A nonlinear mathematical model for a bicycle

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SUMMARY. This paper concerns the mathematical modelling of a bicycle taking account of the presence of a trail. It is well known that the bicycle is a classical example of a nonholonomic system. First we consider the kinematics of rolling of a bicycle and we derive the full Lagrangian for a bicycle with trail. Then we write the equation of motion for a simplified model with zero caster angle, following the constrained Lagrangian point of view.

1 INTRODUCTION

In the recent past, a great interest has been devoted towards bicycle modelling as it is a mechanical system characterized by nonholonomic constraints. The bicycle is probably the most common mode of transportation in the world, next only to walking, and, starting from some pioneering papers at the end of the nineteen century, [1], many researchers have tried to find proper equations to describe the dynamic of this system.

Mainly, it is possible to distinguish between two different approaches: the first obtains the motion equations using the Newton's laws [2], while the second studies the system from a Lagrangian or Hamiltonian point of view [3, 4]. So far, the greatest part of the existing literature has been devoted to models with lots of simplifications, even if these have been capable to explain the dynamical characteristics of the bicycle. For example, linearized equations of motion are commonly introduced in order to cope more easily with the problem, [2].

In this paper we consider a model with seven degrees of freedom, in order to study a system which is quite similar to the real one. Moreover, this choose allows us to introduce the nonholonomic constraints due to the rolling motion in a more natural way. In fact, as the system has seven degrees of freedom, introducing four constraints we can express four coordinates in function of three. In section 3 we consider the kinematics of our model and we study the pitch angle as function of the bicycle parameters. Then we derive the full Lagrangian for a bicycle with trail and without offset. Finally we write the equation of motion for a simplified model with zero caster angle, following the constrained Lagrangian point of view.

2 THE BICYCLE MODEL

2.1 *Description of the* 7*-dimensional configuration space*

In this model it is assumed that the bicycle traverses a flat ground, so that the system is subject to some symmetries. As result, we can choose just seven coordinates in order to describe completely the dynamical system. In particular, we consider the coordinates (x, y) of the rear wheel contact point in the horizontal plane with respect to the inertial reference frame (X, Y, Z), where the Z-axis is perpendicular to the ground plane in the direction opposite to gravity. Clearly, the third coordinate of this contact point can be assume always zero.

Then we consider the yaw angle θ, which is taken about the vertical Z-direction. This angle is zero when the plane of the rear wheel passes through the X-axis. We adopt the right-hand rule, so the angle is positive for counter-clockwise rotations. The roll angle α is the angle that the bicycle's plane of symmetry makes with the vertical direction, so, roughly speaking, it is a rotation about the x-axis of the local reference frame of the rear wheel. In this paper we take $\alpha \mapsto -\alpha$ in order to obtain a positive angle when the bicycle leans to the left.

Furthermore, we take the steering angle ψ about the steering axis, which is tilted backward by the caster angle ε, the latter due to the presence of the trail. As the roll angle, even the steering angle can assume values in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We observe that, consequently to the previous choice about the α angle, the angles ψ and α itself are positive when the bicycle is turning to the left.

Finally, we introduce a rear and front wheel rotation angles χ_r and χ_f in order to consider the inertia moments and the mass of both wheels. So, the configuration space is $Q = SE(2) \times S^1 \times S^1 \times S^1 \times S^1$.

Regarding the nonholonomic constraints, the rear and front wheel contacts with the ground are constrained to have velocities parallel to the line of intersection of their respective wheel planes and the ground plane, but free to rotate about the axis passing through the wheel-ground contact and parallel to the z-axis of the local reference frame.

To sum up, we choose seven generalized coordinates in order to model our system, and then we introduced two nonholonomic constraints that can be written each one as a system of two equation, as it is showed in the following sections.

2.2 *Physical parameters in our model*

In addition to the seven generalized coordinates, the bicycle model presents some time-independent parameters that identify the bicycle itself, as shown in figure 1.

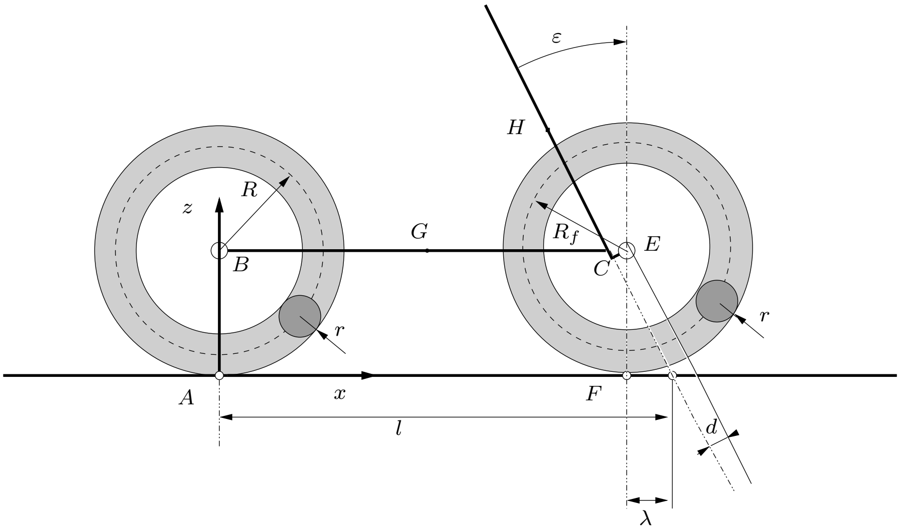

Figure 1: Geometry of the bicycle

In fact, the shape of the bicycle is completely determinated by the radii of the two wheels R and R_f respectively, the wheelbase l, the steering axis length $2h$, the offset d and the caster angle ε. This one depends on the trail by the relation

$$\lambda = R_f \tan \varepsilon - \frac{d}{\cos \varepsilon},$$

and it plays an important role in the motion of the bicycle. In fact, even if these are not time-dependent coordinates of the system, by changing these parameters it is possible to obtain different motion of the bicycle. In particular, the caster angle has a remarkable influence on the motion, and the results obtained are completely different as the angle ε is close to zero or to $-\frac{\pi}{2}$. It is clear that this angle is negative according to the right-hand rule.

At the beginning, we consider a complete bicycle model, with toroidal tires and offset. Then, in order to cope with more simple equations, we assume $d = 0$ and $r = 0$, where r is the wheel crown radius.

3 THE KINEMATICS OF BICYCLE

The kinematic study of bicycle is the first step to obtain the equations of motion for the system. To find the relation between the generalized coordinates chosen before, it is necessary to write down the rotation matrices for the four rigid body which compose the system. In this paper, we assume *alias* transformation, so it is turned the coordinate system.

3.1 *Rear wheel*

First we consider the rear wheel. Without considering the proper rotation of the rear wheel, the rotation matrix associated to the body is

$$\mathcal{R}_1 = \mathcal{R}_x(-\alpha)\mathcal{R}_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\alpha\sin\theta & \cos\alpha\cos\theta & -\sin\alpha \\ -\sin\alpha\sin\theta & \sin\alpha\cos\theta & \cos\alpha \end{pmatrix}, \tag{1}$$

where $\mathcal{R}_{x_i}(\alpha_i)$ is the rotation about axis x_i of an angle α_i. In fact, the rear wheel rotation angle χ_r plays its role when we write the angular velocities of the bodies. Hence, if B is the center of the rear wheel, as shown in figure 1, the expression of the vector $(B-A)$ in the local reference frame is

$$(B-A)_S = (R + r\cos\alpha)\vec{k}_S - r\sin\alpha\vec{\jmath}_S,$$

so it can be expressed in the inertial reference frame by substituting the local unit vectors with their expression respect to the inertial frame, which can be taken from the rows of the rotation matrix (1). We obtain

$$(B-A)_\Sigma = -R\sin\alpha\sin\theta\vec{\imath} + R\sin\alpha\cos\theta\vec{\jmath} + (R\cos\alpha + r)\vec{k},$$

where the coordinates of A respect to the origin of the inertial reference frame are $(A-O)_\Sigma = x\vec{\imath} + y\vec{\jmath}$.

In this way, it is possible to write down the coordinates of each point of the four bodies in the local and the inertial reference frame. In particular, we are interested to obtain the coordinates of the center of mass of each body, in order to express the kinetic energy of the system.

3.2 *Rear frame*

Passing to the rear frame, we introduce an auxiliary angle μ, which represents the pitch angle. In fact, it is common knowledge that when the steering axis is turned, the center of the front wheel goes down, so that the rear frame turns down too.

The fact is that the pitch angle depends on the other coordinates, so it will be expressed in the following sections as a function $\mu = \mu(\theta, \psi, \alpha)$. In order to obtain this relation, we write the rotation matrix of the rear frame with the angle μ

$$\mathcal{R}_2 = \mathcal{R}_y(\mu)\mathcal{R}_1 =$$

$$= \begin{pmatrix} \cos\mu\cos\theta + \sin\mu\sin\alpha\sin\theta & \cos\mu\sin\theta - \sin\mu\sin\alpha\cos\theta & -\sin\mu\cos\alpha \\ -\cos\alpha\sin\theta & \cos\alpha\cos\theta & -\sin\alpha \\ \sin\mu\cos\theta - \cos\mu\sin\alpha\sin\theta & \sin\mu\sin\theta + \cos\mu\sin\alpha\cos\theta & \cos\mu\cos\alpha \end{pmatrix}, \tag{2}$$

so that the coordinates of any point of the rear frame can be expressed as

$$a\,\vec{\imath}_S = a(\cos\mu\cos\theta + \sin\mu\sin\alpha\sin\theta)\vec{\imath} + a(\cos\mu\sin\theta - \sin\mu\sin\alpha\cos\theta)\vec{\jmath} - a\sin\mu\cos\alpha\,\vec{k} \tag{3}$$

in the local and the inertial reference frame respectively, where $a \in [0, l]$. In this way, we can express the coordinates of the point C of intersection of the wheelbase with the steering axis, using the relation (3) with $a = l$. Next we find another way to express these coordinates.

3.3 *Front frame and front wheel*

The rotation matrix associated with the front frame and the front wheel is slightly more complicated respect to those seen before. In fact, it is necessary to multiply four rotation matrices, i.e. a θ-rotation, an α-rotation, an $-\varepsilon$-rotation and a ψ-rotation. Thus we obtain the matrix

$$\mathcal{R}_3 = \begin{pmatrix} \mathrm{c}_\psi(\mathrm{c}_\varepsilon\,\mathrm{c}_\theta - \mathrm{s}_\varepsilon\,\mathrm{s}_\alpha\,\mathrm{s}_\theta) - \mathrm{s}_\psi\,\mathrm{c}_\alpha\,\mathrm{s}_\theta & \mathrm{c}_\psi(\mathrm{c}_\varepsilon\,\mathrm{s}_\theta + \mathrm{s}_\varepsilon\,\mathrm{s}_\alpha\,\mathrm{c}_\theta) + \mathrm{s}_\psi\,\mathrm{c}_\alpha\,\mathrm{c}_\theta & \mathrm{s}_\varepsilon\,\mathrm{c}_\psi\,\mathrm{c}_\alpha - \mathrm{s}_\psi\,\mathrm{s}_\alpha \\ \mathrm{s}_\psi(\mathrm{s}_\varepsilon\,\mathrm{s}_\alpha\,\mathrm{s}_\theta - \mathrm{c}_\varepsilon\,\mathrm{c}_\theta) - \mathrm{c}_\psi\,\mathrm{c}_\alpha\,\mathrm{s}_\theta & -\mathrm{s}_\psi(\mathrm{c}_\varepsilon\,\mathrm{s}_\theta + \mathrm{s}_\varepsilon\,\mathrm{s}_\alpha\,\mathrm{c}_\theta) + \mathrm{c}_\psi\,\mathrm{c}_\alpha\,\mathrm{c}_\theta & -\mathrm{s}_\varepsilon\,\mathrm{s}_\psi\,\mathrm{c}_\alpha - \mathrm{c}_\psi\,\mathrm{s}_\alpha \\ -\mathrm{s}_\varepsilon\,\mathrm{c}_\theta - \mathrm{c}_\varepsilon\,\mathrm{s}_\alpha\,\mathrm{s}_\theta & -\mathrm{s}_\varepsilon\,\mathrm{s}_\theta + \mathrm{c}_\varepsilon\,\mathrm{s}_\alpha\,\mathrm{c}_\theta & \mathrm{c}_\varepsilon\,\mathrm{c}_\alpha \end{pmatrix}, \tag{4}$$

which can be used to express the local unit vectors respect to the inertial frame. However, it is convenient to introduce another rotation matrix with three auxiliary angles, in order to express in a direct way the front wheel nonholonomic constraints. Let $\widetilde{\theta}$, $\widetilde{\alpha}$ and $\widetilde{\mu}$ be the auxiliary angles introduced for the front part, so that we can write the rotation matrix

$$\widetilde{\mathcal{R}} = \begin{pmatrix} \cos\widetilde{\mu}\cos\widetilde{\theta} + \sin\widetilde{\mu}\sin\widetilde{\alpha}\sin\widetilde{\theta} & \cos\widetilde{\mu}\sin\widetilde{\theta} - \sin\widetilde{\mu}\sin\widetilde{\alpha}\cos\widetilde{\theta} & -\sin\widetilde{\mu}\cos\widetilde{\alpha} \\ -\cos\widetilde{\alpha}\sin\widetilde{\theta} & \cos\widetilde{\alpha}\cos\widetilde{\theta} & -\sin\widetilde{\alpha} \\ \sin\widetilde{\mu}\cos\widetilde{\theta} - \cos\widetilde{\mu}\sin\widetilde{\alpha}\sin\widetilde{\theta} & \sin\widetilde{\mu}\sin\widetilde{\theta} + \cos\widetilde{\mu}\sin\widetilde{\alpha}\cos\widetilde{\theta} & \cos\widetilde{\mu}\cos\widetilde{\alpha} \end{pmatrix}. \tag{5}$$

Now it is evident that the director cosines of matrices (4) and (5) are the same, therefore we have nine relations such as

$$\sin\widetilde{\alpha} = \sin\varepsilon\sin\psi\cos\alpha + \cos\psi\sin\alpha,$$

which will be applied in the following sections.

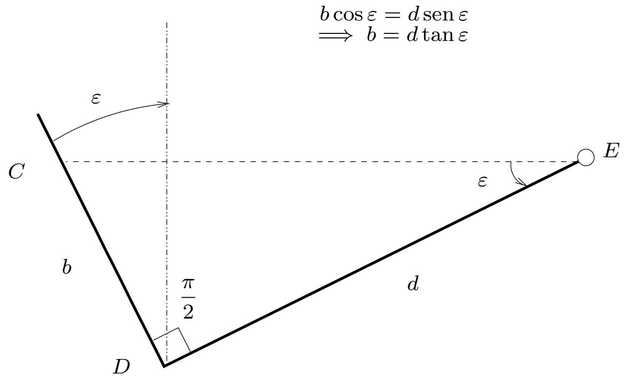

Figure 2: Geometry of the bicycle: the offset

Now we want to express the coordinates of the point C with respect to the front part of the bicycle. We consider the front wheel-ground contact point, whose coordinates respect to the inertial frame are (z, w). Then, referring to figure 2, we have

$$\begin{aligned}(E-F)_\Sigma = {} & R_f(\cos\varepsilon\sin\varepsilon\cos\psi\cos\theta - \sin^2\varepsilon\cos\psi\sin\alpha\sin\theta - \sin\varepsilon\sin\psi\cos\alpha\sin\theta + \\ & -\cos\varepsilon\sin\varepsilon\cos\theta - \cos^2\varepsilon\sin\alpha\sin\theta)\vec{\imath} + \\ & + R_f(\cos\varepsilon\sin\varepsilon\cos\psi\sin\theta + \sin^2\varepsilon\cos\psi\sin\alpha\cos\theta + \sin\varepsilon\sin\psi\cos\alpha\cos\theta + \\ & -\cos\varepsilon\sin\varepsilon\sin\theta + \cos^2\varepsilon\sin\alpha\cos\theta)\vec{\jmath} + \\ & + R_f(\sin^2\varepsilon\cos\psi\cos\alpha - \sin\varepsilon\sin\psi\sin\alpha + \cos^2\varepsilon\cos\alpha)\vec{k} + r\vec{k}\,,\end{aligned}$$

$$\begin{aligned}(D-E)_\Sigma = {} & d(\sin\varepsilon\cos\psi\sin\alpha\sin\theta - \cos\varepsilon\cos\psi\cos\theta + \sin\psi\cos\alpha\sin\theta)\vec{\imath} + \\ & + d(-\sin\varepsilon\cos\psi\sin\alpha\cos\theta - \cos\varepsilon\cos\psi\sin\theta - \sin\psi\cos\alpha\cos\theta)\vec{\jmath} + \\ & + d(\sin\psi\sin\alpha - \sin\varepsilon\cos\psi\cos\alpha)\vec{k}\,,\end{aligned}$$

and

$$\begin{aligned}(C-D)_\Sigma = {} & b(-\sin\varepsilon\cos\theta - \cos\varepsilon\sin\alpha\sin\theta)\vec{\imath} + \\ & + b(-\sin\varepsilon\sin\theta + \cos\varepsilon\sin\alpha\cos\theta)\vec{\jmath} + b\cos\varepsilon\cos\alpha\,\vec{k} = \\ & = d(-\tan\varepsilon\sin\varepsilon\cos\theta - \sin\varepsilon\sin\alpha\sin\theta)\vec{\imath} + \\ & + d(-\tan\varepsilon\sin\varepsilon\sin\theta + \sin\varepsilon\sin\alpha\cos\theta)\vec{\jmath} + d\sin\varepsilon\cos\alpha\,\vec{k}\,.\end{aligned}$$

Therefore, the vector $(C-O)_\Sigma$ can be expressed in two different ways, as either $(C-D)+(D-E)+$

$(E-F)+(F-O)$ or $(C-B)+(B-A)+(A-O)$. Equating these expression yields

$$\begin{aligned}
&\begin{pmatrix} z \\ w \\ 0 \end{pmatrix} + R_f \begin{pmatrix} \mathrm{c}_\varepsilon \, \mathrm{s}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{c}_\theta - \mathrm{s}_\varepsilon^2 \, \mathrm{c}_\psi \, \mathrm{s}_\alpha \, \mathrm{s}_\theta - \mathrm{s}_\varepsilon \, \mathrm{s}_\psi \, \mathrm{c}_\alpha \, \mathrm{s}_\theta - \mathrm{c}_\varepsilon \, \mathrm{s}_\varepsilon \, \mathrm{c}_\theta - \mathrm{c}_\varepsilon^2 \, \mathrm{s}_\alpha \, \mathrm{s}_\theta \\ \mathrm{c}_\varepsilon \, \mathrm{s}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{s}_\theta - \mathrm{s}_\varepsilon^2 \, \mathrm{c}_\psi \, \mathrm{s}_\alpha \, \mathrm{c}_\theta + \mathrm{s}_\varepsilon \, \mathrm{s}_\psi \, \mathrm{c}_\alpha \, \mathrm{c}_\theta - \mathrm{c}_\varepsilon \, \mathrm{s}_\varepsilon \, \mathrm{s}_\theta + \mathrm{c}_\varepsilon^2 \, \mathrm{s}_\alpha \, \mathrm{c}_\theta \\ \mathrm{s}_\varepsilon^2 \, \mathrm{c}_\psi \, \mathrm{c}_\alpha - \mathrm{s}_\varepsilon \, \mathrm{s}_\psi \, \mathrm{s}_\alpha + \mathrm{c}_\varepsilon^2 \, \mathrm{c}_\alpha + r/R_f \end{pmatrix} \\
&+d \begin{pmatrix} \mathrm{s}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{s}_\alpha \, \mathrm{s}_\theta - \mathrm{c}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{c}_\theta + \mathrm{s}_\psi \, \mathrm{c}_\alpha \, \mathrm{s}_\theta \\ - \mathrm{s}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{s}_\alpha \, \mathrm{c}_\theta - \mathrm{c}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{s}_\theta - \mathrm{s}_\psi \, \mathrm{c}_\alpha \, \mathrm{c}_\theta \\ \mathrm{s}_\psi \, \mathrm{s}_\alpha - \mathrm{s}_\varepsilon \, \mathrm{c}_\psi \, \mathrm{c}_\alpha \end{pmatrix} + d \begin{pmatrix} -\tan\varepsilon \, \mathrm{s}_\varepsilon \, \mathrm{c}_\theta - \mathrm{s}_\varepsilon \, \mathrm{s}_\alpha \, \mathrm{s}_\theta \\ -\tan\varepsilon \, \mathrm{s}_\varepsilon \, \mathrm{s}_\theta + \mathrm{s}_\varepsilon \, \mathrm{s}_\alpha \, \mathrm{c}_\theta \\ \mathrm{s}_\varepsilon \, \mathrm{c}_\alpha \end{pmatrix} = \\
&= \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + R \begin{pmatrix} -\sin\alpha\sin\theta \\ \sin\alpha\cos\theta \\ \cos\alpha + r/R \end{pmatrix} + l \begin{pmatrix} \cos\mu\cos\theta + \sin\alpha\sin\mu\sin\theta \\ \cos\mu\sin\theta - \sin\alpha\sin\mu\cos\theta \\ \cos\alpha\sin\mu \end{pmatrix} ,
\end{aligned} \tag{6}$$

so that we have three expressions for the auxiliary variables z, w and μ.

4 THE PITCH ANGLE AND ITS ANALYSIS

As we have already seen at the end of the last section, there is a relation between the pitch angle μ and the other generalized coordinates. In fact, according to what we anticipated before, holds the relation

$$\begin{aligned}
\sin\mu = {} & \frac{R_f}{l}(\sin\varepsilon\sin\psi\tan\alpha - \sin^2\varepsilon\cos\psi - \cos^2\varepsilon) + \\
& + \frac{d}{l}(\sin\varepsilon\cos\psi - \sin\psi\tan\alpha - \sin\varepsilon) + \frac{R}{l} \, .
\end{aligned} \tag{7}$$

First of all, we observe that the pitch angle is function of the angle α and ψ, but is θ-independent, i.e. it is independent of the direction towards the bicycle is moving.

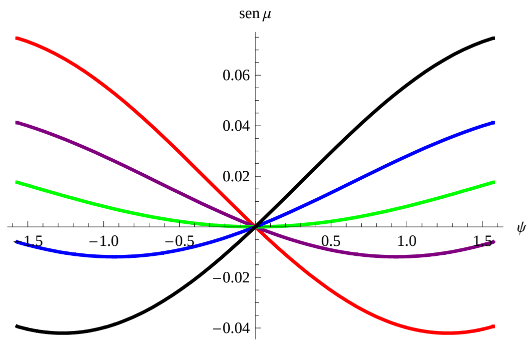

Figura 3: Plot of $\sin\mu$ as function of ψ, with α from $-\frac{\pi}{4}$ (red) to $\frac{\pi}{4}$ (black).

In addition to this, by expression (7) we can evaluate the pitch angle for different values of the roll angle. In particular, we plot the pitch angle and its cosine and sine, assuming the rear and the front wheel radii equal, i.e $R_f = R = 0.35m$, and the others physical parameters with the following values: $d = 0.5m$, $\varepsilon = 0.1\pi rad$ and $l = 1.02m$. In figure 3 it is plotted $\sin\mu$ as function of the steering angle ψ for different values of the roll angle: in particular the latter assumes its values in the range $[-\frac{\pi}{4}, \frac{\pi}{4}]$ with step of $\frac{\pi}{8}$.

It is clear that the sine of the pitch angle is quite small, even if it can assume slightly bigger values as d tends to zero. However, the error made by substituting the angle with its sine is acceptable, as it is shown in figure 4.

On the other hand, $\cos\mu$ is very close to one, as shown in figure 5, so that we can assume $\cos\mu = 1$ with a maximum error of the 0.278%.

According to this analysis, it is possible to consider μ a small angle. However, in the following sections we continue to consider completely the pitch angle, in order to obtain the whole expression of the Lagrangian functional. Although, in order to deal with shorter expression, we now assume the offset $d = 0$, the radius of the tori $r = 0$ and the two wheels with $R_f = R$.

5 THE FRONT WHEEL CONTACT POINT

Before studying the angular velocities of the system in order to write the kinetic energies of the four bodies, we want to spend some words about the position of the front wheel contact point. In fact, as we have already done with the pitch angle, it is possible to express the coordinates of the front wheel contact

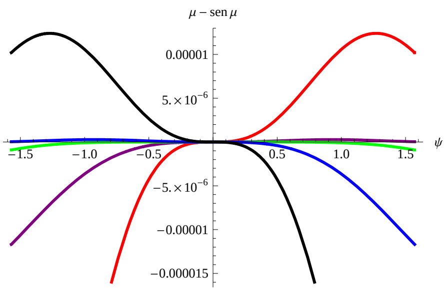

Figure 4: Plot of $\mu - \sin\mu$ as function of ψ, with α from $-\frac{\pi}{4}$ (red) to $\frac{\pi}{4}$ (black).

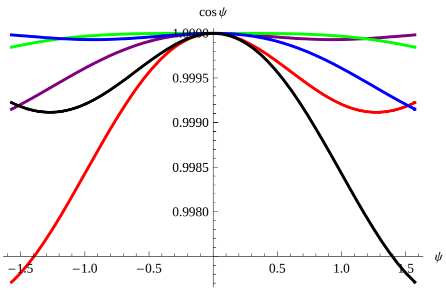

Figure 5: Plot of $\cos\mu$ as function of ψ, with α from $-\frac{\pi}{4}$ (red) to $\frac{\pi}{4}$ (black).

point by the vectorial relation (6). In fact, the first and the second scalar expression gives

$$\begin{aligned}
z = {}& x - R\sin\alpha\sin\theta + l\cos\mu\cos\theta + l\sin\alpha\sin\mu\sin\theta + d(\tan\varepsilon\sin\varepsilon\cos\theta + \\
& + \sin\varepsilon\sin\alpha\sin\theta - \sin\varepsilon\cos\psi\sin\alpha\sin\theta + \cos\varepsilon\cos\psi\cos\theta - \sin\psi\cos\alpha\sin\theta) + \\
& + R_f(\sin^2\varepsilon\cos\psi\sin\alpha\sin\theta - \cos\varepsilon\sin\varepsilon\cos\psi\cos\theta + \cos^2\varepsilon\sin\alpha\sin\theta + \\
& + \sin\varepsilon\sin\psi\cos\alpha\sin\theta + \cos\varepsilon\sin\varepsilon\cos\theta)
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
w = {}& y + R\sin\alpha\cos\theta + l\cos\mu\sin\theta - l\sin\alpha\sin\mu\cos\theta + d(\tan\varepsilon\sin\varepsilon\sin\theta + \\
& - \sin\varepsilon\sin\alpha\cos\theta + \sin\varepsilon\cos\psi\sin\alpha\cos\theta + \cos\varepsilon\cos\psi\sin\theta + \sin\psi\cos\alpha\cos\theta) + \\
& + R_f(\cos\varepsilon\sin\varepsilon\sin\theta - \sin^2\varepsilon\cos\psi\sin\alpha\cos\theta - \cos\varepsilon\sin\varepsilon\cos\psi\sin\theta + \\
& - \cos^2\varepsilon\sin\alpha\cos\theta - \sin\varepsilon\sin\psi\cos\alpha\cos\theta).
\end{aligned} \tag{9}$$

By these expression we can plot the position of the front wheel contact point. In particular, in figure 6 and 7 we have the plot of such position along the X-coordinate and the Y-coordinate, respectively. We assume $x = y = 0$ for the rear wheel contact point, and also $\theta = 0$. The others parameters are equal to those chosen in the last section.

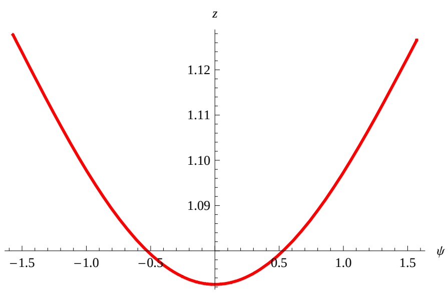

Figure 6: Plot of z as function of ψ, with $\alpha = \frac{\pi}{6}$.

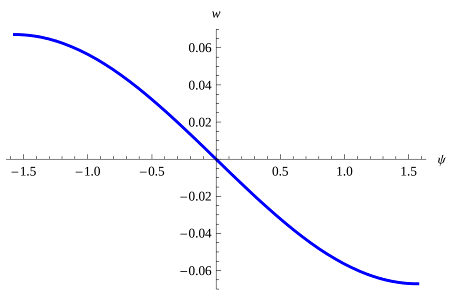

Figure 7: Plot of w as function of ψ, with $\alpha = \frac{\pi}{6}$.

Whereas, in figure 8 we have the front wheel contact point coordinates, so that it is clear how the point moves while the steering angle is changing.

6 THE ANGULAR VELOCITIES

We now turn to write the angular velocities for the four bodies of the bicycle. In order to obtain the right expressions, we use the following relation

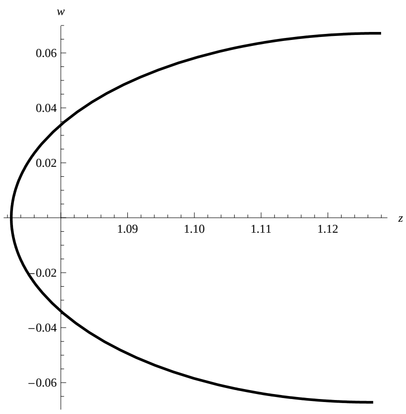

Figura 8: Position of the front contact point, with $\alpha = \frac{\pi}{6}$ and $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\vec{\omega}_S = \left(\frac{d\vec{\jmath}_S}{dt} \wedge \vec{k}_S\right)\vec{\imath}_S + \left(\frac{d\vec{k}_S}{dt} \wedge \vec{\imath}_S\right)\vec{\jmath}_S + \left(\frac{d\vec{\imath}_S}{dt} \wedge \vec{\jmath}_S\right)\vec{k}_S, \tag{10}$$

where the local unit vectors are obtain by the rotation matrices introduced before. In particular, as anticipated, the rotation matrices for the rear and front wheels are given by pre-multiplying the matrices $\mathcal{R}_1$ and $\mathcal{R}_3$ respectively by a matrix $\mathcal{R}_y(\chi)$ and a matrix $\mathcal{R}_y(\chi_a)$.

So, applying relation (10), we have

$$\vec{\omega}_{1S} = \begin{pmatrix} -\dot{\alpha} \\ \dot{\chi} - \dot{\theta}\sin\alpha \\ \dot{\theta}\cos\alpha \end{pmatrix} \qquad \vec{\omega}_{2S} = \begin{pmatrix} -\dot{\alpha}\cos\mu - \dot{\theta}\cos\alpha\sin\mu \\ \dot{\mu} - \dot{\theta}\sin\alpha \\ \dot{\theta}\cos\alpha\cos\mu - \dot{\alpha}\sin\mu \end{pmatrix}$$

$$\vec{\omega}_{3S} = \begin{pmatrix} -\dot{\alpha}\cos\varepsilon\cos\psi + \dot{\theta}(\sin\varepsilon\cos\psi\cos\alpha - \sin\psi\sin\alpha) \\ -\dot{\alpha}\cos\varepsilon\sin\psi - \dot{\theta}(\sin\varepsilon\sin\psi\cos\alpha + \cos\psi\sin\alpha) \\ \dot{\theta}\cos\varepsilon\cos\alpha + \dot{\alpha}\sin\varepsilon + \dot{\psi} \end{pmatrix}$$

$$\vec{\omega}_{4S} = \begin{pmatrix} -\dot{\alpha}\cos\varepsilon\cos\psi + \dot{\theta}(\sin\varepsilon\cos\psi\cos\alpha - \sin\psi\sin\alpha) \\ -\dot{\alpha}\cos\varepsilon\sin\psi - \dot{\theta}(\sin\varepsilon\sin\psi\cos\alpha + \cos\psi\sin\alpha) + \dot{\chi}_a \\ \dot{\theta}\cos\varepsilon\cos\alpha + \dot{\alpha}\sin\varepsilon + \dot{\psi} \end{pmatrix},$$

where the rear wheel is the body 1, the rear frame the body 2, the front frame the body 3 and the front wheel the body 4.

7 THE NONHOLONOMIC CONSTRAINTS

The nonholonomic constraints associated with the rear wheel are expressed by

$$\begin{cases} \dot{x}\cos\theta + \dot{y}\sin\theta = R\dot{\chi} \\ \dot{x}\sin\theta - \dot{y}\cos\theta = 0 \end{cases}$$

while those associate with the front one are expressed by

$$\begin{cases} \dot{z}\cos\widetilde{\theta} + \dot{w}\sin\widetilde{\theta} = R\dot{\chi}_a \\ \dot{z}\sin\widetilde{\theta} - \dot{w}\cos\widetilde{\theta} = 0 \end{cases},$$

where $\dot{z}$ and $\dot{w}$ are obtained deriving expressions (8) and (9), and using the auxiliary angle introduced with the rotation matrix (5). In particular, we have the following equalities:

$$\sin\widetilde{\theta} = \frac{\cos\psi\cos\alpha\sin\theta + \sin\psi(\cos\varepsilon\cos\theta - \sin\varepsilon\sin\alpha\sin\theta)}{\sqrt{1 - \sin^2\varepsilon\sin^2\psi\cos^2\alpha - \cos^2\psi\sin^2\alpha - 2\sin\varepsilon\sin\psi\cos\psi\sin\alpha\cos\alpha}}$$

and

$$\cos\widetilde{\theta} = \frac{\cos\psi\cos\alpha\cos\theta - \sin\psi(\cos\varepsilon\sin\theta + \sin\varepsilon\sin\alpha\cos\theta)}{\sqrt{1 - \sin^2\varepsilon\sin^2\psi\cos^2\alpha - \cos^2\psi\sin^2\alpha - 2\sin\varepsilon\sin\psi\cos\psi\sin\alpha\cos\alpha}}.$$

As the constraints imposed on the rear wheel give a relation on $\dot{x}$ and $\dot{y}$, those associated with the front wheel give a relation on $\dot{\theta}$ and $\dot{\chi}_a$, after having expressed the auxiliary angles as function of the generalized coordinates. In particular, we obtain

$$\begin{cases} \dot{\theta} = h(\psi, \alpha, \mu, \dot{\chi}, \dot{\psi}, \dot{\alpha}, \dot{\mu}) \\ g(\psi, \alpha)\dot{\chi}_a = f(\psi, \alpha, \mu, \dot{\chi}, \dot{\psi}, \dot{\alpha}, \dot{\mu}, \dot{\theta}) \end{cases},$$

where these functions are completely nonlinear. However, we can observe that in conclusion we have four relations which allow to reduce the generalized velocities from seven to three. It is clear that we choose the natural velocities that can be normally controlled by the rider, i.e. the rear wheel velocity, the roll angle and the steering angle variations.

8 THE LAGRANGIAN FUNCTIONAL

Once obtained the relations necessary to express the coordinates of the centers of mass of the four body, we can write the kinetic energy of the system and its potential energy. Hence, let h be the coordinate of the front frame center of mass in the local reference frame, and let m_1, m_2, m_3 and m_4 be the masses of the four bodies. In addition to this, we introduce the inertia tensors of the bodies in order to write the part of the kinetic energy due to rotation:

$$\sigma(B) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_1 \end{pmatrix} \qquad \sigma(G) = \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_4 \end{pmatrix}$$
$$\sigma(E) = \begin{pmatrix} I_5 & 0 & 0 \\ 0 & I_6 & 0 \\ 0 & 0 & I_6 \end{pmatrix} \qquad \sigma(H) = \begin{pmatrix} I_7 & 0 & 0 \\ 0 & I_7 & 0 \\ 0 & 0 & I_8 \end{pmatrix}.$$

Finally, we can write the complete Lagrangian functional for a bicycle with trail and without offset.

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}\Big[&A(\dot{x}\cos\theta + \dot{y}\sin\theta)^2 + AR^2(\dot{\alpha}^2 + \dot{\theta}^2\sin^2\alpha) + B(\dot{\mu}^2 + \dot{\alpha}\sin^2\mu + \dot{\theta}^2\sin^2\alpha\sin^2\mu + \dot{\theta}^2\cos^2\mu) + m_4h^2 \\
&(\dot{\theta}^2\sin^2\epsilon + \dot{\theta}^2\cos^2\epsilon\sin^2\alpha + \dot{\alpha}^2\cos^2\epsilon)\Big] + AR\Big[-\dot{\alpha}\cos\alpha(\dot{x}\sin\theta - \dot{y}\cos\theta) - \dot{\theta}\sin\alpha(\dot{x}\cos\theta + \dot{y}\sin\theta)\Big] \\
&+ \Big[C(\dot{\mu}\cos\mu\sin\alpha + \dot{\alpha}\cos\alpha\sin\mu - \dot{\theta}\cos\mu) + m_4h(\dot{\theta}\sin\epsilon - \dot{\alpha}\cos\epsilon\cos\alpha)\Big](\dot{x}\sin\theta - \dot{y}\cos\theta) \\
&+ \Big[C(-\dot{\mu}\sin\mu + \dot{\theta}\sin\alpha\sin\mu) - m_4h\dot{\theta}\cos\epsilon\sin\alpha\Big](\dot{x}\cos\theta + \dot{y}\sin\theta) + D\sin\mu(\dot{\alpha}^2 - \dot{\theta}^2\sin^2\alpha + \dot{\mu}\dot{\theta}\sin\alpha) \\
&+ D\dot{\alpha}\dot{\theta}\cos\alpha\cos\mu + m_4hR(\dot{\alpha}^2\cos\epsilon - \dot{\theta}\dot{\alpha}\sin\epsilon\cos\alpha + \dot{\theta}^2\cos\epsilon\sin^2\alpha - B(\dot{\mu}\dot{\theta}\sin\alpha + \dot{\alpha}\dot{\theta}\cos\alpha\sin\mu\cos\mu) \\
&+ m_4hl\Big[\dot{\theta}\sin\epsilon(\dot{\alpha}\cos\alpha\sin\mu + \dot{\mu}\sin\alpha\cos\mu) + \cos\epsilon\sin\mu(\dot{\theta}\dot{\mu}\sin\alpha - \dot{\alpha}^2) - \dot{\theta}^2\cos\epsilon\sin\mu\sin^2\alpha \\
&+ \dot{\theta}\dot{\alpha}\cos\epsilon\cos\alpha\cos\mu - \dot{\theta}^2\sin\epsilon\cos\mu\Big] + m_4h^2\dot{\theta}\dot{\alpha}\cos\epsilon\sin\epsilon\cos\alpha + \frac{1}{2}(\dot{\alpha}^2 + \dot{\theta}^2\cos^2\alpha)I_1 + \frac{1}{2}(\dot{\theta}\sin\alpha + \dot{\chi})^2 I_2 \\
&+ \frac{1}{2}(\dot{\alpha}\cos\mu + \dot{\theta}\cos\alpha\sin\mu)^2 I_3 + \frac{1}{2}\Big[(\dot{\mu} - \dot{\theta}\sin\alpha)^2 + (\dot{\theta}\cos\alpha\cos\mu - \dot{\alpha}\sin\mu)^2\Big] I_4 \\
&+ \frac{1}{2}\Big[(\dot{\alpha}\cos\epsilon\cos\psi + \dot{\theta}\sin\psi\sin\alpha - \dot{\theta}\sin\epsilon\cos\psi\cos\alpha)^2 + (\dot{\alpha}\sin\epsilon + \dot{\theta}\cos\epsilon\cos\alpha + \dot{\psi})^2\Big] I_5 \\
&+ \frac{1}{2}\Big[(\dot{\alpha}\cos\epsilon\sin\psi - \dot{\theta}\sin\epsilon\sin\psi\cos\alpha - \dot{\theta}\cos\psi\sin\alpha + \dot{\chi}_a)^2\Big] I_6 + \frac{1}{2}\Big[(\dot{\alpha}^2\cos^2\epsilon + \dot{\theta}^2\sin^2\epsilon\cos^2\alpha \\
&+ \dot{\theta}^2\sin^2\alpha - 2\dot{\theta}\dot{\alpha}\sin\epsilon\cos\epsilon\cos\alpha)\Big] I_7 + \frac{1}{2}\Big[(\dot{\alpha}\sin\epsilon + \dot{\theta}\cos\epsilon\cos\alpha + \dot{\psi}\Big]^2 I_8 \\
&- AgR\cos\alpha + gC\cos\alpha\sin\mu - m_4gh\cos\epsilon\cos\alpha
\end{aligned}$$

where $A = m_1 + m_2 + m_3 + m_4$, $B = m_2\frac{l^2}{4} + m_3l^2 + m_4l^2$, $C = m_2\frac{l}{2} + m_3l + m_4l$, $D = m_2R\frac{l}{2} + m_3Rl + m_4Rl$.

It is clear that the expression given above is quite long and it is difficult to deal with it, even because we have to write also the constrained Lagrangian, imposing the constraints on the Lagrangian function.

9 THE SIMPLIFIED MODEL AND ITS EQUATIONS OF MOTION

In order to obtain the equations of motion of the bicycle, we assume that the caster angle is zero, i.e $\varepsilon = 0$, and thus also $\mu = 0$. This simplification allows us to write the equations of motion.

Fist of all, we introduce the following change of variables:

$$\begin{cases} \dot{x}\sin\theta - \dot{y}\cos\theta = v_\perp \\ \dot{x}\cos\theta + \dot{y}\sin\theta = v_r \end{cases},$$

so that we can rewrite the nonholonomic constraints as

$$\begin{cases} v_r = R\dot{\chi} \\ v_\perp = 0 \\ \dot{\theta} = \dfrac{R}{l}\dot{\chi}\tan\psi \\ \dot{\chi}_a = \dfrac{\dot{\chi}}{\cos\psi} \end{cases}. \tag{11}$$

Taking account of the simplified Lagrangian with $\epsilon = 0$, and imposing the constraints, the constrained Lagrangian has the following form.

$$\begin{aligned}
\mathcal{L}_c =& \frac{1}{2}\left[Av_r^2 + AR^2(\dot{\alpha}^2 + \frac{v_r^2}{l^2}\tan^2\psi\sin^2\alpha) + \left(B + m_4h^2\sin^2\alpha\right)\frac{v_r^2}{l^2}\tan^2\psi\right] + \frac{v_r^2}{l}\left(m_4h - AR\right)\tan\psi\sin\alpha \\
&+\frac{v_r}{l}\left(D + m_4hl\right)v_r\dot{\alpha}\tan\psi\cos\alpha + m_4hR\left(\dot{\alpha}^2 + \frac{v_r^2}{l^2}\tan^2\psi\sin^2\alpha\right) + \frac{1}{2}\left(\dot{\alpha}^2 + \frac{v_r^2}{l^2}\tan^2\psi\cos^2\alpha\right)I_1 \\
&+\frac{1}{2}\left(\frac{v_r}{l}\tan\psi\sin\alpha + \frac{v_r}{R}\right)^2 I_2 + \frac{1}{2}\dot{\alpha}^2 I_3 + \frac{1}{2}\frac{v_r^2}{l^2}\tan^2\psi I_4 + \frac{1}{2}\left[\left(\dot{\alpha}\cos\psi + \frac{v_r}{l}\tan\psi\sin\psi\sin\alpha\right)^2\right. \\
&\left.+\left(\frac{v_r}{l}\tan\psi\cos\alpha + \dot{\psi}\right)^2\right] I_5 + \frac{1}{2}\left(\dot{\alpha}\sin\psi - \frac{v_r}{l}\sin\psi\sin\alpha + \frac{v_r}{R\cos\psi}\right)^2 I_6 \\
&+\frac{1}{2}\left[(\dot{\alpha}^2 + \frac{v_r^2}{l^2}\tan^2\psi\sin^2\alpha\right] I_7 + \frac{1}{2}\left(\frac{v_r}{l}\tan\psi\cos\alpha + \dot{\psi}\right)^2 I_8 - AgR\cos\alpha - m_4gh\cos\alpha\,.
\end{aligned}$$

The equations of motion are obtained by the Lagrange equations for nonholonomic systems [5]:

$$\frac{d}{dt}\frac{\partial\mathcal{L}_c}{\partial\dot{r}^\alpha} - \frac{\partial\mathcal{L}_c}{\partial r^\alpha} + A^a_\alpha\frac{\partial\mathcal{L}_c}{\partial s^a} = -\frac{\partial\mathcal{L}}{\partial\dot{s}^b}B^b_{\alpha\beta}\dot{r}^\beta\,,$$

where the terms due to the matrix A follow from the Ehresmann connection, while the matrix B is representative of the curvature of the connection. In this case the Ehresmann connection is given by the relation

$$\begin{pmatrix}\dot{\theta}\\ v_\perp \\ \dot{\chi}_a\end{pmatrix} + \begin{pmatrix}0 & -\frac{\tan\psi}{l} & 0\\ 0 & 0 & 0\\ 0 & -\frac{1}{R\cos\psi} & 0\end{pmatrix}\begin{pmatrix}\dot{\alpha}\\ v_r\\ \dot{\psi}\end{pmatrix} = 0$$

and we have the following coefficients different from zero

$$A^1_2 = -\frac{\tan\psi}{l} \qquad A^3_2 = -\frac{1}{R\cos\psi}\,,$$

so that the only components of the curvature that don't vanish are

$$B^1_{23} = -\frac{1}{l\cos^2\psi} \qquad B^1_{32} = \frac{1}{l\cos^2\psi}$$
$$B^3_{23} = -\frac{\sin\psi}{R\cos^2\psi} \qquad B^3_{32} = \frac{\sin\psi}{R\cos^2\psi}.$$

Finally, after some boring calculations, the equations of motion are the following.
First equation:

$$\begin{aligned}
&\left[AR^2 + 2m_4hR + I_1 + I_3 + I_7 + I_5\cos^2\psi + I_6\sin^2\psi\right]\ddot{\alpha} + 2\sin\psi\cos\psi(I_6 - I_5)\,\dot{\psi}\,\dot{\alpha} \\
&+\frac{v_r}{l}\left[\left((D + m_4hl)\cos\alpha + \frac{l}{R}I_6\right)\frac{1}{\cos^2\psi} + 2\sin\psi\cos\psi\sin\alpha(I_6 - I_5)\right]\dot{\psi} + \frac{v_r}{l}\tan\psi\sin\alpha(I_5 + I_8)\dot{\psi} \\
&+\frac{\dot{v}_r}{l}\left[(D + m_4hl)\tan\psi\cos\alpha + \sin^2\psi\sin\alpha(I_5 - I_6) + \frac{l}{R}I_6\tan\psi\right] \\
&-\frac{v_r^2}{l^2}\left(AR^2 + 2m_4h^2 + 2m_4hR + I_1 - I_2 + I_5 - I_7 + I_8\right)\tan^2\psi\sin\alpha\cos\alpha
\end{aligned}$$

$$+\frac{v_r^2}{lR}\left(AR^2-m_4hR-I_2+I_6\right)\tan\psi\cos\alpha-\frac{v_r^2}{l^2}\tan^2\psi\sin^2\psi\sin\alpha\cos\alpha I_5-\frac{v_r^2}{l^2}\sin^2\psi\sin\alpha\cos\alpha I_6$$
$$-(AR+m_4h)g\sin\alpha=0\,.$$

Second equation:

$$\dot v_r\left(A+\frac{I_2}{R^2}+B+I_4+\sin^2\alpha\left(AR^2+m_4h^2+2m_4hR+I_2+I_5\sin^2\psi+I_7\right)+\cos^2\alpha\left(I_1+I_5+I_8\right)\right)\frac{\tan^2\psi}{l^2}$$
$$+\dot v_r\left(\frac{2}{lR}\tan\psi\sin\alpha\left(m_4hR+I_2-I_6-AR^2\right)+\frac{I_6}{l^2}\sin^2\psi\sin^2\alpha+\frac{I_6}{R^2\cos^2\psi}\right)$$
$$-\frac{v_r}{l^2}\left(\dot\psi\frac{\tan\psi}{\cos^2\psi}\sin^2\alpha+2\dot\alpha\tan^2\psi\sin\alpha\cos\alpha\right)\left(AR^2+m_4h^2+2m_4hR+I_2+I_5\sin^2\psi+I_7\right)$$
$$+2I_5\frac{v_r}{l^2}\dot\psi\tan\psi\sin^2\psi\sin^2\alpha+\frac{v_r}{l^2}\dot\psi\frac{\tan\psi}{\cos^2\psi}(B+I_4)+\frac{v_r}{lR}\dot\psi\frac{\sin\alpha}{\cos^2\psi}\left(m_4hR+I_2-I_6-AR^2\right)$$
$$+\frac{v_r}{lR}\dot\alpha\tan\psi\cos\alpha\left(m_4hR+I_2-I_6-AR^2\right)+\left(\ddot\alpha\cos\alpha-\dot\alpha^2\cos\alpha\right)\tan\psi\left(\frac{D}{l}+m_4h\right)$$
$$+\frac{v_r}{l^2}\left(\dot\psi\frac{\tan\psi}{\cos^2\psi}\cos^2\alpha-2\dot\alpha\tan^2\psi\cos\alpha\sin\alpha\right)(I_1+I_5+I_8)$$
$$+\left(\ddot\alpha\sin^2\psi\sin\alpha+2\dot\alpha\dot\psi\sin\psi\cos\psi\sin\alpha+\dot\alpha^2\sin^2\psi\cos\alpha\right)\frac{I_5-I_6}{l}$$
$$+\left(\ddot\psi\cos\alpha-\dot\alpha\dot\psi\sin\alpha\right)\tan\psi\frac{I_5+I_8}{l}+2I_6\frac{v_r}{l^2}\left(\dot\psi\cos\psi\sin\psi\sin^2\alpha+\dot\alpha\sin^2\psi\cos\alpha\sin\alpha\right)$$
$$+\frac{I_6}{R}\left(\ddot\alpha\tan\psi+\dot\alpha\dot\psi\frac{1}{\cos^2\psi}\right)+\frac{I_6}{R^2}v_r\dot\psi\frac{\tan\psi}{\cos^2\psi}$$
$$-\frac{I_6}{l^2}v_r\dot\psi\tan\psi\sin^2\alpha+\frac{I_6-I_5}{l}\dot\alpha\dot\psi\tan\psi\sin\alpha-\frac{I_6}{R}\dot\alpha\dot\psi\tan^2\psi+\frac{v_r}{Rl}I_6\dot\psi\tan^2\psi\sin^2\alpha\;=\;0\,.$$

Third equation:

$$(I_5+I_8)\left(\ddot\psi+\frac{\dot v_r}{l}\tan\psi\cos\alpha+2\frac{v_r\dot\psi}{l}\frac{\cos\alpha}{\cos^2\psi}-\frac{v_r\dot\alpha}{l}\tan\psi\sin\alpha\right)-2\left[\left(AR^2+m_4h^2+2m_4hR+I_7\right)\right.$$
$$\left.\sin^2\alpha-(B+I_4)\right]\frac{v_r^2}{l^2}\frac{\tan\psi}{\cos^2\psi}\sin^2\alpha+2\left(AR-m_4h\right)\frac{v_r^2}{l}\frac{\sin\psi}{\cos^2\psi}-2D\frac{v_r\dot\alpha}{l}\frac{\cos\alpha}{\cos^2\psi}$$
$$-2m_4hv_r\dot\alpha\frac{\cos\alpha}{\cos^2\psi}-2I_1\frac{v_r^2}{l^2}\frac{\cos^2\alpha}{\cos^2\psi}-2\left(R\tan\psi\sin\alpha+l\right)\frac{v_r^2}{lR}\frac{\sin\alpha}{\cos^2\psi}I_2+\left(\dot\alpha\cos\psi+\frac{v_r}{l}\tan\psi\sin\psi\sin\alpha\right)$$
$$\left(\dot\alpha\sin\psi-\frac{v_r}{l}\frac{\sin\psi}{\cos^2\psi}\sin\alpha-\frac{v_r}{l}\sin\psi\sin\alpha\right)I_5-\left(2\frac{v_r}{l}\tan\psi\cos\alpha+\dot\psi\right)\frac{v_r}{l}\frac{\cos\alpha}{\cos^2\psi}(I_5+I_8)$$
$$+\left(\dot\alpha\sin\psi-\frac{v_r}{l}\sin\psi\sin\alpha+\frac{v_r}{R\cos\psi}\right)\left(\frac{v_r}{l}\cos\psi\sin\alpha-\dot\alpha\cos\psi-\frac{v_r\tan\psi}{R\cos\psi}\right)I_6+\frac{v_r^2}{l^2}\frac{\tan\psi}{\cos^2\psi}\sin^2\alpha$$
$$\left(I_5\sin^2\psi+I_6\cos^2\psi\right)+\left(I_5-I_6-I_6\frac{v_r}{R}\tan\psi\right)\frac{v_r\dot\alpha}{l}\tan\psi\sin\alpha-I_6\frac{v_r^2}{Rl}\frac{\sin\alpha}{\cos^2\psi}+I_6\frac{v_r\dot\alpha}{R}\tan^2\psi$$
$$+I_6\frac{v_r^2}{R^2}\frac{\tan\psi}{\cos^2\psi}\;=\;0.$$

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