Some General Theorems of Incremental Thermoelectroelasticity

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SUMMARY. From [1] we present a uniqueness theorem for the solutions to the initial boundary value problem, a generalized Hamilton principle, and a theorem of reciprocity of work for incremental thermoelectroelasticity with initial fields.

1 INTRODUCTION

The increasing wide use in sensing and actuation has attracted much attention towards theories about materials exhibiting couplings between elastic, electric, magnetic and thermal fields.

Nowacki [2, 3] proved a uniqueness theorem for the solutions of the initial boundary value problems, a generalized Hamilton principle and a theorem of reciprocity of work, in linear thermopiezoelectricity referred to a natural state, i.e., with no initial fields.

Li [4] generalized the uniqueness and reciprocity theorems for linear thermo-electro-magneto-elasticity referred to a natural state.

Aouadi [5] established a reciprocal theorem for a linear generalized theory of thermo-magnetoelectroelasticity, referred to a natural configuration,with a thermal relaxation time.

Iesan [6] uses the Green-Naghdi theory of continuum thermomechanics to derive a linear theory of thermoelasticity with internal structure where in particular a uniqueness result holds.

Related works on thermoelasticity and thermoelectromagnetism can be found e.g. [7] to [11].

The classical linear theory of thermopiezoelectricity assumes infinitesimal deviations of the field variables from a reference state, where there are no initial mechanical and electric fields. In order to describe the response of thermoelectroelastic materials in the presence of initial fields one needs the theory for infinitesimal fields superposed on initial fields, and this can only be derived from the fully nonlinear theory of thermoelectroelasticity. Yang [13] derived, from the equations of nonlinear thermoelectroelasticity given in Tiersten [12], the equations for infinitesimal incremental fields superposed on finite biasing fields in a thermoelectroelastic body with no assumption on the biasing fields.

Here we present the results of Montanaro [1], where the aforementioned three Nowacki's theorems [2], [3] are extended to incremental thermoelectroelasticity with initial fields.

The incremental theory [13] is used here, and we rewrite from this paper, with the same notations, some formulae and results on constitutive equations of incremental thermoelectroelasticity.

In the uniqueness theorem below we assume that in the initial state entropy does not depend on time and temperature is uniform. For the theorem of reciprocity of work below we assume that in the initial state both entropy and temperature fields do not depend on time.

2 EQUATIONS OF NONLINEAR THERMOELECTROELASTICITY

2.1 *Balance laws and constitutive equations*

The thermoelectroelastic body under consideration in the reference configuration occupies a region V with boundary surface S. Its motion is described by

$$y_i = y_i(X_L,\, t)\,,$$

where y_i denotes the present coordinates and X_L the reference coordinates of material points with respect to the same Cartesian coordinate system.

Let K_{Lj}, ρ_o, f_j, Δ_L, ρ_E, θ, η, Q_L and γ respectively denote the first Piola-Kirchoff stress tensor, the mass density in the reference configuration, the body force per unit mass, the reference electric displacement vector, the free charge density per unit undeformed volume, the absolute temperature, the entropy per unit mass, the reference heat flux vector, and the body heat source per unit mass. Then we have the following equations of motion, electrostatics, and heat conduction written in material form with respect to the reference configuration:

$$K_{Li,L} + \rho_o f_i = \rho_o \ddot{y}_i\,, \tag{1}$$

$$\Delta_{L,L} = \rho_E\,, \tag{2}$$

$$\rho_o \theta \dot{\eta} = -Q_{L,L} + \rho_o \gamma\,, \tag{3}$$

The above equations are adjoined by constitutive relations defined by the specification of the free energy ψ and heat flux Q_L:

$$\psi = \psi(E_{MN},\, W_M,\, \theta)\,, \qquad Q_L = Q_L(E_{MN},\, W_M,\, \theta,\, \Theta_M) \tag{4}$$

where

$$E_{MN} = (y_{j,\,M} y_{j,\,N} - \delta_{MN})/2\,, \qquad W_M = -\phi_{,\,M}\,, \qquad \Theta_M = \theta_{,\,M} \tag{5}$$

are the finite strain tensor, the reference electric potential gradient, and the reference temperature gradient; of course, δ_{MN} is the Kronecker delta, and ϕ is the electric potential. Hence, by using ψ the constitutive relations (4) of [13] are deduced for K_{Li}, Δ_L, η; here we rewrite them from [13]:

$$\begin{aligned} K_{Li} &= y_{i,\,A} \rho_o \frac{\partial \psi}{\partial E_{AL}} + J X_{L,\,j}\, \varepsilon_o (E_j E_i - \frac{1}{2} E_i E_i \delta_{ji})\,, \\ \Delta_L &= \varepsilon_o J X_{L,\,j} E_j - \rho_o \frac{\partial \psi}{\partial W_L}\,, \qquad \eta = -\frac{\partial \psi}{\partial \theta}\,, \end{aligned} \tag{6}$$

with $E_i = -\phi_{,\,i}$. Recall that the heat-flux constitutive relation $(4)_2$ is restricted by

$$Q_L \Theta_L \leq 0\,. \tag{7}$$

Note that, in particular, $(4)_2$ includes the case in which Q_M is linear in Θ_L, that is,

$$Q_M = -\kappa_{ML}(\theta,\, W_A)\, \Theta_L\,. \tag{8}$$

2.2 *The initial boundary value problem for a thermoelectroelastic body*

To describe the corresponding boundary conditions to add to the field equations (1)-(3), three partitions (S_{i1}, S_{i2}), $i = 1, 2, 3$, of the boundary surface $S = \partial\mathcal{B}$ can be assigned. For mechanical boundary conditions, deformation $\tilde{y}_i$ and traction $\tilde{t}_i$ per unit undeformed area are prescribed, respectively, on S_{11} and S_{12}; for electric boundary conditions, electric potential $\tilde{\phi}$ and surface-free charge $\tilde{\Delta}$ per unit undeformed area are prescribed, respectively, on S_{21} and S_{22}; while for thermic boundary conditions, temperature $\tilde{\theta}$ and normal heat flux $\tilde{Q}$ per unit undeformed area are prescribed, respectively,on S_{31} and S_{32}. Hence, we can write

$$y_i = \tilde{y}_i \quad \text{on} \quad S_{11}\,, \qquad K_{Li}N_L = \tilde{K}_i \quad \text{on} \quad S_{12} \qquad ('\text{mechanical}')\,, \tag{9}$$

$$\phi = \tilde{\phi} \quad \text{on} \quad S_{21}\,, \qquad \Delta_L N_L = -\tilde{\Delta} \quad \text{on} \quad S_{22}\,, \qquad ('\text{electric}') \tag{10}$$

$$\theta = \tilde{\theta} \quad \text{on} \quad S_{31}\,, \qquad Q_L N_L = \tilde{Q} \quad \text{on} \quad S_{32} \qquad ('\text{thermic}')\,, \tag{11}$$

where $\mathbf{N} = (N_L)$ is the unit exterior normal on S and

$$S_{i1} \cup S_{i2} = S\,, \quad S_{i1} \cap S_{i2} = \emptyset \quad (i = 1, 2, 3)\,. \tag{12}$$

Let

$$\mathcal{A} := \Big(f_i,\, \rho_E,\, \gamma,\, \tilde{y}_i,\, \tilde{K}_i,\, \tilde{\phi},\, \tilde{\Delta},\, \tilde{\theta},\, \tilde{Q} \Big)\,. \tag{13}$$

The initial boundary value problem is then stated as: given an external action $\mathcal{A}$, to find the solution $(\phi,\, \theta,\, y_i)$ in $\mathcal{B}$ to the constitutive relations (6) and field equations (1)-(3), which satisfies the boundary conditions (9)-(11) and assigned initial conditions.

3 INITIAL AND INCREMENTAL FIELDS

In incremental theories three configurations are distinguished: the reference, initial and present configuration.

In the reference state the body is undeformed and free of all fields. A generic point at this state is denoted by **X** with rectangular coordinates X_N. The mass density in the reference configuration is denoted by ρ_o.

In the initial state the body is deformed finitely under the action of a prescribed initial action. The position of the material point associated with $\mathbf{X}$ is given by $y^o_\alpha = y^o_\alpha(\mathbf{X},\, t)$, with the Jacobian of the initial configuration denoted by $J_o = det(y^o_{\alpha,\,L})$. The initial fields

$$y^o_\alpha = y^o_\alpha(\mathbf{X},\, t), \quad \phi^o = \phi^o(\mathbf{X},\, t), \quad \theta^o = \theta^o(\mathbf{X},\, t) \tag{14}$$

satisfy the equations of nonlinear thermoelectroelasticity (1)-(11) under the prescribed action. The electric potential, electric field and temperature field are denoted by $\phi^o(\mathbf{X},\, t)$, $W^o_\alpha = -\phi^o_{,\alpha}$ and $\theta^o(\mathbf{X},\, t)$, respectively. The solution to the initial state problem is assumed known.

To the deformed body at the initial configuration, infinitesimal deformations, electric, and thermal fields are applied. The present position of the material point associated with $\mathbf{X}$ is given by $y_i(\mathbf{X},\, t)$, with electric potential $\phi(\mathbf{X},\, t)$ and temperature $\theta(\mathbf{X},\, t)$.

The fields $y_i(\mathbf{X},\, t)$, $\phi(\mathbf{X},\, t)$, $\theta(\mathbf{X},\, t)$ satisfy (1) under the action of the external action (13).

Let ε be a small number. The incremental process $\varepsilon(y^1,\, \phi^1,\, \theta^1)$ for $(y,\, \phi,\, \theta)$ superposed to the initial process $(y^o,\, \phi^o,\, \theta^o)$ is assumed to be infinitesimal and, therefore, we write:

$$y_i = \delta_{i\alpha}(y^o_\alpha + \varepsilon y^1_\alpha)\,, \qquad \phi = \phi^o + \varepsilon\phi^1\,, \qquad \theta = \theta^o + \varepsilon\theta^1\,, \tag{15}$$

Corresponding to (15), any other quantity $\mathcal{Q}$ in the present state can be written as

$$\mathcal{Q} \cong \mathcal{Q}^o + \varepsilon \mathcal{Q}^1 \,, \tag{16}$$

where, due to nonlinearity, higher powers of ε may arise. We want to derive equations governing the incremental process $\left(\mathbf{u} := \mathbf{y}^1 \,,\; \phi^1 \,,\; \theta^1\right)$. From (15) and (16), we can further write:

$$E_{KL} \cong E^o_{KL} + \varepsilon E^1_{KL} \,, \quad W_L \cong W^o_L + \varepsilon W^1_L \,, \quad \Theta_L \cong \Theta^o_L + \varepsilon \Theta^1_L \,, \tag{17}$$

where

$$\begin{aligned} E^o_{KL} = (y^o_{\alpha,\,K} y^o_{\alpha,\,L} - \delta_{KL})/2 \,, \qquad E^1_{KL} = (y^o_{\alpha,\,K} y^1_{\alpha,\,L} + y^o_{\alpha,\,L} y^1_{\alpha,\,K})/2 \,, \\ W^o_L = -\phi^o_{,L} \,, \qquad W^1_L = -\phi^1_{,L} \,, \qquad \Theta^o_L = \theta^o_{,L} \,, \qquad \Theta^1_L = \theta^1_{,L} \,. \end{aligned} \tag{18}$$

Substituting (15)-(18) into the constitutive relations (1), Yang (2003) obtains the expressions:

$$K_{Mi} \cong \delta_{i\alpha}(K^o_{M\alpha} + \varepsilon K^1_{M\alpha}) \,, \quad \Delta_M \cong \Delta^o_M + \varepsilon \Delta^1_M \,, \quad \eta \cong \eta^o + \varepsilon \eta^1 \,, \quad Q_M \cong Q^o_M + \varepsilon Q^1_M \,,$$

where

$$K^1_{M\alpha} = G_{M\alpha L\gamma} u_{\gamma,\,L} + R_{LM\alpha} \phi^1_{,L} - \rho_o \Lambda_{M\alpha} \theta^1 \,, \tag{19}$$

$$\Delta^1_M = R_{MN\gamma} u_{\gamma,\,N} - L_{MN} \phi^1_{,N} + \rho_o P_M \theta^1 \,, \tag{20}$$

$$\eta^1 = \Lambda_{M\gamma} u_{\gamma,\,M} - P_M \phi^1_{,M} + \alpha \theta^1 \,, \tag{21}$$

$$Q^1_M = -\kappa_{MN\alpha} u_{\alpha,\,N} - \kappa^E_{MN} \phi^1_{,N} - \kappa_M \theta^1 - \kappa_{MN} \theta^1_{,N} \,. \tag{22}$$

In (19)-(22), $G_{M\alpha L\gamma} = G_{L\gamma M\alpha}$ are the effective elastic constants, $R_{LM\alpha}$ are the effective piezoelectric constants, $\Lambda_{M\alpha}$ are the effective thermoelatic constants, $L_{MN} = L_{NM}$ are the effective dielectric constants, P_M are the effective pyrolectric constants, α is related with the specific heat. Their expressions are given by Eqs. (14), (15) in [13].

3.1 Restriction on the incremental heat flux

Condition (7) on the heat flux $(4)_2$, together with the condition $Q^o_L = 0 \quad \text{for} \quad \Theta^o_L = 0$, implies an analogous restriction on the incremental heat flux (22), that is,

$$Q^1_L \Theta^1_L \leq 0 \,. \tag{23}$$

3.2 Incremental field equations

By substituting (15)-(22) into (1), we find the governing equations for the incremental fields

$$K^1_{M\alpha,\,M} \,+\, \rho_o\, f^1_\alpha \,=\, \rho_o\, \ddot{u}_\alpha \,, \tag{24}$$

$$\Delta^1_{M,\,M} \,=\, \rho^1_E \,, \tag{25}$$

$$\rho_o\,(\theta^o \dot{\eta}^1 \,+\, \theta^1 \dot{\eta}^o) \,=\, -Q^1_{M,\,M} \,+\, \rho_o\, \gamma^1 \,. \tag{26}$$

Introducing the constitutive relations (19)-(22) into the incremental equations of motion (24), the equation of the electric field (25), and the heat equation (26), for $f^1_\alpha = 0$ we have

$$G_{M\alpha L\gamma} u_{\gamma,\,LM} + R_{LM\alpha} \phi^1_{,LM} - \rho_o \Lambda_{M\alpha} \theta^1_{,M} \,=\, \rho_o\, \ddot{u}_\alpha \,, \tag{27}$$

$$R_{MN\gamma}u_{\gamma,\,NM} - L_{MN}\phi^1_{,\,NM} + \rho_o P_M \theta^1_M \,=\, \rho^1_E\,, \tag{28}$$

$$\rho_o\theta^o\Big(\Lambda_{M\gamma}\dot u_{\gamma,\,M} - P_M\dot\phi^1_{,\,M} + \alpha\dot\theta^1\Big) + \rho_o\theta^1\dot\eta^o = \tag{29}$$

$$= \kappa^E_{MN}\phi^1_{,\,NM} + \kappa_M\theta^1_{,\,M} + \kappa_{MN}\theta^1_{,\,NM} + \kappa_{MN\alpha}u_{\alpha,\,NM} + \rho_o\gamma^1. \tag{30}$$

4 UNIQUENESS THEOREM OF THE SOLUTION OF THE INCREMENTAL DIFFERENTIAL EQUATIONS

In [1] the proof in [3] is generalized when there are initial fields with $\dot\eta^o = 0$ and $\Theta^o_L = 0$ (uniform initial temperature field θ^o). Substituting the constitutive relations (19) into the fundamental energy equation

$$\int_{V^o}\Big(f^1_\alpha - \rho_o\dot v_\alpha\Big)v_\alpha\,dV + \int_{S^o}\tilde K_\alpha\,v_\alpha\,dS \,=\, \int_{V^o}K^1_{M\alpha}\,\dot u_{\alpha,\,M}\,dV \tag{31}$$

we obtain

$$\int_{V^o}\Big(f^1_\alpha - \rho_o\dot v_\alpha\Big)v_\alpha\,dV + \int_{S^o}\tilde K_\alpha\,v_\alpha\,dS \,=\, \int_{V^o}\Big(G_{M\alpha L\gamma}u_{\gamma,\,L} + R_{LM\alpha}\phi^1_{,\,L} - \rho_o\Lambda_{M\alpha}\theta^1\Big)\,\dot u_{\alpha,\,M}\,dV\,, \tag{32}$$

thus

$$\frac{d}{dt}\Big(\mathcal{W}+\mathcal{K}\Big) \,=\, \int_{V^o}f^1_\alpha\,v_\alpha\,dV + \int_{S^o}\tilde K_\alpha\,v_\alpha\,dS + \int_{V^o}\Big(\rho_o\Lambda_{M\alpha}\theta^1 - R_{LM\alpha}\phi^1_{,\,L}\Big)\,\dot u_{\alpha,\,M}\,dV\,, \tag{33}$$

where $\mathcal{W}$ is the work of deformation and $\mathcal{K}$ is the kinetic energy:

$$\mathcal{W} \,=\, \frac{1}{2}\int_{V^o}G_{M\alpha L\gamma}\,u_{\alpha,\,M}\,u_{\gamma,\,L}\,dV\,, \qquad \mathcal{K} \,=\, \frac{1}{2}\int_{V^o}\rho_o\,v_\alpha v_\alpha\,dV\,. \tag{34}$$

Let

$$\mathcal{E} \,=\, \frac{1}{2}\,L_{KM}\int_{V^o}W^1_M W^1_K\,dV\,, \quad \mathcal{P} \,=\, \frac{\alpha}{2\theta^o}\int_{V^o}\rho_o\theta^1\,\theta^1\,dV\,, \quad \chi_\phi = \frac{\kappa^E_{ML}}{\theta^o}\int_{V^o}\theta^1_{,\,M}\phi^1_{,\,L}dV\,, \tag{35}$$

$$\chi = \frac{\kappa_M}{\theta^o}\int_{V^o}\theta^1_{,\,M}\theta^1\,dV\,, \quad \chi_\theta = \frac{\kappa_{ML}}{\theta^o}\int_{V^o}\theta^1_{,\,M}\theta^1_{,\,L}dV\,, \quad \chi_u = \frac{\kappa_{ML\alpha}}{\theta^o}\int_{V^o}\theta^1_{,\,M}u_{\alpha,\,L}dV\,.$$

By some manipulations we arrive at the modified energy balance

$$\begin{aligned}
&\frac{d}{dt}\Big(\mathcal{W}+\mathcal{K}+\mathcal{P}+\mathcal{E}+\rho_o P_K\int_{V^o}\theta^1 W^1_K\,dV\Big) + \big(\,\chi\,+\,\chi_\theta\,+\,\chi_\phi\,+\,\chi_U\,\big) \,= \\
&= \int_{V^o}f^1_\alpha\,v_\alpha\,dV \,+\, \int_{S^o}\tilde K_\alpha\,v_\alpha\,dS \,+ \\
&+ \frac{\kappa^E_{ML}}{\theta^o}\int_{S^o}\theta^1\phi^1_{,\,L}N_M\,dS \,+\, \frac{\kappa_L}{\theta^o}\int_{S^o}\theta^1 N_L\,dS \,+\, \frac{\kappa_{ML}}{\theta^o}\int_{S^o}\theta^1\theta^1_{,\,L}N_M\,dS \,+ \\
&+ \frac{1}{\theta^o}\int_{V^o}\rho_o\,\theta^1\gamma^1\,dV \,-\, \int_{S^o}\dot\Delta^1_K N_K\phi^1\,dS\,.
\end{aligned} \tag{36}$$

The energy balance (36) makes possible the proof of the uniqueness of the solution.

We assume that two distinct solutions $(u_i',\ \phi^{1\prime},\ \theta^{1\prime})$ and $(u_i'',\ \phi^{1\prime\prime},\ \theta^{1\prime\prime})$ satisfy Eqs.(24)-(26) and the appropriate boundary and initial conditions. Their difference

$$(\hat{u}_i = u_i' - u_i'', \quad \hat{\phi} = \phi^{1\prime} - \phi^{1\prime\prime}, \quad \hat{\theta} = \theta^{1\prime} = \theta^{1\prime\prime})$$

therefore satisfies the homogeneous equations (24)-(26) and the homogeneous boundary and initial conditions. Equation (36) holds for $(\hat{u}_i,\ \hat{\phi},\ \hat{\theta})$.

In view of the homogeneity of the equations and the boundary conditions, the right-hand side of Eq.(36) vanishes. Hence

$$\frac{d}{dt}\Big(\mathcal{W} + \mathcal{K} + \mathcal{P} + \mathcal{E} + \rho_o P_K \int_{V^o} \theta^1 W_K^1 \, dV\Big) \; = \; -\big(\,\chi \, + \, \chi_\theta \, + \, \chi_\phi \, + \, \chi_u\,\big) \; \leq \; 0\,, \tag{37}$$

where the last inequality is true since by (22), (36) and (23) we have

$$-\big(\,\chi \, + \, \chi_\theta \, + \, \chi_\phi \, + \, \chi_u\,\big) \; = \; \frac{1}{\theta^o}\int_{V^o} Q_M^1 \Theta_M^1 \, dV\,. \tag{38}$$

The integral in the left-hand side of Eq.(37) vanishes at the initial instant, since the functions $\hat{u}_i,\, \hat{\phi},\, \hat{\theta}$ satisfy the homogeneous initial conditions. On the other hand, by the inequality in (37) the left-hand side is either negative or zero.

Now we assume $(i - iii)$ below; note that (iii) is the sufficient condition of J. Ignaczak, written in [3] on pages 176-177.

(i) The initial deformation y_α^o realizes that the tensor $G_{M\alpha L\gamma}$ is positive-definite, so that $\mathcal{W} \geq 0$ by (34).

(ii) The tensor L_{KN} is positive-definite so that, by (35), $\mathcal{E} \geq 0$.

(iii) L_{IJ} is a known positive-definite symmetric tensor, $g_I = \rho_o P_I$ is a vector, and $c = \rho_o \alpha / 2\theta^o > 0$; consider the function

$$A(\theta^1,\, W_L) = \; (\theta^1)^2 + 2\theta^1 g_I W_I^1 + L_{IJ} W_I^1 W_J^1$$

A is nonnegative for every real pair $(\theta^1,\, W_k^1)$, provided

$$|g_I| \leq c\lambda_m$$

where λ_m is the smallest positive eigenvalue of the tensor L_{IJ}.

Under these three assumptions, (37) implies

$$\hat{u}_{i,\,L} = 0, \qquad \hat{\theta} = 0, \qquad \hat{W}_L = 0\,,$$

which imply the uniqueness of the solutions of the incremental thermoelectroelastic equations, i.e.,

$$u_i' = u_i'', \qquad \theta^{1\prime} = \theta^{1\prime\prime}, \qquad W_I^{1\prime} = W_I^{1\prime\prime}\,.$$

Moreover, from the constitutive relations we have that

$$K_{I\alpha}^{1\ \prime} = K_{I\alpha}^{1\ \prime\prime}, \qquad \Delta_L^{1\prime} = \Delta_L^{1\prime\prime}, \qquad \eta^{1\prime} = \eta^{1\prime\prime}\,.$$

5 ON THE GENERALIZED HAMILTON'S PRINCIPLE

The free energy, electric enthalpy, and potential of the heat flow are respectively defined by

$$\psi^1 \,=\, \frac{1}{2} G_{M\alpha L\gamma} u_{\alpha,\,M} u_{\gamma,\,L} + R_{LM\alpha} \phi^1_{,\,L} u_{\alpha,\,M} - \rho_o \theta^1 \Big[\Lambda_{M\alpha} u_{\alpha,\,M} - P_M \phi^1_{,\,M} + \frac{\alpha}{2} \theta^1 \Big] , \quad (39)$$

$$H^1 \,=\, \psi^1 - \frac{1}{2} L_{AB} W^1_A W^1_B \,=\, \psi^1 - \frac{1}{2} L_{AB} \Phi^1_{,\,A} \Phi^1_{,\,B} \,, \qquad \Gamma \,=\, Q^1_M \, \theta^1_{,\,M} \,. \quad (40)$$

Extending Eqs.(36)-(38) of [3], we define two functionals

$$\Pi \,=\, \int_{V^o} \Big(H^1 + \rho_o \eta^1 \theta^1 - f^1_\alpha u_\alpha \Big) dV \,-\, \int_{S^o} \Big(\tilde{K}^1_\alpha u_\alpha - \tilde{\Delta}^1 \phi^1 \Big) dS \quad (41)$$

and

$$\Psi \,=\, \int_{V^o} \Big(\Gamma - \rho_o (\eta^1 \theta^o \dot{\theta}^1 + \eta^1 \dot{\theta}^o \theta^1 + \eta^o \theta^1 \dot{\theta}^1 + \gamma^1 \theta^1) \Big) dV \,+\, \int_{S^o} \theta^1 \tilde{Q} \, dS \,. \quad (42)$$

The generalized Hamilton's principle has the form

$$\delta \int_{t_1}^{t_2} \Big(\mathcal{K} - \Pi \Big) \, dt \,=\, 0 \,, \qquad \delta \int_{t_1}^{t_2} \Psi \, dt \,=\, 0 \,. \quad (43)$$

The virtual processes $(\delta u_\alpha, \, \delta\theta^1, \, \delta\phi^1)$ of the body must be compatible with the conditions restricting the process of the body and must satisfy the conditions

$$\delta u_\alpha(\mathbf{x}, \, t_1) = \delta u_\alpha(\mathbf{x}, \, t_2) = 0, \; \delta\theta^1(\mathbf{x}, \, t_1) = \delta\theta^1(\mathbf{x}, \, t_2) = 0, \; \delta\phi^1(\mathbf{x}, \, t_1) = \delta\phi^1(\mathbf{x}, \, t_2) = 0.$$

Hence, performing the variations in the second of Eqs.(43) and observing that
$\delta H^1 \,=\, K^1_{M\alpha} \delta u_{\alpha,\,M} - \rho_o \eta^1 \delta\theta^1 + \Delta^1_L \delta\Phi^1_{,\,L}$, and

$$\int_{t_1}^{t_2} \Big(\mathcal{K} - \Pi \Big) \, dt \,=\, \int_{t_1}^{t_2} dt \Big[\int_{V^o} \Big(\frac{\rho_o}{2} \dot{u}_\alpha \dot{u}_\alpha \,-\, H^1 \,-\, \rho_o \eta^1 \theta^1 \,+\, f^1_\alpha u_\alpha \Big) dV \,+\, \int_{S^o} \Big(\tilde{K}^1_\alpha u_\alpha - \tilde{\Delta}^1 \phi^1 \Big) dS \Big],$$

we have

$$\delta \int_{t_1}^{t_2} \Big(\mathcal{K} - \Pi \Big) \, dt \,=\, \int_{t_1}^{t_2} dt \Big[\int_{V^o} \Big(- \rho_o \ddot{u}_\alpha \delta u_\alpha \,-\, K^1_{M\alpha} \delta u_{\alpha,\,M} - \Delta^1_L \delta\Phi^1_{,\,L} \,+\, f^1_\alpha \delta u_\alpha \Big) dV$$
$$+ \int_{S^o} \Big(\tilde{K}^1_\alpha \delta u_\alpha - \tilde{\Delta}^1 \delta\phi^1 \Big) dS \Big] \,=\, 0 \,. \quad (44)$$

Since the variations δu_α and $\delta\phi^1$ are arbitrary, it is easy to show that *Eq.(44) is equivalent to the equations governing the incremental motion and electric field, completed by the appropriate boundary conditions*, that are written above.

Thus (i) *the variational equation (43)*$_2$ *performed with* $\delta u_\alpha \,=\, 0 \,=\, \delta\phi^1$ *is equivalent to* (j) *the entropy balance (26) and* (jj) *the boundary condition for the heat flow*

$$Q^1_L N_L \,=\, \tilde{Q} \,, \qquad (\mathbf{x} \in S) \quad (45)$$

if and only if $\kappa_L \,=\, 0$.

Alternatively, (ii) by performing the variation (43)$_2$ with all the variations $\delta u_\alpha, \;\delta\phi^1, \;\delta\theta^1$ arbitrary, we deduce that

the variational equation (43)$_2$ *is equivalent to the entropy balance (26) and the boundary condition for the heat flow (45) if and only if* $\kappa_L = 0, \quad \kappa^E_{ML} = 0, \quad \kappa_{ML\alpha} = 0$.

6 THEOREM OF RECIPROCITY OF WORK

The theorem of reciprocity of work is extended by following and developing some steps in [3] on pages 179-182. We assume that the body is homogeneous and moreover that the initial state is static, so that in particular $\dot{\theta}^o = 0,\ \dot{\eta}^o = 0$. Here we do not assume that θ^o is uniform. The Laplace transform of functions $\nu = \nu(\mathbf{x},\, t)$,

$$\overline{\nu}(\mathbf{x},\, p) \,=\, \int_0^\infty e^{-pt}\nu(\mathbf{x},\, t)\, dt\,, \tag{46}$$

will be used below. Consider two sets of causes $\mathcal{A}^1,\ \mathcal{A}^{1\prime}$ for incremental processes, and respective effects $(u_\alpha,\, \phi,\, \theta),\ (u'_\alpha,\, \phi',\, \theta')$. Starting from the equations of motion

$$K^1_{L\alpha,L} + \rho_o f_\alpha = \rho_o \ddot{u}_\alpha\,, \qquad K^{1\prime}_{L\alpha,L} + \rho_o f'_\alpha = \rho_o \ddot{u}'_\alpha\,, \tag{47}$$

taking their Laplace transform, multiplying each by $\overline{\theta^o}$, then multiplying the first by $\overline{u}'_\alpha$ and the second by $\overline{u}_\alpha$, and making the difference of their integrals over the instantaneous region V, assuming that the initial conditions for the displacements are homogeneous, and performing some very lengthy algebraic manipulations, we obtain

$$\frac{1}{p}\Big[\int_{S^o}\Big(\overline{\theta^{1\prime}}\,\overline{Q^1_L} - \overline{\theta^1}\,\overline{Q^{1\prime}{}_L}\Big)N_L\, dS - \int_{V^o}\Big(\overline{\theta^{1\prime}_{,L}}\,\overline{Q^1_L} - \overline{\theta^1_{,L}}\,\overline{Q^{1\prime}{}_{,L}}\Big)\, dV\Big] \tag{48}$$

$$+pP_M\int_{V^o}\overline{\rho}_o\overline{\theta^o}\Big(-\overline{\theta^{1\prime}W^1_M} + \overline{\theta^1 W^{1\prime}_M}\Big)dV$$

$$+p\Big[\int_{V^o}\overline{\theta^o}\Big(\overline{F}_\alpha\overline{u}'_\alpha - \overline{F}'_\alpha\overline{u}_\alpha\Big)dV + \int_{S^o}\overline{\theta^o}\Big(\overline{K}^1_{L\alpha}\overline{u}'_\alpha - \overline{K}^{1'}_{L\alpha}\overline{u}_\alpha\Big)N_L dS$$

$$+\int_{S^o}\overline{\theta^o}\Big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\Big)N_L dS - \int_{V^o}(\overline{\theta^o})_{,L}\Big(\overline{\Delta^1{}_L}\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}{}_L}\,\overline{\phi^1}\Big)\, dV$$

$$-\int_{V^o}\overline{\theta^o}\rho_o P_L\Big(\overline{\theta^1}\,\overline{W^{1\prime}_L} - \overline{\theta^{1\prime}}\,\overline{W^1_L}\Big)dV - \int_{V^o}\overline{(\theta^o)_{,L}}\Big(\overline{K^1_{L\alpha}}\overline{u'_\alpha} - \overline{K^{1\prime}{}_{L\alpha}}\overline{u_\alpha}\Big)dV\Big] \tag{49}$$

$$+\int_{V^o}\rho_o\Big(\overline{\theta^1}\overline{\frac{\gamma^{1\prime}}{\theta^o}} - \overline{\theta^{1\prime}}\overline{\frac{\gamma^1}{\theta^o}}\Big)dV \,=\, 0\,. \tag{50}$$

The latter is the final form of the *theorem of reciprocity of work*, containing all causes and effects. It generalizes Eq.(62) of [3], and reduces exactly to the latter in case of vanishing initial fields, that is, when the initial configuration is natural.

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