algorithm appeared to be the most efficient of the at that time available tools for the solution of large scale boundary value problems whenever the solution needed to be evaluated at a limited number of points. The procedure was then extended, a few years later, to two dimensional acoustic scattering in [2] where the author described a similar procedure capable to reduce the CPU time requirements of the algorithm to $N^{4/3}$. In both papers no connection with the BEM was introduced.

It took almost ten years for scientific community to realise the potential capability of coupling the FMM with the BEM. A comprehensive review of the fundamentals of FMM and FMM accelerated Boundary Integral Equation Method (BIEM) with reference to the Laplace and Helmholtz equations is surveyed in [3]. With conventional BIEM it is not possible to solve beyond several thousands of unknowns with a desktop computer. Actually, methods of solution of problems of the size of more than 10^8 unknowns (which roughly correspond to 10^6 unknowns in the BEM context) are investigated in FEM with massively parallel computers. With fast multipole accelerated BIEM, problems of the size of 10^6 unknowns can be handled even in desktop computers. However, the use

of the FMM has increased the complexity in implementations of the BEM: the structure of the code changes completely and the pre-processor stage becomes more important than in the conventional approach. An interesting introduction to the Fast Multipole Boundary Element Method (FMBEM) for potential problems is presented in [4]: the structure of a FMBEM program along with the details of the method with reference to the Laplace equation is presented.

An adaptive FMBEM for 3D acoustic wave problems is investigated in [5] where the Burton-Miller formulation is applied to overcome the non-uniqueness difficulties. The adaptive approach is demonstrated to be several times faster than the non-adaptive FMBEM while maintaining the accuracy of the BEM.

To the authors' knowledge [6] represents the unique application of the FMBEM to 2D acoustic problems. The FMM is used to accelerate the construction of the influence matrix in the BEM. The approach is of non-adaptive type and the number of floating-point operations is reduced from $O(N^2)$ to $O(N\,log^a N)$ where a is a small constant independent on N.

This paper intends to present a FMBEM for two-dimensional acoustics aimed at noise control. The iterative solver GMRES is adopted to improve the overall computational efficiency. A numerical example is shown to demonstrate the accuracy and potentials for solving large-scale problems. After this Introduction, the integral equations which govern the 2D acoustic problem are presented along with the main relations of the FMM. Afterward, the algorithm and the integration schemes are detailed. Finally, a numerical example is investigated in order to measure the reliability of the procedure.

2 THE FAST MULTIPOLE BOUNDARY INTEGRAL RELATIONS

The propagation of time-harmonic acoustic waves in a homogeneous isotropic acoustic medium (either finite or infinite) is described by the Helmholtz equation:

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0 \tag{1}$$

under the boundary conditions:

$$p(\mathbf{x}) = \overline{p}(\mathbf{x}) \qquad \mathbf{x} \in \Gamma_1 \tag{2a}$$

$$q(\mathbf{x}) = p(\mathbf{x})_{,n} = \overline{q}(\mathbf{x}) \qquad \mathbf{x} \in \Gamma_2 \tag{2b}$$

where p is the acoustic pressure, $k = \omega/c$ with $\omega =$ angular frequency and $c =$ sound velocity, comma indicates partial derivative, $\Gamma_1 \cup \Gamma_2 = \Gamma$, Γ is the boundary of the domain Ω under analysis, $n = n(\mathbf{x})$ is the outward normal to the boundary in $\mathbf{x}$, q is the flux and the barred quantities indicate given values.

The boundary value problem described by the above equations can be transformed into the following integral representation [7]-[8]:

$$c(\boldsymbol{\xi})p(\boldsymbol{\xi}) + \int_\Gamma q^*(\boldsymbol{\xi},\mathbf{x})p(\mathbf{x})d\Gamma(\mathbf{x}) - \int_\Gamma p^*(\boldsymbol{\xi},\mathbf{x})q(\mathbf{x})d\Gamma(\mathbf{x}) = 0 \tag{3}$$

where $c(\boldsymbol{\xi})$ occurs in the limiting process from the internal point to the boundary point, being equal to 0.5 if the tangent line to the boundary at $\boldsymbol{\xi}$ is continuous. The fundamental solutions p^* and q^* are given by:

$$p^*(\boldsymbol{\xi},\mathbf{x}) = \frac{i}{4}H_0^{(1)}(kr) \tag{4a}$$

$$q^*(\boldsymbol{\xi},\mathbf{x}) = -\frac{ik}{4}H_1^{(1)}(kr)r_{,n} \tag{4b}$$

where $r = \| \mathbf{x} - \boldsymbol{\xi} \|$ is the distance between the collocation point $\boldsymbol{\xi}$ and the field point $\mathbf{x}$, $H_0^{(1)}$ and $H_1^{(1)}$ are the Hankel function of the first kind, 0^{th} and 1^{st} order respectively.

The conventional BEM numerical procedure is based on two steps: first, the discretisation of the boundary Γ, second, the collocation of the eq (3) in each node in order to build a final (square) system of equations in the unknowns either p or q on the boundary. In the present contribution constant elements are adopted: with such a choice some integrals can be performed analytically. The discretised equation collocated at node $\boldsymbol{\xi}_i$ can be written as:

$$c(\boldsymbol{\xi}_i)p(\boldsymbol{\xi}_i) + \sum_{j=1}^{N} p_j \int_{\Gamma_j} q^*(\boldsymbol{\xi}_i, \mathbf{x}) d\Gamma(\mathbf{x}) = \sum_{j=1}^{N} q_j \int_{\Gamma_j} p^*(\boldsymbol{\xi}_i, \mathbf{x}) d\Gamma(\mathbf{x}) \tag{5}$$

The procedure requires the evaluation of either the integral of p^* or the integral of q^* on each boundary element.

For convenience, the complex notation is introduced, i.e. the collocation and field points are replaced by their complex representation:

$$\boldsymbol{\xi} = z_0 = \xi_1 + i\,\xi_2 \tag{6a}$$

$$\mathbf{x} = z = x_1 + i\,x_2 \tag{6b}$$

with $i = \sqrt{-1}$. With such an assumption it is simple to show that the fundamental solutions in $\boldsymbol{\xi}, \mathbf{x}$ coincide with their expression in complex notation:

$$p^*(\boldsymbol{\xi}, \mathbf{x}) = p^*(z_0, z) \tag{7a}$$

$$q^*(\boldsymbol{\xi}, \mathbf{x}) = q^*(z_0, z) \tag{7b}$$

The FMM relations intervene in the evaluation of integrals involved in the eq (5). The multipole expansion is the key point in reducing the CPU time which is necessary to evaluate each integral. If $F(z_0, z)f$ indicates either $p^*(z_0, z)q$ or $q^*(z_0, z)p$, the following *local expansion* can be obtained:

$$\int_{\Gamma_j} F(z_0, z)\, f d\Gamma(z) = \frac{i}{4} \sum_{p=-\infty}^{\infty} (-1)^p L_{-p}(z_L) I_p(z_0 - z_L) \tag{8}$$

where:

$$I_p(z) = (-i)^p J_p(kr) e^{ip\theta} \tag{9}$$

r, θ are the polar coordinates of z and J_p stands for the Bessel function of the p^{th} order.

The coefficients L_{-p} are given by the following *M2L translation*:

$$L_l(z_L) = \sum_{k=-\infty}^{\infty} O_{k+l}(z_L - z_C) P_{-k}(z_C) \tag{10}$$

where $\| z_0 - z_L \| < \| z_C - z_L \|$ must be satisfied and:

$$O_m(z) = i^m H_m^{(1)}(kr) e^{im\theta} \tag{11}$$

The term $P_k(z_C)$ is called *moment about* z_C, it is independent from the collocation point z_0 and it only needs to be computed once. Its expression is given as follows:

$$P_k(z_C) = M_k(z_C) = q \int_{\Gamma_j} I_k(z - z_C) d\Gamma(z) \tag{12a}$$

$$P_k(z_C) = N_k(z_C) = p \int_{\Gamma_j} \frac{\partial I_k(z - z_C)}{\partial n} d\Gamma(z) \tag{12b}$$

The point z_C is assumed to be located close to Γ_j so that $\max_{z \in \Gamma_i} \parallel z - z_C \parallel < \parallel z_0 - z_C \parallel$ holds. The series expansion eq (10) involving P_k can be truncated to $nexp$ terms with a good approximation if $nexp$ is set larger than kr_{max} (see [2] for details).

If the point z_C is moved to a new location $z_{C'}$, the following ***M2M translation*** is obtained:

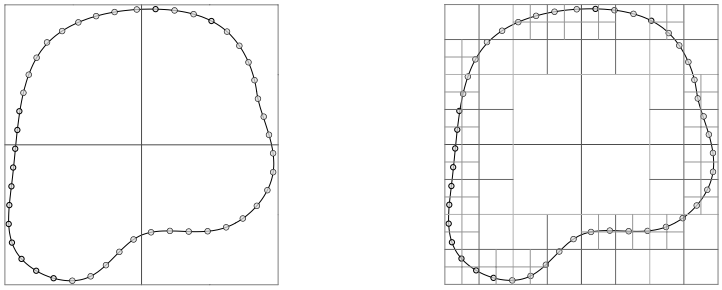
$$P_p(z_{C'}) = \sum_{m=-\infty}^{\infty} I_{p-m}(z_C - z_{C'}) P_m(z_C) \tag{13}$$

Analogously, if the point for local expansion is moved from z_L to $z_{L'}$, the following ***L2L expansion*** is obtained:

$$L_l(z_L) = \sum_{k=-\infty}^{\infty} I_{l-k}(z_{L'} - z_L) L_k(z_{L'}) \tag{14}$$

3 THE ALGORITHM

The adaptive procedure starts from a square containing the entire boundary and then repeatedly divides it and the successive sub-cells into four sub-squares until a fixed maximum number of boundary elements is contained in each cell. In Fig. 1 the subdivision process up to the last level is depicted with the allowed maximum number of elements set to one. The last cells of the division process (in the figure the cells containing one element) are also called *leaves*.

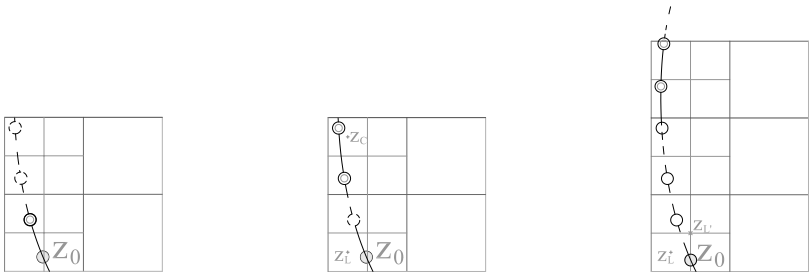


(a) (b)

Figure 1: Square division up to level 1 (a) and up to level 4 (b)

For a given collocation node z_0, the integral over the entire boundary is determined in different way on the basis of the distance $\parallel z - z_0 \parallel$. If the integration element is *close* (where *close* means

in one of the cells surrounding it, see Fig. 2a) to the collocation node, the integral contribution is determined directly as in the conventional BEM. In the present contribution such integrals are evaluated analytically and presented in the successive section. On the other hand, if the position of the integration element with respect to the collocation node is as depicted in Fig. 2b, i.e. the integration element belongs to the collocation node's parents, the eq (8) is applied via the *M2L translation*. Finally, the contribution from far cells (depicted in Fig. 2c) is obtained by the local expansion via the *L2L translation*.



(a) (b) (c)

Figure 2: Direct (a), M2L (b) and L2L (c) integrations

The elements of the matrix A of the final system of equations $A\mathbf{x} = \mathbf{b}$ are not stored. The product $A\mathbf{x}_i$ is evaluated iteratively until the solution converges within a given tolerance. The GMRES method is adopted. It was first proposed by [9] in order to solve large, sparse and nonsymmetric linear systems. The routine implemented in the paper allows the use of different preconditioning to accelerate the solution process.

4 THE ANALYTICAL INTEGRATION

The FMBEM procedure requires the evaluation of some integrals. Some of them are involved in the direct integration whereas two integrals are necessary to evaluate the moments Eq. (12). The term H_{ii} involving the fundamental solution q^* when the source node belongs to the integration element is zero as constant elements are adopted (i.e. $r_{,n} = 0$).

The diagonal term G_{ii} can be determined analytically by using the procedure presented in [10]:

$$\begin{aligned}\int_{\Gamma_{AB}} p^*(\boldsymbol{\xi}, \mathbf{x}) d\Gamma(\mathbf{x}) &= \frac{i}{4} l_{AB} \left[H_0^{(1)}(k\frac{l_{AB}}{2}) \right. \\ &\left. + \frac{\pi}{2} \left(\hat{H}_0(k\frac{l_{AB}}{2}) H_1^{(1)}(k\frac{l_{AB}}{2}) - \hat{H}_1(k\frac{l_{AB}}{2}) H_0^{(1)}(k\frac{l_{AB}}{2}) \right) \right] \quad (15)\end{aligned}$$

where $\hat{H}_\nu(z)$ denotes the Struve function of order ν. A similar expression can be obtained when the source point does not belong to the integration element but it lies on the line AB.

The off-diagonal terms can be analytically evaluated only in the case $kr \leq 2$. The case $kr > 2$ can only be solved numerically. As a matter of fact, in the FMBEM context, a boundary discretisation which satisfies $6 - 8$ elements for wavelength would never require such a numerical integration.

The expression of G_{ij} and H_{ij} for $kr \leq 2$ are given by:

$$\int_{\Gamma_j} p^*(\boldsymbol{\xi}, \mathbf{x}) d\Gamma(\mathbf{x}) = -\frac{1}{2\pi} \sum_{i=1}^{7} \left[\frac{A_i}{2} P_{1,i} - B_i P_{2,i} \right]_{\boldsymbol{\xi}_A}^{\boldsymbol{\xi}_B} \tag{16a}$$

$$\int_{\Gamma_j} q^*(\boldsymbol{\xi}, \mathbf{x}) d\Gamma(\mathbf{x}) = \frac{ik}{2\pi} \left[\eta \sum_{i=1}^{7} \left(\frac{D_i}{2} P_{1,i} + E_i P_{2,i-1} \right) \right]_{\boldsymbol{\xi}_A}^{\boldsymbol{\xi}_B} \tag{16b}$$

where the terms involved in the above expressions are reported in [11].

The integral expressing the *moment* is also evaluated analytically. No contributions are available in the scientific literature. The M_k term is analytically determined by a new procedure developed by the authors which starts from the Graf addition theorem (see for instance [12]):

$$H_\nu^{(1)}(\tilde{\omega}) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(Z) J_m(z) e^{im\phi} \tag{17}$$

where $\tilde{\omega}$, z and Z form a triangle in which ϕ is the angle between z and Z and ψ is the angle between $\tilde{\omega}$ and Z (see Fig. 3).

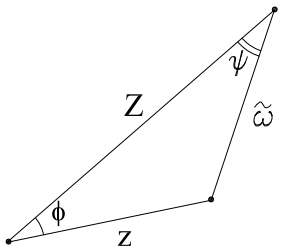


Figure 3: Graf's theorem.

By the application of the above theorem it is possible to transform the integral involved in M_k into the sum of integrals involving the Bessel functions $J_m(t)$ only. Such integrals can be evaluated analytically by well-known expressions.

5 NUMERICAL RESULTS

In order to demonstrate the accuracy of the proposed procedure, a numerical example, for which the analytical solution is available, is presented. The results refer to the wave propagation inside a cylinder of radius $R = 1$ for different values of the wave number. The analytical solution is available both for pressure and for flux given on the boundary. The number of boundary elements is always fixed in order to have $6 - 10$ elements per wavelength. Table 1 presents the comparison between analytical and numerical values on the boundary.

For the same cylinder a comparison of the CPU time required by the entire computation versus the number of degrees of freedom (DOFs) is depicted in Fig. 4. It is well clear how effective

		0.5	1.0	2.0	5.0
$\overline{p}_\Gamma = 1$	Analytic	0.129	0.575	5.15	9.22
	FMBEM	0.129	0.578	5.12	9.1
$\overline{q}_\Gamma = 1$	Analytic	7.75	1.74	0.194	0.108
	FMBEM	7.75	1.73	0.194	0.106

Table 1: Comparison between analytical and FMBEM solution on the boundary.

the proposed procedure is with respect to the conventional approach. It must be underlined that no preconditioner was used in the above calculations. Therefore, further improvements in the CPU time can be obtained if a suitable preconditioner is adopted.

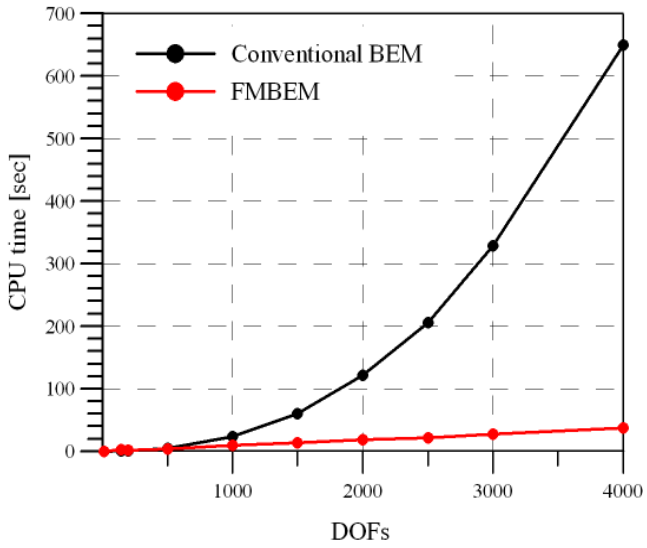


Figure 4: CPU time versus DOFs.

6 CONCLUSIONS

In this paper an adaptive fast multipole boundary element method for solving 2D acoustic wave problems is presented. The procedure allows a great save in the computational time. Numerical analyses are still in progress.

7 ACKNOWLEDGEMENTS

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