Some remarks on a system modeling the cosmic rays balance.

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Keywords: Cosmic rays, Evolution systems.

SUMMARY. We report a model constituted by three integro-differential equations for the densities of nucleons, high-energy mesons and low-energy mesons diffusing in the Earth's atmosphere after the collision of the primary cosmic rays with the air particles. We neglect the transformation of mass into energy, since it is usually considered not relevant, and we study the problem by referring to the theory of evolution equations in the Banach space $L^1(I, dE) \cap L^1(I, EdE)$, where E is the energy and $I \subset (0, \infty)$; the norms have the meaning of the total number of nucleons and of their total energy, respectively.

We solve the problem making some assumptions to ensure the existence, uniqueness and positivity of the solutions representing the three densities and to satisfy some physical requests, like positivity and homogeneity of functions, and to ensure the bound of the number of particles during the collisions.

We note that in literature one finds empirical expressions for the energy distribution functions and for the inverses of mean free paths which do not satisfy all of such assumptions.

1 INTRODUCTION

In this paper we report a model constituted by three integro-differential equations for the densities of nucleons, high-energy mesons and low-energy mesons diffusing in the Earth's atmosphere after the collision of the primary cosmic rays with the air particles.

Several systems of balance equations are reported in literature, see [1, 2, 3, 4], to describe the mentioned densities. The assumption that the secondary particles, produced after the collisions in air of primary particles with air molecules, have a lower energy with respect to the incident ones holds (the transformation of mass into energy is considered not to be relevant).

The quoted authors treat a couple of transport equations, one for the nucleons and the other for the mesons. Say $E' \in (0, \infty)$ the energy of the incident particle and $E \in (0, E')$ the energy of the emerging particle, after a collision. Say $f(E, E')$ the energy distribution function, $\varphi(x, E')$ the density of the incident particles having energy E' at the atmospheric depth $x > 0$, and $\sigma(E')$ the inverse of mean free path of the incident particle having energy E'. Integrals having the structure $\int_E^\infty f(E, E')\sigma(E')\varphi(x, E')dE'$ represent the positive gain contributions to the densities due to those particles emerging, after a collision, with energy E. The negative loss contributions are due to collisions and/or decay.

The quoted authors give empirical expressions for the initial distribution of the nucleons, at the null atmospheric depth $x = 0$, i.e. it is assumed to be a given function of the form $N_0 E^{-2.7}$, and for the inverse of mean free path: $\sigma(E) = \sigma_0 E^a$ with $a = 0$ or $a = 0.06$, or rather $\sigma(E) = \sigma_0(1 + a \ln(E/\epsilon))$, E, ϵ in GeV, or $\sigma(E) = \sigma_0(1 + \overline{a} \ln^2(E/\overline{\epsilon}))$, $E, \overline{\epsilon}$ in GeV, see [2, 5]. The energy distribution functions $f(E, E')$ are assumed to be homogeneous of degree -1, so that the formulation is independent from the energy unit. Guessed expressions for the energy distributions $f(E, E')$ are given, see also [5, 6]. An example is given by the energy distribution for pions Π: $f_{N\Pi}(E, E') = \frac{5}{3}(\frac{E'}{A})^\alpha[1 - (\frac{E'}{A})^{\alpha'} \frac{E}{E'}]^4 \frac{1}{E}$ with α, $\alpha' \ll 1$, see [6].

In a previous work, see [7], we studied the quoted system of two transport equations allowing the transformation of mass into energy. Under suitable assumptions (the physical quantities were assumed essentially bounded, non-negative, and the energy distributions were assumed homogeneous of degree -1) we proved the existence, positivity and uniqueness of solutions for the densities of nucleons and mesons, referring to the theory of evolution problems in the Banach space $L^1((0,\infty), dE)$.

In this work we neglect the transformation of mass into energy, since it is usually considered not relevant, and we study the problem presented above by referring to the theory of evolution equations in the Banach space $L^1(I, dE) \cap L^1(I, EdE)$, with $I \subset (0, \infty)$. Both spaces $L^1(I, dE)$ and $L^1(I, EdE)$ are considered because of their physical meaning (i.e., the norms have the meaning of the total number of nucleons and of their total energy, respectively). Moreover, referring to [8], we consider two groups of mesons: the high energy ones, whose energy is $E \in (E_1, \infty)$, $E_1 > 0$, and the low energy ones, whose energy is $E \in (0, E_1)$. The aim of this distinction is to emphasize or disregard the importance of the decay of mesons. Therefore, the system is composed by three transport equations.

We make some assumptions to ensure the existence, uniqueness and positivity of the solutions representing the three densities and to satisfy some physical requests, like positivity and homogeneity of functions, and to ensure that the number of particles is bounded and the total energy does not increase during the collisions.

We note that in literature one finds empirical expressions for the energy distribution functions and for the inverse of mean free path which do not satisfy all of such assumptions, see for instance [5, 6].

2 THE MODEL

Let $N(x, E)$ be the density of nucleons, $M_h(x, E)$ the density of high energy mesons and $M_l(x, E)$ the density of low energy mesons, at the atmospherical depth $x > 0$. We recall that the atmospheric depth x is measured in g $\cdot$ cm^{-2}, the energy $E > 0$ in eV, the densities $N(x, E)$, $M_h(x, E)$ and $M_l(x, E)$ are measured in (eV)$^{-1}$cm^{-2}s^{-1}sr, the inverse of mean free path of nucleons $\sigma(E)$ and the one of mesons $\sigma_M(E)$ are measured in g^{-1}cm^2, $f_{ij}(E, E')$, with $i, j \in \{N, h, l\}$, are energy distribution functions, measured in (eV)$^{-1}$.

The integro-differential balance equations to be considered for $x > 0$ are written as follows:

$$\frac{\partial}{\partial x} N(x, E) = -\sigma(E) N(x, E) + \int_E^\infty f_{NN}(E, E')\sigma(E') N(x, E') dE', \tag{1}$$

for $E \in (0, \infty)$, with $N(x = 0, E) = G(E)$;

$$\begin{aligned} \frac{\partial}{\partial x} M_h(x, E) = &- \sigma_M(E) M_h(x, E) + \int_E^\infty f_{hh}(E, E')\sigma_M(E') M_h(x, E') dE' \\ &+ \int_E^\infty f_{Nh}(E, E')\sigma(E') N(x, E') dE', \end{aligned} \tag{2}$$

for $E \in (E_1, \infty)$, with $M_h(x = 0, E) = 0$;

$$\begin{aligned} \frac{\partial}{\partial x} M_l(x, E) = &- \sigma_M(E) M_l(x, E) - \frac{b}{Ex} M_l(x, E) + \int_E^{E_1} f_{ll}(E, E')\sigma_M(E') M_l(x, E') dE' \\ &+ \int_{E_1}^\infty f_{hl}(E, E')\sigma_M(E') M_h(x, E') dE' + \int_E^\infty f_{Nl}(E, E')\sigma(E') N(x, E') dE', \end{aligned} \tag{3}$$

for $E \in (0, E_1)$, with $M_l(x = 0, E) = 0$.

The second term in the right hand side in the third equation takes into account the decay of low energy mesons ($b > 0$ is a constant measured in eV), see [8].

We recall that we neglect the transformation of mass into energy, therefore the total energy does not increase. It is also physically consistent to assume that the total number of particles cannot explode to infinity. For physical reasons, we assume that all the densities of particles are nonnegative and we suppose that the inverse of mean free paths and the energy distributions functions are positive almost everywhere in their domain. The energy distribution functions are supposed to be homogeneous of degree -1. We list all these physical requests in Section 3 together with the mathematical ones.

3 THE SOLUTION OF THE EVOLUTION PROBLEM WHEN THE INVERSES OF MEAN FREE PATHS ARE ESSENTIALLY BOUNDED

We look at the system (1)-(3) together with the initial conditions like an evolution problem with respect to x in the Banach space $L^1(I, (1 + E)dE)$, with $I \subset (0, \infty)$. Say $J = (0, \infty)$, $J_h = (E_1, \infty)$ and $J_l = (0, E_1)$. We assume that

(1) $\sigma \in L^\infty(J)$, $\operatorname{ess\,sup}_{E \in J} |\sigma(E)| = \overline{\sigma}$, $\sigma(E) > 0$ a.e. $E \in J$;

(2) $\sigma_M \in L^\infty(J)$, $\operatorname{ess\,sup}_{E \in J} |\sigma_M(E)| = \overline{\sigma}_M$, $\sigma_M(E) > 0$ a.e. $E \in J$;

(3) f_{NN} homogeneous of degree -1, $f_{NN}(E, E') > 0$ a.e. $(E, E') \in (0, E') \times J$ and $\int_0^{E'} f_{NN}(E, E')dE \le c_{NN} < \infty$ a.e. $E' \in J$, $\int_0^{E'} f_{NN}(E, E')EdE \le E'$ a.e. $E' \in J$;

(4) f_{Nh} homogeneous of degree -1, $f_{Nh}(E, E') > 0$ a.e. $(E, E') \in J_h \times (E, \infty)$ and $\int_{E_1}^{E'} f_{Nh}(E, E')dE \le c_{Nh} < \infty$ a.e. $E' \in J_h$;

(5) f_{Nl} homogeneous of degree -1, $f_{Nl}(E, E') > 0$ a.e. $(E, E') \in J_l \times (E, \infty)$ and $\int_0^{E_1} f_{Nl}(E, E')dE \le c_{Nl} < \infty$ a.e. $E' \in J_h$ and $\int_0^{E'} f_{Nl}(E, E')dE \le c'_{Nl} < \infty$ a.e. $E' \in J_l$;

(6) f_{hh} homogeneous of degree -1, $f_{hh}(E, E') > 0$ a.e. $(E, E') \in (E_1, E') \times J_h$ and $\int_{E_1}^{E'} f_{hh}(E, E')dE \le c_{hh} < \infty$ a.e. $E' \in J_h$;

(7) f_{ll} homogeneous of degree -1, $f_{ll}(E, E') > 0$ a.e. $(E, E') \in (0, E') \times J_l$ and $\int_0^{E'} f_{ll}(E, E')dE \le c_{ll} < \infty$ a.e. $E' \in J_l$;

(8) f_{hl} homogeneous of degree -1, $f_{hl}(E, E') > 0$ a.e. $(E, E') \in J_l \times J_h$ and $\int_0^{E_1} f_{hl}(E, E')dE \le c_{hl} < \infty$ a.e. $E' \in J_h$.

Remark 1. Integrating both members of (1) with weight E, we deduce that $\int_0^{E'} f_{NN}(E, E')EdE \le E'$ a.e. $E' \in (0, \infty)$, since energy does not increases when x changes. This condition is physically consistent, but it is not actually used in the mathematical proofs.

Remark 2. The condition $\int_0^{E'} f_{NN}(E, E')dE \le c_{NN}$ a.e. $E' \in (0, \infty)$ represents the boundedness of the total number of nucleons produced as a consequence of the collision of a nucleon and it is used in the following mathematical treatment.

The following linear operators are bounded (here I denotes the identity operator):

(a) σI, with domain $L^1(J,(1+E)dE)$ and range in $L^1(J,(1+E)dE)$, and with norm $\|\sigma I\| \leq \operatorname{ess\,sup}_{E\in J}\sigma(E) \doteqdot \overline{\sigma}$;

(b) $\sigma_M I$, with domain $L^1(J_h,(1+E)dE)$ and range in $L^1(J_h,(1+E)dE)$, and with norm $\|\sigma_M I\| \leq \operatorname{ess\,sup}_{E\in J_h}\sigma_M(E) \doteqdot \overline{\sigma}_{Mh} \leq \overline{\sigma}_M$;

(c) $\sigma_M I$, with domain $L^1(J_l,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, and with norm $\|\sigma_M I\| \leq \operatorname{ess\,sup}_{E\in J_l}\sigma_M(E) \doteqdot \overline{\sigma}_{Ml} \leq \overline{\sigma}_M$.

(d) K_{NN}, with domain $L^1(J,(1+E)dE)$ and range in $L^1(J,(1+E)dE)$, acting as $K_{NN}N(E) = \int_E^\infty f_{NN}(E,E')\,\sigma(E')\,N(E')\,dE'$, with norm $\|K_{NN}\| \doteqdot k_{NN} \leq \overline{\sigma}\,c_{NN}$;

(e) K_{Nh}, with domain $L^1(J,(1+E)dE)$ and range in $L^1(J_h,(1+E)dE)$, acting as $K_{Nh}N(E) = \int_E^\infty f_{Nh}(E,E')\,\sigma(E')\,N(E')\,dE'$, with norm $\|K_{Nh}\| \doteqdot k_{Nh} \leq \overline{\sigma}\,c_{Nh}$;

(f) K_{Nl}, with domain $L^1(J,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $K_{Nl}N(E) = \int_E^\infty f_{Nl}(E,E')\sigma(E')N(E')dE'$, with norm $\|K_{Nl}\| \doteqdot k_{Nl} \leq \overline{\sigma}\overline{c}_{Nl}$, where $\overline{c}_{Nl} = \max\{c_{Nl}, c'_{Nl}\}$;

(g) K_{hh}, with domain $L^1(J_h,(1+E)dE)$ and range in $L^1(J_h,(1+E)dE)$, acting as $K_{hh}M_h(E) = \int_E^\infty f_{hh}(E,E')\,\sigma_M(E')\,M_h(E')\,dE'$ with norm $\|K_{hh}\| \doteqdot k_{hh} \leq \overline{\sigma}_{Mh}\,c_{hh}$;

(h) K_{ll}, with domain $L^1(J_l,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $K_{ll}M_l(E) = \int_E^{E_1} f_{ll}(E,E')\,\sigma_M(E')\,M_l(E')\,dE'$, with norm $\|K_{ll}\| \doteqdot k_{ll} \leq \overline{\sigma}_{Ml}\,c_{ll}$;

(i) K_{hl}, with domain $L^1(J_h,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $K_{hl}M_h(E) = \int_{E_1}^\infty f_{hl}(E,E')\,\sigma_M(E')\,M_h(E')\,dE'$, with norm $\|K_{hl}\| \doteqdot k_{hl} \leq \overline{\sigma}_{Mh}\,c_{hl}$.

Proofs of the above estimates are easily obtained, thus they are omitted. The positivity assumptions are not essential to obtain estimates, but they have physical meaning.

The theory of semigroups of bounded linear operators in Banach spaces allows us to state the following theorems for the abstract formulation of problem (1)-(3), see [9, 10, 11]:

Theorem 3. *The abstract differential problem*

$$\dot{N}(x) = (-\sigma I + K_{NN})\,N(x), \quad N(x=0) = G \in L^1(J,(1+E)dE) \tag{4}$$

has one and only one strongly continuously differentiable solution $[0,\infty) \ni x \mapsto N(x) \in L^1(J,(1+E)dE)$ *given by:*

$$N(x) = \sum_{n=0}^{\infty} \frac{x^n(-\sigma I + K_{NN})^n G}{n!}. \tag{5}$$

The solution is nonnegative if $G(E) \geq 0$ *a.e.* $E \in J$.

Theorem 4. *The abstract differential problem*

$$\dot{M}_h(x) = (-\sigma_M I + K_{hh})\,M_h(x) + K_{Nh}N(x), \quad M_h(x=0) = 0, \tag{6}$$

where $N(x)$ *is the solution of problem* (4)*, has one and only one strongly continuously differentiable solution* $[0,\infty) \ni x \mapsto M_h(x) \in L^1(J_h,(1+E)dE)$ *given by*

$$M_h(x) = \int_0^x Z_{hh}(x-r)K_{Nh}N(r)dr \tag{7}$$

where

$$Z_{hh}(x) \doteqdot \sum_{n=0}^{\infty} \frac{x^n(-\sigma_M I + K_{hh})^n}{n!}.$$

The solution is nonnegative if $G(E) \geq 0$ *a.e.* $E \in J$.

One has to consider the following operator:

(j) $\beta(x)$, $x > 0$, with domain $D(\beta(x)) \subset L^1(J_l, (1+E)dE)$ and range in $L^1(J_l, (1+E)dE)$, acting as $\beta(x)M_l(E) \doteqdot -\frac{b}{Ex} M_l(E)$, where $b > 0$ is a constant measured in eV. The domain $D(\beta(x)) \doteqdot D_\beta$ of this operator does not depend on $x > 0$; it is dense in $L^1(J_l, (1+E)dE)$ since continuous functions with compact support in J_l belong to D_β. Moreover $\beta(x)$ is a closed operator, since for $\lambda > 0$ the resolvent operator $[\lambda I - \beta(x)]^{-1}$, with domain $L^1(J_l, (1+E)dE)$, has norm $\left\|[\lambda I - \beta(x)]^{-1}\right\| \leq \lambda^{-1}$ (a similar proof can be found in [7]). Therefore, $\beta(x) \in \mathcal{G}(1, 0; L^1(J_l, (1+E)dE))$.

Theorem 5. *The abstract differential problem*

$$\dot{M}_l(x) = [\beta(x) - \sigma_M I + K_{ll}]\, M_l(x) + K_{hl} M_h(x) + K_{Nl} N(x)\,, \quad M_l(0) = 0\,, \tag{8}$$

where $N(x)$ *is the solution of problem* (4) *and* M_h *is the one of problem* (6)*, has one and only one strongly continuously differentiable solution* $[0, \infty) \ni x \mapsto M_l(x) \in D_\beta$ *given by*

$$M_l(x) = \int_0^x V(x, r)\, [K_{hl} M_h(r) + K_{Nl} N(r)]\, dr\,. \tag{9}$$

where the family $\{V(x, s) : 0 \leq s \leq x\,,\ x > 0\}$ *is the evolution system generated by the linear unbounded operator* $\beta(x) - \sigma_M I + K_{ll}$ *in the space* $L^1(J_l, (1+E)dE)$ *and solves, for* $\varphi \in L^1(J_l, (1+E)dE)$*, the equation:*

$$V(x, s)\, \varphi = \mathcal{V}(x, s)\, \varphi + \int_s^x \mathcal{V}(x, r)\, K_{ll}\, V(r, s)\, \varphi\, dr,$$

where $\{\mathcal{V}(x, s)\}$ *is the evolution system generated by the unbounded linear operator* $\{\beta(x) - \sigma_M I\,;\ x > 0\}$ *and it is given by:*

$$\big(\mathcal{V}(x, s)\, \varphi\big)(E) = e^{-\sigma_M(E)\,(x-s)} \left(\frac{s}{x}\right)^{\frac{b}{E}} \varphi(E),$$

if $0 \leq s \leq x\ ,\ x > 0$.

The proof is analogous to that given in [7].

4 THE SOLUTION OF THE EVOLUTION PROBLEM WHEN THE INVERSES OF MEAN FREE PATHS ARE LOCALLY ESSENTIALLY BOUNDED

In literature one finds inverses of mean free paths that do not satisfy the hypotheses given in Section 3. For instance, in [2] one finds the following particular case in which $\sigma(E) = \sigma_0 E^a$, where σ_0 is positive, and drawings of graphics where $a = 0.03$, $a = 0.04$, $a = 0.06$ and $a = 0.10$. It is also foreseen the case $a = 0$. The best value is considered to be $a = 0.06$. In [5], one finds $\sigma(E) = \sigma_0(1 + a\ln(E/\epsilon))$ and graphics are drawn for $\sigma_0 = (1/96.4)\mathrm{g}^{-1}\mathrm{cm}^2$, $a = 0.027$, $\epsilon = 20$

GeV, E measured in GeV and for $\sigma_0 = (1/80)\mathrm{g}^{-1}\mathrm{cm}^2$, $a = 0.037$, $\epsilon = 20$ GeV, E measured in GeV. In the same article another expression for $\sigma(E)$ is given: $\sigma(E) = \sigma_0(1 + \overline{a}\ln^2(E/\overline{\epsilon}))$, where $\sigma_0 = (1/96.4)\mathrm{g}^{-1}\mathrm{cm}^2$, $\overline{a} = 4.975 \times 10^{-3}$, $\overline{\epsilon} = 10$ GeV, E measured in GeV. In [6] we find again a power-law function: $\sigma(E) = \sigma_0\,(E/B)^a$ where $\sigma_0 = (1/96.40)\ \mathrm{g}^{-1}\mathrm{cm}^2$, $B = 20$ GeV, $a = 0.027$ or, in a different energy region, the best fit for $\sigma(E)$ is for $\sigma_0 = (1/80)\ \mathrm{g}^{-1}\mathrm{cm}^2$, $B = 1$ TeV, $a = 0.06$.

It is clear that those functions do not satisfy assumptions in Section 3. As a consequence, we have to change hypotheses on σ and σ_M to obtain inverses of mean free paths agreeing with those presented in literature. We assume σ and σ_M belonging to $L^\infty_{loc}(J)$ for almost every $E \in J$. The assumptions on the energy distribution functions given in the previous Section still hold, and we also assume the following bounds: $\operatorname{ess\,sup}_{E'\in J}\left\{\frac{\sigma(E')}{1+E'}\int_0^{E'} f_{NN}(E,E')(1+E)dE\right\} \doteqdot \gamma < 1$; $\operatorname{ess\,sup}_{E'\in J_h}\left\{\frac{\sigma(E')}{1+E'}\int_{E_1}^{E'} f_{Nh}(E,E')(1+E)dE\right\} \doteqdot \gamma'$; $\operatorname{ess\,sup}_{E'\in J_h}\left\{\frac{\sigma_M(E')}{1+E'}\int_{E_1}^{E'} f_{hh}(E,E')(1+E)dE\right\} \doteqdot \gamma_h < 1$; $\operatorname{ess\,sup}_{E'\in J_l}\left\{\frac{\sigma_M(E')}{1+E'}\int_0^{E'} f_{ll}(E,E')(1+E)dE\right\} \doteqdot \gamma_l < 1$; $\operatorname{ess\,sup}_{E'\in J_h}\left\{\frac{\sigma_M(E')}{1+E'}\times \int_0^{E_1} f_{hl}(E,E')\,(1+E)\,dE\right\} \doteqdot \gamma_h'$; $\operatorname{ess\,sup}_{E'\in J_l}\left\{\frac{\sigma(E')}{1+E'}\int_0^{E'} f_{Nl}(E,E')\,(1+E)\,dE\right\} \doteqdot \gamma_l'$; $\operatorname{ess\,sup}_{E'\in J_h}\left\{\frac{\sigma(E')}{1+E'}\int_0^{E_1} f_{Nl}(E,E')\,(1+E)\,dE\right\} \doteqdot \gamma_l''$.

Consider the following linear operators:

(a') Σ, with domain $D(\Sigma) \doteqdot D \subset L^1(J,(1+E)dE)$ and range in $L^1(J,(1+E)dE)$, acting as $\Sigma N(E) \doteqdot \sigma(E)N(E)$;

(b') Σ_M, with domain $D_h(\Sigma_M) \doteqdot D_h \subset L^1(J_h,(1+E)dE)$ and range in $L^1(J_h,(1+E)dE)$, acting as $\Sigma_M M_h(E) \doteqdot \sigma_M(E)M_h(E)$;

(c') Σ_M, with domain $D_l(\Sigma_M) \doteqdot D_l \subset L^1(J_l,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $\Sigma_M M_l(E) \doteqdot \sigma_M(E)M_l(E)$;

(d') K_{NN}, with domain $D(K_{NN}) = D(\Sigma) \doteqdot D \subset L^1(J,(1+E)dE)$ and range in $L^1(J,(1+E)dE)$, acting as $K_{NN}N(E) = \int_E^\infty f_{NN}(E,E')\,\sigma(E')\,N(E')\,dE'$;

(e') K_{Nh}, with domain $D(K_{Nh}) = D(\Sigma) \doteqdot D \subset L^1(J,(1+E)dE)$ and range in $L^1(J_h,(1+E)dE)$, acting as $K_{Nh}N(E) = \int_E^\infty f_{Nh}(E,E')\,\sigma(E')\,N(E')\,dE'$;

(f') K_{Nl}, with domain $D(K_{Nl}) = D(\Sigma) \doteqdot D \subset L^1(J,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $K_{Nl}N(E) = \int_E^\infty f_{Nl}(E,E')\,\sigma(E')\,N(E')\,dE'$;

(g') K_{hh}, with domain $D(K_{hh}) = D_h(\Sigma_M) \doteqdot D_h \subset L^1(J_h,(1+E)dE)$ and range in $L^1(J_h,(1+E)dE)$, acting as $K_{hh}M_h(E) = \int_E^\infty f_{hh}(E,E')\,\sigma_M(E')\,M_h(E')\,dE'$;

(h') K_{ll}, with domain $D(K_{ll}) = D_l(\Sigma_M) \doteqdot D_l \subset L^1(J_l,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $K_{ll}M_l(E) = \int_E^{E_1} f_{ll}(E,E')\,\sigma_M(E')\,M_l(E')\,dE'$;

(i') K_{hl}, with domain $D(K_{hl}) = D_h(\Sigma_M) \doteqdot D_h \subset L^1(J_h,(1+E)dE)$ and range in $L^1(J_l,(1+E)dE)$, acting as $K_{hl}M_h(E) = \int_{E_1}^\infty f_{hl}(E,E')\,\sigma_M(E')\,M_h(E')\,dE'$.

***Lemma* 6.** *The operator $A \doteqdot -\Sigma + K_{NN}$ belongs to $\mathcal{G}(1,\gamma;L^1(J,(1+E)dE))$.*

Proof. The domain D of A is dense in $L^1(J,(1+E)dE)$ because continuous functions with compact support in J belong to D. Besides, for every $z>\gamma$ the operator $(zI-A)^{-1}$ from $L^1(J,(1+E)dE)$ to D exists, is linear and bounded and in particular it is such that: $\|(zI-A)^{-1}\| \leq \frac{1}{z-\gamma}$. Indeed, solving equation $(zI-A)N(E)=g(E)$ for every $g \in L^1(J,(1+E)dE)$ and $z>\gamma$, one obtains $N(E)=\frac{g(E)}{z+\sigma(E)}+\frac{1}{z+\sigma(E)}\int_E^\infty f_{NN}(E,E')\sigma(E')N(E')dE'$. Putting $KN(E)=\frac{1}{z+\sigma(E)}\int_E^\infty f_{NN}(E,E')\sigma(E')N(E')dE'$, one has $N(E)=\frac{g(E)}{z+\sigma(E)}+KN(E)$ with $\|KN(E)\| \leq \int_0^\infty \frac{1}{z+\sigma(E)}\int_E^\infty f_{NN}(E,E')\sigma(E')|N(E')|dE'(1+E)dE = \int_0^\infty \sigma(E')|N(E')|\frac{1+E'}{1+E'}dE'\int_0^{E'}\frac{1}{z+\sigma(E)}\times f_{NN}(E,E')(1+E)dE = \int_0^\infty|N(E')|(1+E')dE'\left\{\frac{\sigma(E')}{1+E'}\int_0^{E'}\frac{1}{z+\sigma(E)}f_{NN}(E,E')(1+E)dE\right\} \leq \frac{1}{z}\int_0^\infty|N(E')|(1+E')dE'\left\{\frac{\sigma(E')}{1+E'}\int_0^{E'}f_{NN}(E,E')(1+E)dE\right\} \leq \frac{1}{z}\int_0^\infty|N(E')|(1+E')dE' \times \operatorname{ess\,sup}_{E'\in J}\left\{\frac{\sigma(E')}{1+E'}\int_0^{E'}f_{NN}(E,E')(1+E)dE\right\} = \frac{1}{z}\gamma\int_0^\infty|N(E')|(1+E')dE' = \frac{1}{z}\gamma\|N(E)\| = \xi\|N(E)\|$ where one has put $\xi \doteqdot \frac{1}{z}\gamma$. Note that it results $\xi<\gamma<1$. Therefore, it is $\|K\| \leq \xi<\gamma<1$. Being $N(E)=\sum_{n=0}^\infty K^n\frac{g(E)}{z+\sigma(E)}$, one has $\|N(E)\| \leq \sum_{n=0}^\infty\|K\|^n\left\|\frac{g(E)}{z+\sigma(E)}\right\| \leq \sum_{n=0}^\infty\xi^n\left\|\frac{g(E)}{z+\sigma(E)}\right\| = \frac{1}{1-\xi}\left\|\frac{g(E)}{z+\sigma(E)}\right\| \leq \frac{1}{1-\xi}\frac{\|g(E)\|}{z} = \frac{1}{z-\gamma}\|g(E)\|$ for every $g \in L^1(J,(1+E)dE)$ and $z>\gamma$. Therefore $\|(zI-A)^{-1}g(E)\| \leq \frac{1}{z-\gamma}\|g(E)\|$ for every $g \in L^1(J,(1+E)dE)$ and $z>\gamma$. To conclude, it also implies that the operator A is closed and so A belongs to $\mathcal{G}(1,\gamma;L^1(J,(1+E)dE))$. □

Referring to the theory of semigroups of linear operators in Banach spaces, see [9, 10, 11], the following theorem can be stated:

Theorem 7. *The abstract differential problem*

$$\dot{N}(x)=(-\Sigma+K_{NN})\,N(x), \quad N(x=0)=G\in D \tag{10}$$

has one and only one strongly continuously differentiable solution $[0,\infty)\ni x\mapsto N(x)\in D$ *given by:*

$$N(x)=\mathcal{Z}(x)G, \tag{11}$$

where

$$\mathcal{Z}(x)\doteqdot \mathrm{e}^{\gamma x}\lim_{n\to\infty}\left\{\left(I-\frac{x}{n}A_1\right)^{-1}\right\}^n$$

is the semigroup generated by $A\doteqdot(-\Sigma+K_{NN})\in\mathcal{G}(1,\gamma;L^1(J,(1+E)dE))$*, being* $A_1\doteqdot A-\gamma I$*. The solution is nonnegative if* $G(E)\geq 0$ *a.e.* $E\in J$*.*

Similarly, one has (the proof is analogous to that given for Lemma 6):

***Lemma* 8.** *The operator* $A_h\doteqdot-\Sigma_M+K_{hh}$ *belongs to* $\mathcal{G}(1,\gamma_h;L^1(J_h,(1+E)dE))$*.*

Theorem 9. *The abstract differential problem*

$$\dot{M}_h(x)=(-\Sigma_M+K_{hh})\,M_h(x)+K_{Nh}N(x), \quad M_h(x=0)=0, \tag{12}$$

where $N(x)$ *is the solution of problem* (10)*, has a unique strict solution* $[0,\infty)\ni x\mapsto M_h(x)\in D_h$ *given by*

$$M_h(x)=\int_0^x Z_h(x-r)K_{Nh}N(r)dr \tag{13}$$

where

$$Z_h(x) \doteqdot \mathrm{e}^{\gamma_h x} \lim_{n\to\infty} \left\{ \left(I - \frac{x}{n} A_2 \right)^{-1} \right\}^n$$

and $A_2 \doteqdot A_h - \gamma_h I$.

Proof. It follows from Lemma 8 (see [9]). In addition, being (12) a non-homogeneous evolution problem, one has to prove that the source term $K_{Nh}N(x)$ is strongly continuous. To prove the strong continuity, one evaluates the norm: $\|K_{Nh}N(x+t) - K_{Nh}N(x)\| \le \int_{E_1}^{\infty}\int_{E}^{\infty} f_{Nh}(E,E') \times \sigma(E')\,|N(x+t,E') - N(x,E')|\; dE'(1+E)dE \le \int_{E_1}^{\infty} |N(x+t,E') - N(x,E')|\,(1+E')dE' \times \operatorname{ess\,sup}_{E'\in J_h} \left\{ \frac{\sigma(E')}{1+E'} \int_{E_1}^{E'} f_{Nh}(E,E')(1+E)dE \right\} \le \gamma' \|N(x+t) - N(x)\|$ which goes to 0 for $t \to 0$, being N strongly continuous (because it is the strict solution of (10)). $\square$

Consider again the operator given in (j). Note that $D_\beta \cap D_l \neq \emptyset$. Following the same steps of [7], one can prove the following theorem:

Theorem 10. *The abstract differential problem*

$$\dot{M}_l(x) = [\beta(x) - \sigma_M I + K_{ll}]\, M_l(x) + K_{hl} M_h(x) + K_{Nl} N(x)\,, \quad M_l(0) = 0\,, \qquad (14)$$

where $N(x)$ is the solution of problem (10) and M_h is the one of problem (12), has one and only one strongly continuously differentiable solution $[0,\infty) \ni x \mapsto M_l(x) \in D_\beta \cap D_l$ given by

$$M_l(x) = \int_0^x V(x,r)\,[K_{hl} M_h(r) + K_{Nl} N(r)]\, dr\,, \qquad (15)$$

where the family $\{V(x,s)\;:\; 0 \le s \le x\,,\;\; x > 0\}$ is the evolution system generated by the linear unbounded operator $\beta(x) - \Sigma_M + K_{ll}$ in the space $L^1(J_l,(1+E)dE)$ and solves, for $\varphi \in L^1(J_l,(1+E)dE)$, the equation:

$$V(x,s)\,\varphi = \mathcal{V}(x,s)\,\varphi + \int_s^x \mathcal{V}(x,r)\, K_{ll}\, V(r,s)\,\varphi\, dr,$$

where $\{\mathcal{V}(x,s)\}$ is the evolution system generated by the unbounded linear operator $\{\beta(x) - \Sigma_M\;;\; x > 0\}$ and it is given by:

$$\big(\mathcal{V}(x,s)\,\varphi\big)(E) = e^{-\sigma_M(E)\,(x-s)} \left(\frac{s}{x}\right)^{\frac{b}{E}} \varphi(E)\,,$$

if $0 \le s \le x\;,\; x > 0$.

5 CONCLUSIONS

In literature we find several expressions for the inverses of mean free paths and for the energy distribution functions. Astrophysicians evince such expressions from the experimental data: they look for those mathematical expressions that best fit the experimental data. We report some of those expressions. For instance, in [2] it is reported the particular case in which $G(E) = G_0 E^{-2.7}$ where G_0 is a positive constant. It is clear that this initial datum does not belong to $L^1(J,(1+E)dE)$. In [2] it is also assumed that the inverse of mean free path for nucleons is $\sigma(E) = \sigma_0 E^a$, where σ_0 is positive, and the best fit is for $a = 0.06$. This function does not agree with the assumptions of

Section 3. In [2] the energy distribution function for nucleons is $f(E,E') = (1+\beta)\frac{1}{E'}(1-\frac{E}{E'})^{\beta}$. This function, for $\beta \geq 0$, satisfies the hypotheses given in Section 3, but one cannot say the same for the energy distribution functions for mesons: for instance, in [2] the energy distribution function for pions is: $f_{N\Pi}(E,E') = 1.04\frac{E'-E}{E'E}\exp\left(-5\frac{E}{E'}\right)$ and it does not satisfy all the assumptions in Section 3. Now consider the hypotheses of Section 4: in [2] σ is locally essentially bounded on J but the bound requested for $\operatorname{ess\,sup}_{E'\in J}\left\{\frac{\sigma(E')}{1+E'}\int_0^{E'} f(E,E')(1+E)dE\right\} \doteq \gamma < 1$ does not hold. It is clear that using functions as given in [2], one does not obtain solutions in $L^1(I,(1+E)dE)$, with $I = J, J_h, J_l$.

In [5], it is suggested that $\sigma(E) = \sigma_0(1 + a\ln(E/\epsilon))$ and graphics are drawn for $\sigma_0 = (1/96.4)$ g^{-1}cm^2, $a = 0.027$, $\epsilon = 20$ GeV, E measured in GeV and for $\sigma_0 = (1/80)$g^{-1}cm^2, $a = 0.037$, $\epsilon = 20$ GeV, E measured in GeV. In the same article another expression for $\sigma(E)$ is given: $\sigma(E) = \sigma_0(1 + \overline{a}\ln^2(E/\overline{\epsilon}))$, where $\sigma_0 = (1/96.4)$g^{-1}cm^2, $\overline{a} = 4.975\times 10^{-3}$, $\overline{\epsilon} = 10$ GeV, E measured in GeV. The functions $G(E)$ and $f(E,E')$ are the same given in [2]. These functions do not satisfy our assumptions.

In [6], we find again a power-law function: $\sigma(E) = \sigma_0\,(E/B)^a$ where $\sigma_0 = (1/96.40)$ g^{-1}cm^2, $B = 20$ GeV, $a = 0.027$ or, in a different energy region, the best fit for $\sigma(E)$ is for $\sigma_0 = (1/80)$ g^{-1}cm^2, $B = 1$ TeV, $a = 0.06$. For the energy distributions, the following function is suggested: $f_{N\Pi}(E,E') = \frac{5}{3}(\frac{E'}{A})^{\alpha}[1-(\frac{E'}{A})^{\alpha'}\frac{E}{E'}]^4\frac{1}{E}$ where A is the pion production normalization energy, and α, $\alpha' \ll 1$. These functions do not satisfy our assumptions either.

On the other hand, we think that the request we made for solutions being in $L^1(I,(1+E)dE)$, with $I = J, J_h, J_l$, is physically consistent and it is a natural request. Using functions as given in the quoted articles, one does not obtain solutions in $L^1(I,(1+E)dE)$, with $I = J, J_h, J_l$. Such functions could be used if we considered energy sets having the following form: $J' = [E_{min}, E_{max}]$ with $E_{min} > 0$ and $E_{max} < \infty$, but in this way one would exclude low and high energies. Such an assumption seems to be reductive.

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