

# Parameter-dependent normal forms for bifurcation equations of dynamical systems

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**SUMMARY.** Normal form for bifurcation equations of finite-dimensional dynamical systems, are studied. First, the general theory for no-parameter systems is restated in a form amenable to further developments. Then, the presence of bifurcation parameters in the equations is accounted for. The drawbacks affecting two classical methods usually employed in literature are commented. A new method, based on a perturbation expansion, which is free from these difficulties, is proposed, and illustrated by a worked example.

## 1 INTRODUCTION

The main goal of the local bifurcation analysis from equilibrium points consists in classifying the long-time behavior of a dynamical system in the parameter space. The task requires obtaining bifurcation equations capturing the asymptotic essential dynamics. The bifurcation equations are usually built-up via the Center Manifold Theory [1,2], that permits to reduce the dimension of the original system to that of the critical (generalized) eigenspace of the associated linear system. These equations describe the asymptotic motion of the nonlinear system on an invariant manifold which is tangent to the critical subspace. Once the bifurcation equations have been obtained, and in order to make their analysis easier, they are conveniently transformed according to the Normal Form Theory [1,2].

The normal form of a nonlinear ordinary differential equation is the simplest form it can assume under a suitable change of coordinates. Of course, the best occurrence would be to transform the nonlinear equation in a linear one. This is indeed possible whenever the eigenvalues of the Jacobian matrix are non-resonant, i.e. none of them can be expressed as a linear combination, with positive integers, of the eigenvalues. However, this favorable circumstance never occurs in bifurcation analysis, since the Jacobian matrix admits zero and/or purely imaginary eigenvalue, for which resonance always exists; therefore, bifurcation equations cannot be linearized. In spite of this, the equations can be strongly simplified, by removing all the non-resonant terms and leaving only the unmovable resonant terms.

The procedure of elimination of the non-resonant terms is often described in literature through successive changes of coordinates able to (partially) remove, in turn, quadratic, cubic and higher-order terms [1-3]. A nicer illustration of the procedure is presented in [4], in the framework of a perturbation method, where a unique change of coordinates is introduced, and linear perturbation equations are solved in chain. The previous treatments, however, are devoted to systems in which no parameters appear in the equation. In contrast, the main interest of bifurcation analysis just relies on equations in which parameters do appear, able to span the neighborhood of the bifurcation point. To account for parameters, it is suggested in literature to consider them as

dummy state-variables, constant in time, similarly to that done in the Center Manifold Theory [1,3]. The procedure, although simple in principle, leads to cumbersome calculations, since it increases the dimension of the dynamical system.

An alternative algorithm is suggested in [2], and illustrated with reference to a simple Hopf bifurcation. There, the parameters are considered as fixed quantities, included in the coefficients, and therefore they do not increase the dimension of the system. As a drawback, the procedure leads to linear equations which are *quasi-singular* close to bifurcation, so that resonances become *quasi-resonances*. These latter, if not properly tackled, would entail the occurrence of small divisors both in the coordinates transformation and in the normal form. The problem can be easily overcome in the case in which the Jacobian matrix is diagonalizable, namely in the example illustrated in [2], but, in contrast, is not trivial in the case in which the Jacobian matrix contains a Jordan block.

In this paper, parameter-dependent dynamical systems are considered, close to bifurcation points, at which their Jacobian matrix, admitting non-hyperbolic eigenvalues, is either diagonalizable or non-diagonalizable. A perturbation algorithm is developed to derive parameter-dependent normal forms, without extending the state space, but, in contrast, by keeping unaltered the dimension of the original system. In Sect 2, the background relevant to the Normal Form Theory, concerning the parameter-independent case, is supplied. In Sect 3 a new algorithm accounting for parameter-dependent system is detailed, and an example is worked out in Sect 4.

## 2 PARAMETER-INDEPENDENT NORMAL FORMS

Let us first consider a dynamical system independent of parameters. Its equation of motion, Taylor-expanded around the equilibrium point, reads:

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{f}(\mathbf{x}) \quad (1)$$

where  $\mathbf{f}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^2)$ ,  $\mathbf{x} \in \mathbb{R}^N$  or  $\mathbb{C}^N$ , and  $\mathbf{J}$  is the Jacobian matrix recast in Jordan form. Our goal is to transform Eq (1) into the *normal form*:

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{y}) \quad (2)$$

where  $\mathbf{g}(\mathbf{y}) = \mathcal{O}(|\mathbf{y}|^2)$  is the simplest possible. To this end, we introduce a *near-identity transformation* of coordinates  $\mathbf{x} \rightarrow \mathbf{y}$ , in which  $\mathbf{y}$  differs from  $\mathbf{x}$  only by higher order-quantities:

$$\mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y}) \quad (3)$$

with  $\mathbf{h}(\mathbf{y}) = \mathcal{O}(|\mathbf{y}|^2)$  an unknown function. With Eqs (2) and (3), since  $\dot{\mathbf{x}} = [\mathbf{I} + \mathbf{h}_y(\mathbf{y})]\dot{\mathbf{y}}$ , where  $\mathbf{I}$  the  $N \times N$  identity matrix and  $\mathbf{h}_y(\mathbf{y})$  the Jacobian matrix of  $\mathbf{h}(\mathbf{y})$ , Eq (1) modifies into:

$$\mathbf{h}_y(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \mathbf{h}(\mathbf{y})) - [\mathbf{I} + \mathbf{h}_y(\mathbf{y})]\mathbf{g}(\mathbf{y}) \quad (4)$$

This is a *differential equation for the unknown  $\mathbf{h}(\mathbf{y})$ , for any properly chosen  $\mathbf{g}(\mathbf{y})$* . Since it cannot be solved exactly, we will solve it by series expansions [4]. We assume:

$$\begin{pmatrix} \mathbf{f}(\mathbf{y}) \\ \mathbf{g}(\mathbf{y}) \\ \mathbf{h}(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \mathbf{f}_2(\mathbf{y}) \\ \mathbf{g}_2(\mathbf{y}) \\ \mathbf{h}_2(\mathbf{y}) \end{pmatrix} + \begin{pmatrix} \mathbf{f}_3(\mathbf{y}) \\ \mathbf{g}_3(\mathbf{y}) \\ \mathbf{h}_3(\mathbf{y}) \end{pmatrix} + \dots \quad (5)$$

where  $\mathbf{f}_k(\mathbf{y})$ ,  $\mathbf{g}_k(\mathbf{y})$ ,  $\mathbf{h}_k(\mathbf{y})$  are homogeneous polynomials of degree  $k$  in  $\mathbf{y}$ . By substituting the series expansions (5) in (4), and equating separately to zero terms of the same degree, we obtain the following chain of equations:

$$\begin{aligned} \mathbf{h}_{2,y}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}_2(\mathbf{y}) &= \mathbf{f}_2(\mathbf{y}) - \mathbf{g}_2(\mathbf{y}) \\ \mathbf{h}_{3,y}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}_3(\mathbf{y}) &= \mathbf{f}_3(\mathbf{y}) + \mathbf{f}_{2,y}(\mathbf{y})\mathbf{h}_2(\mathbf{y}) - \mathbf{h}_{2,y}(\mathbf{y})\mathbf{g}_2(\mathbf{y}) - \mathbf{g}_3(\mathbf{y}) \\ \dots\dots\dots \end{aligned} \quad (6)$$

which are *linear in*  $\mathbf{h}_k(\mathbf{y})$ . They can be solved recursively, to furnish  $\mathbf{h}_2(\mathbf{y})$ ,  $\mathbf{g}_2(\mathbf{y})$  at the order-2,  $\mathbf{h}_3(\mathbf{y})$ ,  $\mathbf{g}_3(\mathbf{y})$  at the order-3, and so on. The generic equation (6) can be rewritten as:

$$\mathcal{L}\mathbf{h}_k(\mathbf{y}) = \hat{\mathbf{f}}_k(\mathbf{y}) - \mathbf{g}_k(\mathbf{y}) \quad k = 2, 3, \dots \quad (7)$$

in which  $\mathcal{L}$  is a linear differential operator, whose action is  $\mathcal{L}\mathbf{h}_k(\mathbf{y}) := \mathbf{h}_{k,y}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}_k(\mathbf{y})$ ; moreover  $\hat{\mathbf{f}}_k(\mathbf{y})$  is a *known-term*, since  $\hat{\mathbf{f}}_2(\mathbf{y}) \equiv \mathbf{f}_2(\mathbf{y})$  and  $\hat{\mathbf{f}}_k(\mathbf{y})$  ( $k > 2$ ) is completely determined by the lower-order solutions.

### 2.1 Solving the generic perturbation equation

The three homogeneous polynomials of degree  $k$  appearing in Eq (7), are:

$$\hat{\mathbf{f}}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \alpha_{km} \mathbf{p}_{km}(\mathbf{y}), \quad \mathbf{g}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \beta_{km} \mathbf{p}_{km}(\mathbf{y}), \quad \mathbf{h}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \gamma_{km} \mathbf{p}_{km}(\mathbf{y}) \quad (8)$$

where  $\alpha_{km}, \beta_{km}, \gamma_{km}$  are constants, and  $\{\mathbf{p}_{k1}(\mathbf{y}), \mathbf{p}_{k2}(\mathbf{y}), \dots, \mathbf{p}_{kM_k}(\mathbf{y})\}$  is a basis of  $M_k$  linearly independent vector-valued monomial of degree  $k$  in the  $N$  coordinates  $y_j$ . By substituting Eqs (8) in Eq (7), and equating separately to zero the coefficients of the same monomials, we obtain a linear system of equations, of type:

$$\mathbf{L}_k \boldsymbol{\gamma}_k = \boldsymbol{\alpha}_k - \boldsymbol{\beta}_k \quad (9)$$

in which  $\mathbf{L}_k$  is a  $M_k \times M_k$  matrix; moreover,  $\boldsymbol{\alpha}_k := \{\alpha_{km}\}$ ,  $\boldsymbol{\beta}_k := \{\beta_{km}\}$  and  $\boldsymbol{\gamma}_k := \{\gamma_{km}\}$  are  $M_k$ -vectors. In Eq (9), coefficients  $\boldsymbol{\alpha}_k$  are known, whereas coefficients  $\boldsymbol{\beta}_k$  and  $\boldsymbol{\gamma}_k$  are unknown.

However, since our goal is to render  $\mathbf{g}_k(\mathbf{y})$  the simplest possible, we have some freedoms in choosing  $\boldsymbol{\beta}_k$ . Of course, the best choice would be to take  $\boldsymbol{\beta}_k = \mathbf{0}$  (entailing  $\mathbf{g}_k(\mathbf{y}) = \mathbf{0}$ ); however, this operation is subordinated to the rank of  $\mathbf{L}_k$ . Two cases must be analyzed: (a) *matrix  $\mathbf{L}_k$  is non-singular*; (b) *matrix  $\mathbf{L}_k$  is singular*.

If  $\det \mathbf{L}_k \neq 0$ , we can take  $\boldsymbol{\beta}_k = \mathbf{0}$  and solve (9) for  $\boldsymbol{\gamma}_k$ , to obtain  $\boldsymbol{\gamma}_k = \mathbf{L}_k^{-1} \boldsymbol{\alpha}_k$ . Consequently, from Eqs (8),  $\mathbf{g}_k(\mathbf{y}) = \mathbf{0}$ , and  $\mathbf{h}_k(\mathbf{y})$  is univocally determined. In contrast, if  $\det \mathbf{L}_k = 0$ , in order for Eq (9) can be solved, its know-term  $\boldsymbol{\alpha}_k - \boldsymbol{\beta}_k$  must belong to the range of matrix  $\mathbf{L}_k$  (solvability, or *compatibility condition*). This requirement is fulfilled if the known term is orthogonal to the kernel of the adjoint operator  $\mathbf{L}_k^H$  (transposed and conjugate of  $\mathbf{L}_k$ ). Since this space is spanned by  $H_k$  vectors  $\mathbf{v}_{kh}$  satisfying  $\mathbf{L}_k^H \mathbf{v}_{kh} = \mathbf{0}$  ( $h = 1, 2, \dots, H_k$ ), where  $H_k$  is the geometrical multiplicity of the zero-eigenvalue of  $\mathbf{L}_k$ , compatibility requires:

$$\mathbf{V}_k^H \boldsymbol{\beta}_k = \mathbf{V}_k^H \boldsymbol{\alpha}_k \quad (10)$$

where  $\mathbf{V}_k := [\mathbf{v}_{k1} | \mathbf{v}_{k2} | \dots | \mathbf{v}_{kH_k}]$  is a  $M_k \times H_k$  matrix collecting the vectors  $\mathbf{v}_{kh}$ . Equations (10) entail that only  $M_k - H_k$  (suitable) entries of vector  $\boldsymbol{\beta}_k$  can be taken equal to zero, the remaining having to satisfy compatibility. *Therefore  $H_k$  monomials in  $\mathbf{g}_k(\mathbf{y})$  cannot be removed by the near-identity transformation.*

The previous analysis must, of course, be carried out at each order  $k$ , up-to the maximum order  $K$  considered in the analysis. At each of these orders, the number  $M_k$  of independent vectors  $\mathbf{p}_{km}(\mathbf{y})$  increases, this entailing large-dimensional matrix  $\mathbf{L}_k$  and vectors  $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k, \boldsymbol{\gamma}_k$ . It is likely to occur, for example, that matrix  $\mathbf{L}_k$  is non-singular, but  $\mathbf{L}_{k+1}$  is singular. In this case, we will be able *to remove all nonlinearities of degree  $k$ , but just some of degree  $k+1$* . As final result of the procedure, the vector  $\mathbf{g}(\mathbf{y})$  furnishes the normal form (2) of the original system (1), truncated at the order  $K$ . Moreover, the associated vector  $\mathbf{h}(\mathbf{y})$ , permits to come back to the original variable  $\mathbf{x}$ , via the near-identity transformation (3).

## 2.2 Resonance conditions

An important question concerns the possibility to predict in advance, *on the ground of the sole knowledge of the spectral properties of the Jacobian matrix  $\mathbf{J}$* , if matrix  $\mathbf{L}_k$  is singular or not, and, in addition, *which terms in  $\mathbf{f}_k(\mathbf{x})$  can be removed and which, instead, have to remain in the normal form*. It is possible to show (see, e.g. [1]), that, if the Jacobian matrix  $\mathbf{J}$  is diagonalizable, then  $\mathbf{L}_k$  is also diagonal:

$$\mathbf{L}_k = \text{diag}[\Lambda_{im}] \quad (11)$$

where:

$$\Lambda_{im} := \sum_{j=1}^N (m_j \lambda_j - \lambda_i), \quad i = 1, N \quad (12)$$

and where  $\{m_1, m_2, \dots, m_N\}$  are all the combinations of positive integers such that  $m_1 + m_2 + \dots + m_N = k$ <sup>1</sup>. Equation (11) states that  $\mathbf{L}_k$  is nonsingular (and therefore all nonlinearities of degree  $k$  are removable) if all  $\Lambda_{im} \neq 0$ . In contrast,  $\mathbf{L}_k$  is singular (and therefore some nonlinearities of degree  $k$  are *not* removable) if one or more  $\Lambda_{im}$  are zero (or nearly zero). In this latter case, the monomial associated with  $\Lambda_{im}$  (i.e. the  $m$ -th independent monomial in the  $i$ -th equation) survives in the normal form. The condition  $\Lambda_{im} = 0$  is also referred to as *resonance condition* (of order  $k$ ), since it entails that one of the eigenvalue, e.g.  $\lambda_i$ , is as a linear combination of all the eigenvalues.

It is important to stress that, *when the eigenvalues at an equilibrium point are on the imaginary axis, resonance always occur*, due to the fact that the imaginary eigenvalues are conjugate in pairs. This confirms the well known results that *the flow around a non-hyperbolic equilibrium point cannot be linearized*.

Finally, we mention the fact that, when the Jacobian matrix is not diagonalizable, the matrix  $\mathbf{L}_k$ , appearing in the equation (9), *is no more diagonal*. Nevertheless, it can be proved, that its eigenvalues are still given by (12), so that singularity of  $\mathbf{L}_k$  is due to the occurrence of the resonance conditions  $\Lambda_{im} = 0$ . Therefore, only resonant terms appear in the normal form, but their *a priori* selection is not trivial as in the diagonal case.

### 2.3 Normalization conditions

Equations (9), in the case of interest in which  $\mathbf{L}_k$  is singular, are now analyzed in greater detail. As we have already observed, due to the  $H_k$  compatibility condition (10),  $M_k - H_k$  components of vector  $\boldsymbol{\beta}_k$  can be chosen freely, best if equal to zero. Therefore, normal form is not unique, but it depends on the specific components that are zeroed. In order to state a criterion that determines  $\boldsymbol{\beta}_k$  uniquely, it needs to introduce a *normalization condition* that establishes the space  $\mathcal{B}_k$  of belonging of the vector, namely:

$$\boldsymbol{\beta}_k = \mathbf{B}_k \mathbf{b}_k \quad (13)$$

<sup>1</sup> For example, if  $N=2$ , and  $k=3$ , there are  $M_3=8$  independent monomials in Eq (8); therefore, from Eqs (11), (12):

$$\begin{aligned} \mathbf{L}_3 &= \text{diag}[3\lambda_1 - \lambda_1, 2\lambda_1 + \lambda_2 - \lambda_1, \lambda_1 + 2\lambda_2 - \lambda_1, 3\lambda_2 - \lambda_1; 3\lambda_1 - \lambda_2, 2\lambda_1 + \lambda_2 - \lambda_2, \lambda_1 + 2\lambda_2 - \lambda_2, 3\lambda_2 - \lambda_2] \\ &= \text{diag}[2\lambda_1, \lambda_1 + \lambda_2, 2\lambda_2, 3\lambda_2 - \lambda_1; 3\lambda_1 - \lambda_2, 2\lambda_1, \lambda_1 + \lambda_2, 2\lambda_2] \end{aligned}$$

where  $\mathbf{B}_k$  is a  $M_k \times H_k$  matrix, listing columnwise a basis for  $\mathcal{B}_k$ , and  $\mathbf{b}_k := \{b_{kh}\} (h=1,2,\dots,H_k)$  are the components of  $\boldsymbol{\beta}_k$  in this basis. With this normalization, the compatibility conditions (Eq (10)) become:

$$\mathbf{V}_k^H \mathbf{B}_k \mathbf{b}_k = \mathbf{V}_k^H \boldsymbol{\alpha}_k \quad (14)$$

which is a system of  $H_k$  equations in the  $H_k$  unknowns  $\mathbf{b}_k$ . A common choice consists in requiring  $\boldsymbol{\beta}_k$  has zero-projection onto the range  $\mathcal{R}(\mathbf{L}_k)$ , i.e.  $\boldsymbol{\beta}_k \in \mathcal{K}(\mathbf{L}_k^H)$ , this entailing  $\mathbf{B}_k \equiv \mathbf{V}_k$ ; however, this choice not always give the ‘simplest’ normal form.

Once  $\boldsymbol{\beta}_k$  has been determined, Eqs (9) give  $\infty^{H_k}$  solutions for  $\boldsymbol{\gamma}_k$ , namely:

$$\boldsymbol{\gamma}_k = \hat{\boldsymbol{\gamma}}_k + \mathbf{U}_k \mathbf{c}_k \quad \forall \mathbf{c}_k \quad (15)$$

where  $\mathbf{U}_k := [\mathbf{u}_{k1} | \mathbf{u}_{k2} | \dots | \mathbf{u}_{kH_k}]$  is a  $M_k \times H_k$  matrix collecting the right eigenvectors  $\mathbf{u}_{kh}$  satisfying  $\mathbf{L}_k \mathbf{u}_{kh} = 0$ . The arbitrary constants  $\mathbf{c}_k$  can be chosen by enforcing a second normalization condition, of the type:

$$\mathbf{C}_k^H \boldsymbol{\gamma}_k = \mathbf{0} \quad (16)$$

which requires  $\boldsymbol{\gamma}_k$  is orthogonal to the space spanned by the columns of the  $M_k \times H_k$  matrix  $\mathbf{C}_k$ . Again, one can take  $\mathbf{C}_k \equiv \mathbf{V}_k$ , from which  $\mathbf{c}_k = -\mathbf{V}_k^H \hat{\boldsymbol{\gamma}}_k$ , since  $\mathbf{V}_k^H \mathbf{U}_k = \mathbf{I}$ ; if  $\mathbf{L}_k$  is non-diagonalizable, the same result holds if the proper left eigenvectors in  $\mathbf{V}_k$  are substituted by the higher-order generalized eigenvectors. It should be noticed that, different choices for  $\mathbf{C}_k$  entail *different* coordinate transformations  $\mathbf{h}_k(\mathbf{y})$ , all leading to the same normal form  $\mathbf{g}_k(\mathbf{y})$ ; however, these different choices have repercussions on higher-order terms.

### 3 PARAMETER-DEPENDENT NORMAL FORMS

The previous treatment is devoted to systems *at* bifurcation, in which parameters (if any) assume fixed values. In contrast, the main interest of bifurcation analysis is to study systems *close* to a bifurcation, in which parameters are likely to vary. There exist two approaches in literature to include parameters in Normal Forms. They are shortly commented ahead; then, an alternative method is illustrated.

#### 3.1 Usual approaches

The commonest procedure followed in literature consists in considering the parameters as dummy state-variables, constant in time, (similarly to that done in the Center Manifold Theory [1,3]). Accordingly, the state-space is extended to parameters  $\boldsymbol{\mu}$ , and trivial equations are

appended to the bifurcation equations, that therefore read:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) \\ \dot{\boldsymbol{\mu}} = \mathbf{0} \end{cases} \quad (17)$$

where  $\mathbf{J}$  is the Jacobian matrix at the bifurcation point  $(\mathbf{x}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{0})$ , and bilinear (and higher-order) terms  $\mathbf{J}_{\boldsymbol{\mu}}\boldsymbol{\mu}\mathbf{x}, \dots$  have been shifted into  $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu})$ . Accordingly, the normal form and the near-identity transformation must be taken, respectively, as:

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{y}, \boldsymbol{\mu}), \quad \mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y}, \boldsymbol{\mu}) \quad (18)$$

The procedure, although simple in principle, leads to cumbersome calculations, due to the larger number of independent monomials in the  $(\mathbf{y}, \boldsymbol{\mu})$ -space.

An alternative procedure is suggested in [2]. According to this method, the parameters are incorporated in the coefficients, and therefore they do not increase the dimension of the system. Consequently, Eqs (1)-(3) change into:

$$\dot{\mathbf{x}} = \mathbf{J}(\boldsymbol{\mu})\mathbf{x} + \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}), \quad \dot{\mathbf{y}} = \mathbf{J}(\boldsymbol{\mu})\mathbf{y} + \mathbf{g}(\mathbf{y}; \boldsymbol{\mu}), \quad \mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y}; \boldsymbol{\mu}) \quad (19)$$

and Eq (9) into:

$$\mathbf{L}_k(\boldsymbol{\mu})\boldsymbol{\gamma}_k(\boldsymbol{\mu}) = \boldsymbol{\alpha}_k(\boldsymbol{\mu}) - \boldsymbol{\beta}_k(\boldsymbol{\mu}) \quad (20)$$

The dependence on  $\boldsymbol{\mu}$  of the eigenvalues of  $\mathbf{J}$ , of course entails a similar dependence of the eigenvalues of  $\mathbf{L}_k(\boldsymbol{\mu})$ . Since, due to the resonance,  $\mathbf{L}_k(\mathbf{0})$  is singular for some  $k$ , and since  $\boldsymbol{\mu}$  is small,  $\mathbf{L}_k(\boldsymbol{\mu})$  is nearly-singular, i.e. a *quasi-resonance* occurs. If this occurrence were not properly tackled, small denominators would appear in the normal form, leading to series not uniformly valid. The drawback can be overcome in the diagonal case, by considering the nearly-zero eigenvalues  $\Lambda_{im}(\boldsymbol{\mu})$  of  $\mathbf{L}_k(\boldsymbol{\mu})$  as they were exactly zero, by exploiting the arbitrariness of the quantities  $\boldsymbol{\beta}_k$ 's. However, the problem is not so trivial if the Jacobian matrix is not diagonalizable, and a solution of the problem, available in the literature, is not known to the author.

### 3.2 A perturbation algorithm

A perturbation algorithm is proposed to derive parameter-dependent normal forms. In order to keep unaltered the dimension of the original system, the state space is *not* extended. Equation (20), is again taken into account, but instead to solve it *as is*, the parameters appearing in the equations are rescaled as  $\boldsymbol{\mu} \rightarrow \boldsymbol{\varepsilon}\boldsymbol{\mu}$ , where  $\boldsymbol{\varepsilon} > 0$  is a small bookkeeping parameter. Consequently, after expanding in series all quantities, it follows:

$$\mathbf{L}_k = \mathbf{L}_{k,0} + \varepsilon \mathbf{L}_{k,1} + \varepsilon^2 \mathbf{L}_{k,2} + \dots, \quad \begin{pmatrix} \mathbf{a}_k \\ \mathbf{\beta}_k \\ \boldsymbol{\gamma}_k \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{k,0} \\ \mathbf{\beta}_{k,0} \\ \boldsymbol{\gamma}_{k,0} \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{a}_{k,1} \\ \mathbf{\beta}_{k,1} \\ \boldsymbol{\gamma}_{k,1} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \mathbf{a}_{k,2} \\ \mathbf{\beta}_{k,2} \\ \boldsymbol{\gamma}_{k,2} \end{pmatrix} + \dots \quad (21)$$

where:

$$\begin{aligned} \mathbf{L}_{k,0} &:= \mathbf{L}_k(\mathbf{0}), & \mathbf{L}_{k,1} &:= \left. \frac{d\mathbf{L}_k}{d\boldsymbol{\mu}} \right|_{\boldsymbol{\mu}=\mathbf{0}} \boldsymbol{\mu}, & \mathbf{L}_{k,2} &:= \frac{1}{2} \left. \frac{d^2\mathbf{L}_k}{d\boldsymbol{\mu}^2} \right|_{\boldsymbol{\mu}=\mathbf{0}} \boldsymbol{\mu}^2, \dots \\ \mathbf{a}_{k,0} &:= \mathbf{a}_k(\mathbf{0}), & \mathbf{a}_{k,1} &:= \left. \frac{d\mathbf{a}_k}{d\boldsymbol{\mu}} \right|_{\boldsymbol{\mu}=\mathbf{0}} \boldsymbol{\mu}, & \mathbf{a}_{k,2} &:= \frac{1}{2} \left. \frac{d^2\mathbf{a}_k}{d\boldsymbol{\mu}^2} \right|_{\boldsymbol{\mu}=\mathbf{0}} \boldsymbol{\mu}^2, \dots \end{aligned} \quad (22)$$

and similar. By substituting Eqs (21) and (22) in Eq (20), and equating separately to zero terms with the same power of  $\varepsilon$ , the following chain of perturbation equations is obtained:

$$\begin{aligned} \varepsilon^0 : & \quad \mathbf{L}_{k,0} \boldsymbol{\gamma}_{k,0} = \mathbf{a}_{k,0} - \mathbf{\beta}_{k,0} \\ \varepsilon^1 : & \quad \mathbf{L}_{k,0} \boldsymbol{\gamma}_{k,1} = \mathbf{a}_{k,1} - \mathbf{\beta}_{k,1} - \mathbf{L}_{k,1} \boldsymbol{\gamma}_{k,0} \\ & \dots\dots \\ \varepsilon^n : & \quad \mathbf{L}_{k,0} \boldsymbol{\gamma}_{k,n} = \hat{\mathbf{a}}_{k,n} - \mathbf{\beta}_{k,n} \end{aligned} \quad (23)$$

where  $\hat{\mathbf{a}}_{k,n}$  is a vector known from the previous steps. The generating equation (23<sub>1</sub>) coincides with that of the no-parameter case; therefore, quasi-resonances are ruled out at any orders, since  $\mathbf{L}_{k,0}$  is (exactly) singular. By solving in sequence Eqs (23), and enforcing compatibility and normalization at each step, all the terms of the series (21<sub>2</sub>) for  $\mathbf{\beta}_k$  and  $\boldsymbol{\gamma}_k$  are evaluated up-to the desired order. An example of the procedure is worked out in the next section.

#### 4 AN EXAMPLE: THE DOUBLE-ZERO BIFURCATION

As an example, let us consider the following two-dimensional dynamical system, governing the motion, reduced to the Center Manifold, of a larger system around a double-zero (Takens-Bogdanov) bifurcation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \nu & \mu \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \alpha_{21}x_1^2 + \alpha_{22}x_1x_2 + \alpha_{23}x_2^2 \\ \alpha_{24}x_1^2 + \alpha_{25}x_1x_2 + \alpha_{26}x_2^2 \end{pmatrix} \quad (24)$$

where  $\boldsymbol{\mu} = (\mu, \nu)$  are bifurcation parameters, vanishing at the bifurcation point. To make the example simplest as possible, the coefficients  $\alpha_{2m}$  are assumed to be independent of  $\boldsymbol{\mu}$ .



According to Eqs (18), we assume a normal form and a near-identity transformation as follows:

$$\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{bmatrix} 0 & 1 \\ \nu & \mu \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \beta_{21}(\boldsymbol{\mu})y_1^2 + \beta_{22}(\boldsymbol{\mu})y_1y_2 + \beta_{23}(\boldsymbol{\mu})y_2^2 \\ \beta_{24}(\boldsymbol{\mu})y_1^2 + \beta_{25}(\boldsymbol{\mu})y_1y_2 + \beta_{26}(\boldsymbol{\mu})y_2^2 \end{pmatrix} + \dots \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \gamma_{21}(\boldsymbol{\mu})y_1^2 + \gamma_{22}(\boldsymbol{\mu})y_1y_2 + \gamma_{23}(\boldsymbol{\mu})y_2^2 \\ \gamma_{24}(\boldsymbol{\mu})y_1^2 + \gamma_{25}(\boldsymbol{\mu})y_1y_2 + \gamma_{26}(\boldsymbol{\mu})y_2^2 \end{pmatrix} + \dots \end{aligned} \quad (25)$$

By substituting Eqs (25) in Eq (6<sub>2</sub>) and zeroing separately the coefficients of the three independent monomials in the two equations, we obtain six algebraic equations, of the type (20) (with  $k = 2$ ):

$$\left[ \begin{array}{ccc|ccc} 0 & \nu & 0 & -1 & 0 & 0 \\ 2 & \mu & 2\nu & 0 & -1 & 0 \\ 0 & 1 & 2\mu & 0 & 0 & -1 \\ \hline -\nu & 0 & 0 & -\mu & \nu & 0 \\ 0 & -\nu & 0 & 2 & 0 & 2\nu \\ 0 & 0 & -\nu & 0 & 1 & \mu \end{array} \right] \begin{pmatrix} \gamma_{21}(\boldsymbol{\mu}) \\ \gamma_{22}(\boldsymbol{\mu}) \\ \gamma_{23}(\boldsymbol{\mu}) \\ \gamma_{24}(\boldsymbol{\mu}) \\ \gamma_{25}(\boldsymbol{\mu}) \\ \gamma_{26}(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \alpha_{21} - \beta_{21}(\boldsymbol{\mu}) \\ \alpha_{22} - \beta_{22}(\boldsymbol{\mu}) \\ \alpha_{23} - \beta_{23}(\boldsymbol{\mu}) \\ \alpha_{24} - \beta_{24}(\boldsymbol{\mu}) \\ \alpha_{25} - \beta_{25}(\boldsymbol{\mu}) \\ \alpha_{26} - \beta_{26}(\boldsymbol{\mu}) \end{pmatrix} \quad (26)$$

The matrix  $\mathbf{L}_2(\boldsymbol{\mu})$  appearing in Eq (26) admits the eigenvalues  $\Lambda_{1,2} = (\mu \pm 3\sqrt{\mu^2 + 4\nu})/2$ ,  $\Lambda_{3,4} \equiv \Lambda_{5,6} = (\mu \pm \sqrt{\mu^2 + 4\nu})/2$ , which all tend to zero when  $\boldsymbol{\mu} \rightarrow \mathbf{0}$ ; thus  $\mathbf{L}_2(\boldsymbol{\mu})$  cannot be inverted, if small denominators must be avoided. Therefore, according to the proposed method, we resort to the perturbation equations (23).

The generation equation (23<sub>1</sub>) is obtained by putting  $\boldsymbol{\mu} = \mathbf{0}$  in Eq (26). Since  $\text{Rank}[\mathbf{L}_{2,0}] = 4$ ,  $\mathbf{L}_{2,0}$  has a two-dimensional kernel  $\mathcal{K}(\mathbf{L}_{2,0}) = \text{span}\{\mathbf{u}_{21}, \mathbf{u}_{22}\}$ , where  $\mathbf{u}_{21} = (0, 1, 0, 0, 0, 1)^T$  and  $\mathbf{u}_{22} = (0, 0, 1, 0, 0, 0)^T$ ; moreover,  $\mathcal{K}(\mathbf{L}_{2,0}^T) = \text{span}\{\mathbf{v}_{21}, \mathbf{v}_{22}\}$ , where  $\mathbf{v}_{21} = (2, 0, 0, 0, 1, 0)^T$  and  $\mathbf{v}_{22} = (0, 0, 0, 1, 0, 0)^T$ . In order Eq (23<sub>1</sub>) can be solved, the known term must satisfy two compatibility conditions (Eqs (10)):

$$2(\alpha_{21} - \beta_{21,0}) + (\alpha_{25} - \beta_{25,0}) = 0, \quad \alpha_{24} - \beta_{24,0} = 0 \quad (27)$$

Since  $\beta_{22,0}, \beta_{23,0}, \beta_{26,0}$  are not involved in compatibility, they are taken zero; moreover,  $\beta_{21,0} = 0$ ,  $\beta_{24,0} = \alpha_{24}$ ,  $\beta_{25,0} = \alpha_{25} + 2\alpha_{21}$  is chosen to satisfy Eqs (27); therefore  $\boldsymbol{\beta}_k \in \text{span}\{\mathbf{e}_4, \mathbf{e}_5\}$  is taken as normalization condition. Solution to Eq (23<sub>1</sub>) furnishes  $\boldsymbol{\gamma}_{2,0} = ((\alpha_{22} + \alpha_{26})/2, \alpha_{23} + c_1, c_2, -\alpha_{21}, \alpha_{26}, c_1)^T, \forall c_1, c_2$ ; we chose  $c_1 = c_2 = 0$ , thus

adopting the normalization  $\boldsymbol{\gamma}_k \in \text{span}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5]$ .

Passing to the  $\mathcal{E}$ -order perturbation equation (23<sub>2</sub>), we first evaluate the right hand member. Since  $\boldsymbol{\alpha}_{k,1} = \mathbf{0}$ ,  $\mathbf{L}_{k,1}\boldsymbol{\gamma}_{k,0} = (\alpha_{23}\boldsymbol{\nu}, \alpha_{23}\boldsymbol{\mu}, 0, \alpha_{21}\boldsymbol{\mu} + (\alpha_{26} - \alpha_{22})\boldsymbol{\nu}/2, -\alpha_{23}\boldsymbol{\nu}, 0)^T$  and, according to normalization adopted,  $\boldsymbol{\beta}_{2,1} = (0, 0, 0, \beta_{24,1}, \beta_{25,1}, 0)^T$ , solvability furnishes  $\beta_{24,1} = -\alpha_{21}\boldsymbol{\mu} + (\alpha_{22} - \alpha_{26})\boldsymbol{\nu}/2$ ,  $\beta_{25,1} = -\alpha_{23}\boldsymbol{\nu}$ . Finally, the normalized solution to Eq (23<sub>2</sub>) reads  $\boldsymbol{\gamma}_{2,1} = (-\alpha_{23}\boldsymbol{\mu}/2, 0, 0, \alpha_{23}\boldsymbol{\nu}, 0, 0)^T$ .

By substituting the results obtained in Eqs (25), to within an error of  $\mathcal{O}(|\mathbf{y}|^3, |\boldsymbol{\mu}|^2|\mathbf{y}|^2)$ , we have the following normal form and coordinate transformation:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \boldsymbol{\nu} & \boldsymbol{\mu} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_{24}y_1^2 + \beta_{25}y_1y_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \gamma_{21}y_1^2 + \gamma_{22}y_1y_2 \\ \gamma_{24}y_1^2 + \gamma_{25}y_1y_2 \end{pmatrix} \quad (28)$$

where:

$$\begin{aligned} \beta_{24} &= \alpha_{24} - \alpha_{21}\boldsymbol{\mu} + (\alpha_{22} - \alpha_{26})\boldsymbol{\nu}/2, & \beta_{25} &= [2\alpha_{21} + \alpha_{25} - \alpha_{23}\boldsymbol{\nu}] \\ \gamma_{21} &= (\alpha_{22} + \alpha_{26} - \alpha_{23}\boldsymbol{\mu})/2, & \gamma_{22} &= \alpha_{23}, & \gamma_{24} &= -\alpha_{21} + \alpha_{23}\boldsymbol{\nu}, & \gamma_{25} &= \alpha_{26} \end{aligned} \quad (29)$$

It should be noticed, that parameters only alter the coefficients, not the structure of the two expressions, as a consequence of having used the same normalizations at all orders.

## CONCLUSIONS

By following a perturbation approach, we proposed a new method for evaluating parameter-dependent normal forms of bifurcation equations, both for diagonalizable and non-diagonalizable Jacobian matrices. The method avoids the extension of the state space to parameters, as well as the occurrence of small denominators in the solution, which are drawbacks encountered in classical methods. The efficiency of the procedure was illustrated by an example, relevant to a double-zero bifurcation.

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