An analysis of the implicit filters induced by integral-based conservation laws

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SUMMARY. A theoretical analysis of the effective implicit filtering in Finite Volume (FV) methods for LES is presented. As usual in the LES approach, the equations are explicitely filtered before they are spatial discretized and time integrated. Due to the fact that the spatial discretization introduces a discretization length, *i.e.* a further (implicit) filter, different discretization approaches are discussed. Final aim of the present research activity lies in discussing the fundamental properties of the subgrid scale term.

1 INTRODUCTION

Large Eddy Simulation (LES) of turbulent flows is a methodology based on a formal separation between large (resolved) and small (unresolved) flow scale contributions, obtained by means of the application of a low-pass filtering operator on the governing equations having the aim of reducing the degree of freedom of the problem. However, owing to the non-linearity of the equations, in order for the mathematical problem to be closed, the unresolved flow scales require some Sub-Grid Scales (SGS) modelling procedure. Nevertheless, filtering the variables in LES is often only a formalism in the writing of the equations and practically the discretization of the domain and operators is used as implicit grid-filtering [1, 2, 5]. Moreover, numerical representation of the filtered variables is associated with a finite number of resolved scales and marginal resolution hence any discrete model can induce significant alterations of the resolved scales' dynamic. Thus, while performing LES, the recognizing of the effective implicit filter in use is a critical task. Some authors analysed the suitability of using explicit filtering technique (pre-filtering), despite of their additional computational effort and loss of resolution, but for which one can exactly identify the filter type and control the truncation errors [1, 2].

This paper focuses on the theoretical recognizing of the effective implicit filtering acting while using integral-based Finite Volume (FV) methods for performing LES. Focusing here only on FV-based LES is a choice dictated by the feasibility of such method for problems of engineering interest, especially because simple generalizations are possible also on complex grids. Furthermore, it is worthwhile observing nowadays the introduction of sophisticated SGS modelling (e.g., the dynamic procedure) in commercial CFD codes very common in industrial environments that, however, exploit only FV methodologies. The goal of the present study is to link the formalism of LES filtering on the equations to the volume average over a small domain of linear measure Δ, proper of the FV approach in integral-based formulation thus assuming that the filtering is the exact top-hat filter. For FV-based formulations, although the integral form is quite more complicated to be discretized than the differential counterpart (according to [3] three levels of approximation are required, interpolation, differentiation and integration), it appears to be the most opportune since leading to solve discrete equation models, which allow mass, momentum as well as any conservative quantity, to be *a-priori* conserved, no matter of what the actual accuracy order is in effect. On the contrary, it is well known that other methods such as Spectral Methods (SM) or Finite Difference (FD) ones do not automatically share such property. Many papers discussed the form of the discretization, e.g., divergence form, skew-symmetric, etc., and the resulting numerical errors (discrete approximation plus aliasing) [4, 5], conclusions being sometimes not univocal. Particularly, it seems that the real integral-based FV discretization is someway disregarded in those analyses. A recent paper [5] analysed combined filtering effects in term of the *modified equation*. It appears generally accepted that implicit filtering causes a strong dependence on the type of the adopted spatial discretization (sometimes with "fortuitous" cancellation of the error effects). Of course, implicit filtering is characterized also by a formal lack in a grid-independent LES solution being the DNS the limiting situation for vanishing grid size.

There appear a number of interesting issues for which the discerning of the effective filter shape is relevant in a practical LES application. They are the understanding of the practical scales separation and the consequent Sub-Grid Scale (SGS) modelling, for example in fixing a value in case of the static Smagorinsky eddy viscosity model. Perhaps, also for the dynamic SGS modelling the choice of the test-filter width is influenced by the effective primary filter in effect. Last, it is worthwhile remarking that the comparisons between LES and DNS data are more rational when DNS fields are post-filtered by means of a filter function that mimics the implicit filter in effect during the LES.

2 CONTINUOUS AND DISCRETE 1D SAMPLE PROBLEMS

Consider a smooth spatially periodic function $u : [-\pi, +\pi) \times [0, +\infty) \rightarrow \mathbb{R}$ of space x and time t. As well known, it can written in terms of the Fourier series, which results to be uniformly convergent. The Fourier coefficients $\hat{u}$ are exponentially vanishing for $k \rightarrow \pm\infty$. The function u is assumed satisfying the initial value problem:

$$\begin{cases} \partial_t u + \partial_x f = g \\ u(x, 0) = u_0(x) \quad \text{given,} \end{cases} \tag{1}$$

where $f(u) = u^2/2 - \nu\, \partial_x u$ is the total flux, given by the sum of convective and diffusive (ν the kinematic viscosity coefficient) terms and g is a given forcing term, depending on both x and t but not on the function u itself. The initial data u_0 is also assumed sufficently smooth.

In correspondence with an even positive integer m ($m = 2m'$, with m' positive integer), the application of the spectral cut-off filter (indicated with the subscript "$_c$") acting outside the wavenumber interval $[-m', +m']$ to the function u leads to the new function u_c, or $\tilde{u}$ for shortness. It is defined as:

$$\tilde{u}(x, t \mid m) = \sum_{k=-m'}^{+m'} \hat{u}(k, t)\ \exp(\boldsymbol{i}kx) = \int_{-\pi}^{+\pi} d\xi\ u(\xi, t) D_{m'}(x - \xi)\ , \tag{2}$$

$D_{m'}(y) = \sin(m''y)/[2\pi \sin(y/2)]$ ($m'' = m' + 1/2$) being the Dirichlet kernel of order m'. Provided that the same integer m is used, this filter is idempotent: $\tilde{\tilde{u}} = \tilde{u}$. In the following, the integer m will be related either to the filter width Δ ($m' = [\pi/\Delta]$) or to the length h of a cell in the discrete formulation of the problem ($m' = \pi/h$). Due to the fact that u satisfies the problem (1), the following initial value problem is posed for the filtered function $\tilde{u}$ (2):

$$\begin{cases} \partial_t \tilde{u} + \partial_x \tilde{f}_c = \tilde{g} + s_c \\ s_c = \partial_x (\widetilde{\tilde{u}^2} - \widetilde{u^2})/2 \\ \tilde{u}(x,0) = \tilde{u}_0(x) \ . \end{cases} \tag{3}$$

The problem (3) is obtained by applying the above spectral filter to the corresponding one (1), by producing in this way the subgrid term s_c. It is worth noticing that the flux $f_c(\tilde{u}) = \tilde{u}^2/2 - \nu\, \partial_x \tilde{u}$ calculated in correspondence to the filtered function $\tilde{u}$ appears in the evolution equation: it possesses nonvanishing Fourier components outside the wavenumber interval $[-m', +m']$ in which $\tilde{u}$ is defined, due to the presence of the non-linear term. For this reason, a filtering of f_c is needed in order to reduce its wavenumber support to the proper interval $[-m', +m']$.

Furthermore, associated to u, its spatial mean u_m, or $\overline{u}$ for shortness, on an interval of width Δ ($\Delta' = \Delta/2$) will be also considered:

$$\overline{u}(x,t \mid \Delta) = \int_{-\pi}^{+\pi} d\xi\ u(\xi,t) G(x-\xi \mid \Delta) := \frac{1}{\Delta}\int_{x-\Delta'}^{x+\Delta'} d\xi\ u(\xi,t) = \sum_{k=-\infty}^{+\infty} \widehat{\overline{u}}(k,t \mid \Delta)\ \exp(\boldsymbol{i}kx)\ , \tag{4}$$

where $G(y \mid \Delta) = H(y+\Delta')H(\Delta'-y)/\Delta$, $H(y)$ being the Heaviside function (1 for $y > 0$, 0 for $y < 0$). In general, the filter width Δ can depend on x, so that this filtering does not commute with spatial derivatives. Moreover, if the filter width is constant, the Fourier coefficients $\widehat{\overline{u}}$ are simply related to the corresponding ones of u ($\hat{u}$) in the following way:

$$\widehat{\overline{u}}(k,t \mid \Delta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\xi\ \overline{u}(\xi,t \mid \Delta)\ \exp(-\boldsymbol{i}k\xi) = \frac{\sin(k\Delta')}{k\Delta'}\ \hat{u}(k,t) = \hat{G}(k \mid \Delta)\ \hat{u}(k,t)\ . \tag{5}$$

The above relation clearifies also that this filter is not idempotent, thus: $\overline{\overline{u}} \neq \overline{u}$.

The filtered function (4) satisfies the new Cauchy problem, written according to the weak formulation:

$$\begin{cases} \partial_t \overline{u} + \overline{\partial_x f} = \overline{g} \\ \overline{u}(x,0) = \overline{u}_0(x)\ , \end{cases} \tag{6}$$

which is obtained through filtering the one (1); the evolution equation can be rewritten highlighting the integral-based formulation:

$$\partial_t \overline{u}(x,t \mid \Delta) + \frac{f(x+\Delta',t) - f(x-\Delta',t)}{\Delta} = \overline{g}(x,t \mid \Delta)\ ,$$

the terms in f being unknown. Two approaches will be discussed below in order to close the above problem.

In the first one, the convective term is rewritten as $\partial_x \overline{u}^2/2$, leading to the divergence form of the differential LES problem:

$$\begin{cases} \partial_t \overline{u} + \partial_x f_m = \overline{g} + s_{m,d} + c_{m,d} \\ s_{m,d} = \partial_x (\overline{u}^2 - \overline{u^2})/2 \\ c_{m,d} = \quad (\partial_x \overline{u^2} - \overline{\partial_x u^2})/2 + \\ \qquad\quad +\nu\ \left[\ (\overline{\partial^2_{xx} u} - \partial_x \overline{\partial_x u}) + \partial_x(\overline{\partial_x u} - \partial_x \overline{u})\ \right] \\ \overline{u}(x,0) = \overline{u}_0(x)\ , \end{cases} \qquad \text{DIFFERENTIAL APPROACH} \tag{7}$$

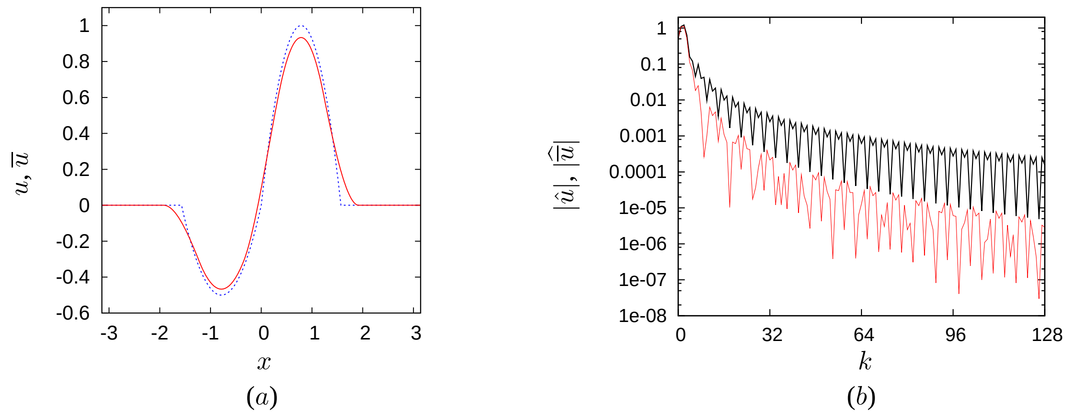

Figure 1: For a sample function $u(x)$ (blue dashed line) its filtered one $\overline{u}(x \mid \Delta)$ (red solid) with $\Delta = \pi\sqrt{5}/10$ is drawn vs. x in (a), while in (b) $|\hat{u}(k)|$ (black line) as well as $|\widehat{\overline{u}}(k \mid \Delta)|$ (red) are drawn vs. k.

in general if Δ depends on x. The flux $f_m(\overline{u}) = \overline{u}^2/2 - \nu\, \partial_x \overline{u}$ is used. Moreover, the subgrid term $s_{m,d}$ and the term $c_{m,d}$ appear. This latter one is due to the commutation between filtering and spatial derivatives: it vanishes for constant filter width Δ.

In the second approach the spatial mean is kept outside the spatial derivative, according to a finite volume approach. The Cauchy problem (6) becomes:

$$\begin{cases} \partial_t \overline{u} + \overline{\partial_x f_m} = \overline{g} + s_{m,i} \\ s_{m,i} = \overline{\partial_x (\overline{u}^2 - u^2)}/2 + \nu\, \overline{\partial^2_{xx}(u - \overline{u})} \\ \overline{u}(x,0) = \overline{u}_0(x)\ , \end{cases} \qquad \text{INTEGRAL APPROACH} \qquad (8)$$

in which only the subgrid term $s_{m,i}$ appears: no further commutation terms are involved.

According to the idea of limiting the support of u in the wavenumber space, the spectral cut-off filter can be applied to the function (4), thus obtaining the new function u_{cm}, or $\widetilde{\overline{u}}$ for shortness:

$$\widetilde{\overline{u}}(x, t \mid m, \Delta) = \sum_{k=-m'}^{+m'} \widehat{\overline{u}}(k, t \mid \Delta)\ \exp(\boldsymbol{i}kx) = \int_{-\pi}^{+\pi} d\xi\ \overline{u}(\xi, t \mid \Delta) D_{m'}(x - \xi)\ . \qquad (9)$$

Notice that $\widetilde{\overline{u}} = \overline{\tilde{u}}$: the spectral cut-off and the spatial mean can be interchanged.

The Cauchy problem which defines the new function $\widetilde{\overline{u}}$ can be obtained from the two different approaches already discussed for $\overline{u}$. In the differential formulation the initial value problem reads:

$$\begin{cases} \partial_t \widetilde{\overline{u}} + \partial_x \tilde{f}_{cm} = \widetilde{\overline{g}} + s_{cm,d} + c_{cm,d} \\ s_{cm,d} = \partial_x \left[\, \widetilde{(\widetilde{\overline{u}})^2} - \widetilde{\overline{u^2}}\, \right]/2 \\ c_{cm,d} = \ (\partial_x \overline{\widetilde{u^2}} - \overline{\partial_x \widetilde{u^2}})/2 + \\ \qquad\quad + \nu \left[\, (\overline{\partial^2_{xx} \tilde{u}} - \partial_x \overline{\partial_x \tilde{u}}) + \partial_x (\overline{\partial_x \tilde{u}} - \partial_x \widetilde{\overline{u}})\, \right] \\ \widetilde{\overline{u}}(x,0) = \widetilde{\overline{u}}_0(x)\ , \end{cases} \qquad \text{DIFFERENTIAL APPROACH} \qquad (10)$$

where the flux $f_{cm}(\widetilde{\widetilde{u}}) = \widetilde{\widetilde{u}}^2/2 - \nu\, \partial_x \widetilde{\widetilde{u}}$ has been introduced. As in the problem (3), due to the presence of the non-linear term, the flux f_{cm} possesses nonvanishing Fourier components outside the wavenumber support of $\widetilde{\widetilde{u}}$, *i.e.* the interval $[-m', +m']$. These components must be cut-off before f_{cm} is used in the evolution equation for $\widetilde{\widetilde{u}}$. A subgrid term $s_{cm,d}$ appears, as well as the term $c_{cm,d}$ which is due to the commutation among filtering and spatial derivatives and vanishes if the filter width Δ is kept constant. A quite different approach is obtained by adopting the integral point of view:

$$\begin{cases} \partial_t \widetilde{\widetilde{u}} + \overline{\partial_x \tilde{f}_{cm}} = \widetilde{\widetilde{g}} + s_{cm,i} \\ s_{cm,i} = \overline{\partial_x (\widetilde{\widetilde{u}}^2 - \widetilde{u^2})/2} - \nu\, \overline{\partial^2_{xx} (\widetilde{\widetilde{u}} - \tilde{u})} \\ \widetilde{\widetilde{u}}(x, 0) = \widetilde{\widetilde{u}}_0(x)\ , \end{cases} \qquad \text{INTEGRAL APPROACH} \tag{11}$$

which can be easily discretized in terms of finite volumes. In the LES problem (11) only the subgrid term $s_{cm,i}$ appears, while the flux $f_{cm} = \widetilde{\widetilde{u}}^2/2 - \nu \partial_x \widetilde{\widetilde{u}}$ is re-filtered by using the spectral cut-off filter, due to the doubling of the wavenumber support induced by the square term.

Finally, the discretization of the Cauchy problems for any filtered function is discussed. The interval $[-\pi, +\pi)$ is divided in an even number n ($n' = n/2$) of equal cells, having width $h = \pi/n'$ ($h' = h/2$). The n mean points: $x_j = -\pi + h' + (j-1)h$ for $j = 1, 2, \ldots, n$ (or $X(n) := \{x_1, x_2, \ldots, x_n\}$ for shortness) will be also used below. In the following, a suitable function of space x and time is semi-discretized on $X(n)$ by considering its values the points $x = x_j$ for $j = 1, \ldots, n$ as n smooth functions of time. Among the above functions, the ones (2, 9) are only considered, having a finite wavenumber support (with $m' = \pi/h = n'$). The so-called *discretized forms* of these functions are trigonometric interpolants (indicated with a *) defined through the values of the functions $\tilde{u}$ and $\widetilde{\widetilde{u}}$ on the nodes $X(n)$, as described below. For the sake of shortness, these sets of functions $\mathrm{I\!R} \to \mathrm{I\!R}^n$ (or n-dimensional column vectors, having time dependent components) are named $\tilde{U} = \tilde{U}[X(n), t]$ and $\widetilde{\widetilde{U}} = \widetilde{\widetilde{U}}[X(n), t \mid \Delta]$.

As a sample case, consider the function $\tilde{u}$. Its interpolant is defined by the following formula:

$$\tilde{u}^\star(x \mid \tilde{U}) = \sum_{k=-n'}^{+n'} \widehat{\tilde{u}}^\star(k, t)\ \exp(\boldsymbol{i}kx)\ , \tag{12}$$

the Fourier coefficients of which ($\widehat{\tilde{u}}^\star$) are evaluated by enforcing that the above function takes the prescribed value $\tilde{u}(x_j, t \mid n)$ in $x = x_j$, for $j = 1, \ldots, n$. In this way, the Fourier coefficients solve the linear system:

$$\begin{cases} \displaystyle\sum_{k=-n'}^{+n'-1} \widehat{\tilde{u}}^\star(k, t) \exp(\boldsymbol{i}kx_j) + \widehat{\tilde{u}}^\star(+n', t) \exp(\boldsymbol{i}n'x_j) = \tilde{u}(x_j, t \mid n) \qquad \text{for } j = 1, 2, \ldots, n \\ \widehat{\tilde{u}}^\star(+n', t) = -\widehat{\tilde{u}}^\star(-n', t)\ , \end{cases} \tag{13}$$

where the second condition expresses the property of the discrete Fourier coefficients: $\widehat{\tilde{u}}^\star(l + pn, t) \equiv (-1)^p \widehat{\tilde{u}}^\star(l, t)$, for $l = 0, \pm 1, \ldots, \pm n'$ and p relative integer. The system (13) leads to the definition of the k-th Fourier coefficient:

$$\widehat{\tilde{u}}^\star(k, t) = \frac{1}{n} \sum_{j=1}^{n} \tilde{u}(x_j, t \mid n)\ \exp(-\boldsymbol{i}kx_j)\ . \tag{14}$$

Notice that a comparison between the above form of the k-th Fourier coefficient (as well as the corrisponding one for $\widetilde{\overline{u}}$) and the corresponding continous one for $\tilde{u}$ (or the one (5) for $\overline{u}$ and then also for $\widetilde{\overline{u}}$) shows that the formula (14) can be obtained by considering a piecewise constant $\tilde{u}$ (or $\overline{u}$) and by evaluating the integral of the exponential on the j-th interval as $\exp(-\boldsymbol{i}kx_j)$ times the width h. By inserting the form (14) of the Fourier coefficients $\widehat{\tilde{u}}^{\star}$ inside the definition (12) of $\tilde{u}^{\star}$ it follows:

$$\tilde{u}^{\star}(x \mid \tilde{U}) = h \sum_{j=1}^{n} \tilde{u}(x_j, t \mid n)\; D_{n'}(x - x_j)\ . \tag{15}$$

As discussed above for the Fourier coefficients, also the discretized form (15) can be obtained from the corresponding continuous one (2) (or from the equation (9) for the function $\widetilde{\overline{u}}$) by using a piecewise constant function $\tilde{u}$ ($\overline{u}$) and by evaluating the integral in ξ of $D_{n'}(x-\xi)$ on the j-th cell as $hD_{n'}(x - x_j)$.

In order to simplify the notations, in the following the n-dimensional vector the components of which are the squares of the corresponding components of U will be named as U^2. With such a notation, the values of $\tilde{u}$ on X (*i.e.*, $\tilde{U}$) satisfy the following non-linear n-dimensional initial value problem (dependences on $X(n)$ are neglected):

$$\begin{cases} \partial_t \tilde{U} + d_x \tilde{F}_c = \tilde{G} + S_c \\ S_c = d_x(\widetilde{\tilde{U}^2} - \widetilde{U^2})/2 \\ \tilde{U}(0) = \tilde{U}_0 \quad \text{given,} \end{cases} \tag{16}$$

obtained by discretizing the corresponding problem (3). In the problem (16) the discretized flux F_c has the form:

$$F_c(\tilde{U}) = \tilde{U}^2/2 - \nu\; d_x \tilde{U}\ , \tag{17}$$

while d_x is a discrete operator approximating the first derivative in space (it will be represented as $\mathcal{D}/h$, $\mathcal{D}$ being an $n \times n$ matrix constant in time). $\tilde{G}$ and S_c are n-dimensional forcing vectors function of time, obtained by semi-discretizing the corresponding terms $\tilde{g}$ (forcing) and s_c (subgrid) defined in the problem (3). Finally, the solution of the problem (16) enables us to define the semi-discretized function of time $\tilde{u}^{\star}$ (12), through the use of its representation (15).

About the continuous formulations, one of the aims of our research is the rewriting of the subgrid terms s_c, $s_{m,d}$, $s_{m,i}$, $s_{cm,d}$ and $s_{cm,i}$ in the problems (3, 7, 8, 10) and (11) in terms of the correction factors on the Fourier coefficients of the product $u(\xi,t)\; u(\eta,t)$. For the sake of shortness, only two relevant intermediate steps of this analysis will be discussed below.

3 EQUIVALENT FILTER FOR THE SUBGRID TERM IN THE PROBLEM (3) FOR $\tilde{u}$

One of the main issues in solving the filtered equations is the definition of the resolved convective term, since it defines that part of wavenumber components in the resolved spectrum. As a matter of fact, the consequent decomposition in resolved and unresolved terms defines also the action of the filtered convective term in the SGS one. In fact, this latter represents the residual of the unfiltered convective term with respect to the resolved one. Generally, the filtered equations defined in spectral space are defined in literature [1] in terms of the resolved term $\partial_x \tilde{u}^2$. Perhaps, while solving the problem only for the resolved components, one must be aware that such a truncation implicitly defines the effective resolved term $\partial_x \widetilde{\tilde{u}^2}$ as well as the corresponding SGS term. In the present section, the shape of the filter $\mathcal{N}_c$ such that:

$$\widetilde{\tilde{u}^2}(x,t \mid m) = \int_S d\xi d\eta \; \tilde{u}(\xi,t \mid m) \; \tilde{u}(\eta,t \mid m) \; \mathcal{N}_c(x-\xi, x-\eta \mid m) \tag{18}$$

is now deduced. By starting from the definition of $\tilde{u}$, the term $\widetilde{\tilde{u}^2}$ is written as:

$$\begin{aligned} \widetilde{\tilde{u}^2}(x,t \mid m) &= \sum_{\substack{p,q=-m' \\ -m' \le p+q \le +m'}}^{+m'} \hat{u}(p,t)\hat{u}(q,t) \; \exp[\boldsymbol{i}(p+q)x] \\ &= \int_S d\xi d\eta \; \tilde{u}(\xi,t \mid m)\tilde{u}(\eta,t \mid m) \; \frac{1}{(2\pi)^2} \sum_{p,q\in\mathcal{L}_m} \exp[\boldsymbol{i}p(x-\xi)] \; \exp[\boldsymbol{i}q(x-\eta)] \end{aligned}$$

where the lattice on which the sum is carried out is indicated with $\mathcal{L}_m$ (see Fig. 2-a). The following form of the filter function $\mathcal{N}_c$ is obtained:

$$\mathcal{N}_c(x,y \mid m) = \frac{1}{8\pi^2} \; \frac{\sin[(x-y)/2]\cos[m''(x-y)] - \sin(x/2)\cos(m''x) + \sin(y/2)\cos(m''y)}{\sin(x/2)\sin(y/2)\sin[(x-y)/2]} \; . \tag{19}$$

It can be shown that $\mathcal{N}_c$ satisfies the important property to be invariant with respect to the convolution with the product of Dirichlet kernes $D_{m'}(\xi)D_{m'}(\eta)$. In Fig. 2 the product of the Dirichlet kernels $D_{m'}(x)D_{m'}(y)$ (b) is compared with the new filter $\mathcal{N}_c(x,y \mid m)$ (c), for the same value of m. From the analysis of Fig. 2, one can observe the difference from the expected filtering effect (b) and

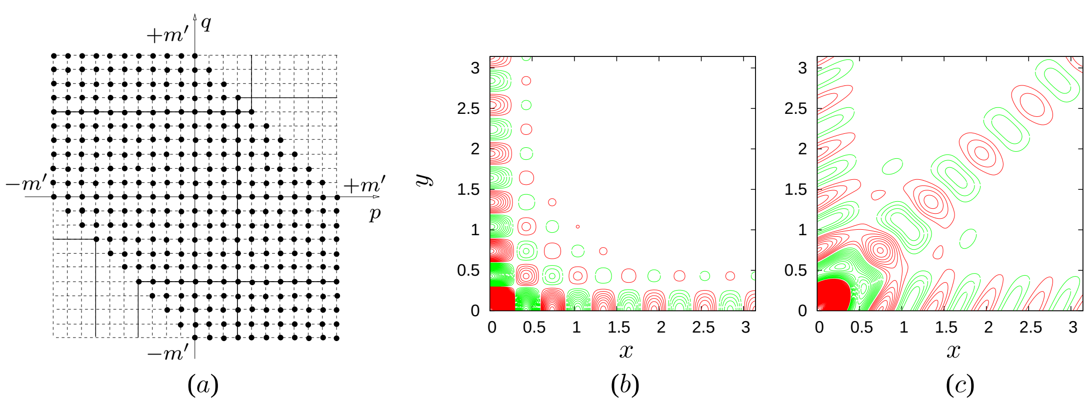

Figure 2: In (a), the wavenumber lattice $\mathcal{L}_m$ is drawn. In (b, c) the level lines of the filters $D_{m'}(x)D_{m'}(y)$ for $m' = 10$ (min: -2.5, max 11.2) and $\mathcal{N}_c(x,y \mid m)$ (min: -1.2, max: 8.3) are drawn. Red lines indicate positive levels and green ones negative, with a step 0.1.

the effective one (c). In particular, contributions from neighbourhoods of the diagonals ($x = \pm y$) are strongly amplified by the effective filter $\mathcal{N}_c$. Let us highlight that Fig. 2-c is representative of the lattice of the same extension and containing exactly the same magnitude of the resolved coefficients in a practical spectral methods (implicitly inducing the sharp cut-off filter). SGS terms are therefore only those outside the lattice. On the other hand, in case one uses local methods such as Finite Volume one, being implicitly induced a smooth transfer function (the top-hat) the coefficients contained in the lattice will be decreased accordingly. As a consequence, one gets in the SGS term part of the resolved spectrum that is somehow still recoverable.

4 THE WRITING OF $\tilde{u}$ IN TERMS OF $\widetilde{\overline{u}}$

The filter (4) enables us to evaluate the local mean $\overline{u}$, by starting from the function u, for any possible choice of the filter width Δ. On the contrary, by starting from the function $\overline{u}$ the recovering of u is possible if and only if $\Delta/(2\pi)$ is not a (positive) rational number. The simple example in which $\Delta/(2\pi) = 1/N$ (with N positive integer) clarifies this statement, even if similar considerations can be carried out for any positive rational number P/N (P positive integer). In correspondence with this choice, the Fourier representation of u is rewritten as:

$$u(x,t) = \underbrace{\sum_{\substack{k=-\infty \\ k/N \text{ not integer}}}^{+\infty} \hat{u}(k,t)\ \exp(\boldsymbol{i}kx)}_{\textit{filtered}} + \underbrace{\sum_{\substack{p=-\infty \\ p\neq 0}}^{+\infty} \hat{u}(pN,t)\ \exp(\boldsymbol{i}pNx)}_{\textit{annihilated}}\ , \tag{20}$$

by separating the filtered contributions (Fourier components having periods larger than the filter width, or smaller but not exactly submultiples of it) from the annihilated ones (components having period which is exactly a submultiple of Δ), which describe the kernel of the mean filtering operator (4). In the first term of equation (20), the Fourier coefficients are easily written in terms of the mean ones through the rule (5):

$$\hat{u}(k,t) = \frac{k\Delta'}{\sin(k\Delta')}\,\widehat{\overline{u}}(k,t) = \frac{\widehat{\overline{u}}(k,t)}{G(k \mid \Delta)}\ , \tag{21}$$

$k\Delta' = \pi k/N$ being never a multiple of π. Notice that $G(k \mid \Delta)$ is positive for $k = 0, \pm 1, \ldots, \pm(N-1)$, while it can change sign outside that interval. Moreover, if the ratio $\Delta/(2\pi)$ is not a rational number, the relation (21) can be applied to each Fourier coefficient, so that it becomes possible to write u in terms of $\overline{u}$:

$$u(x,t) = \sum_{k=-\infty}^{+\infty} \frac{\widehat{\overline{u}}(k,t)}{G(k \mid \Delta)}\ \exp(\boldsymbol{i}kx)\ , \tag{22}$$

the filter kernel being formed by the null function, only.

In the case of the spectral cut-off filtered functions $\tilde{u}(x,t \mid m)$ (4) and $\widetilde{\overline{u}}(x,t \mid m,\Delta)$ (9), $\tilde{u}$ is recovered from $\widetilde{\overline{u}}$ through the action of an equivalent filter that is computed by using the following integral representation of the cosecant:

$$\csc z = \frac{1}{\sin z} = \frac{1}{\pi}\int_0^{+\infty} d\zeta\ \frac{\zeta^{z/\pi}}{\zeta^2+\zeta} \quad \text{for } 0 < \mathrm{Re}(z)/\pi < 1.$$

In order to simplify the discussion, the quantities $\omega = \pi/\Delta$ and $\Omega = [\omega]$ are introduced, so that $\Omega\Delta'$ is smaller than $\pi/2$. By assuming $m' < 2\Omega$ (which implies that $m'/(2\omega) < \Omega/\omega < 1$), the form (22) of u with k bounded between $-m'$ and $+m'$ gives ($\eta = x - \xi$):

$$\tilde{u}(x,t \mid m) \ =$$

$$
\begin{aligned}
&= \widehat{\overline{u}}(0,t\mid\Delta) - \boldsymbol{i}\Delta' \,\Big(\sum_{k=+1}^{+m'} + \sum_{k=-1}^{-m'}\Big)\, \frac{\boldsymbol{i}\;k\;\exp(\boldsymbol{i}\;k\;x)}{\sin(k\Delta')}\,\widehat{\overline{u}}(k,t\mid\Delta) \\
&= \widehat{\overline{u}}(0,t\mid\Delta) - \boldsymbol{i}\Delta' \int_{-\pi}^{+\pi} d\xi\;\widetilde{\overline{u}}(\xi,t\mid m,\Delta)\;\frac{1}{2\pi}\frac{d}{d\eta}\left[\sum_{k=1}^{m'}\frac{\exp(+\boldsymbol{i}k\eta)}{\sin(k\Delta')} - \sum_{k=1}^{m'}\frac{\exp(-\boldsymbol{i}k\eta)}{\sin(k\Delta')}\right] \\
&= \quad \widehat{\overline{u}}(0,t\mid\Delta) + \\
&\quad + \int_{-\pi}^{+\pi} d\xi\;\widetilde{\overline{u}}(\xi,t\mid m,\Delta)\;\frac{1}{4\boldsymbol{i}\pi\omega}\int_0^{+\infty}\frac{d\zeta}{\zeta^2+\zeta}\;\times \\
&\qquad\qquad \times\frac{d}{d\eta}\left\{\sum_{k=0}^{m'}\left[\exp(+\boldsymbol{i}\eta)\zeta^{1/(2\omega)}\right]^k - \sum_{k=0}^{m'}\left[\exp(-\boldsymbol{i}\eta)\zeta^{1/(2\omega)}\right]^k\right\} \\
&= \int_{-\pi}^{+\pi} d\xi\;\widetilde{\overline{u}}(\xi,t\mid m,\Delta)\;\mathcal{F}(x-\xi\mid m,\Delta)\;, \qquad (23)
\end{aligned}
$$

in which the filter function is evaluated (at the present time, numerically) by reducing the domain of integration (in ζ) to the interval $(0,1)$. Samples of the calculation of the filter $\mathcal{F}$ (23) are shown in Fig. 3 for two different choices of m and Δ.

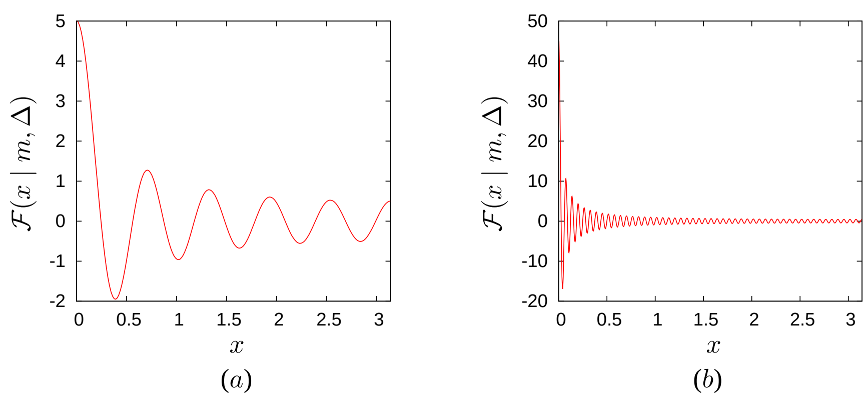

Figure 3: The filter (23) for $m = 20$, $\Delta = \pi/(5\sqrt{2})$ (a) and for $m = 200$, $\Delta = \pi/(50\sqrt{2})$ (b).

Recognizing the previous transfer functions linking the filtered velocities is someway introductory to a closure analysis of the filtered equations. This way, in the framework of the continuous form, it is possible addressing the effect of the truncation in wavenumber space when applied on the top-hat filtered velocity. Such relations express the counterpart of the deconvolution operation applied on the filtered velocity, provided filtering is smooth in wavenumbers space. Simply cut-off truncation of the unfiltered velocity is not suitable to be expressed in the inverse relation. Mainly, such relations represent the comparison terms one could consider for analysing the quality of an SGS closure modelling. They could be used once a model, such as the scale similarity one, is introduced in the filtered equations.

5 CONCLUSIONS

Application of filtering on the continuous equations is not the only and desired operation to consider for deducing the LES equations. The effective filter is consequent to formal and practical operations that need to be assessed. Classical filters are considered for expressing the LES equation in differential divergence form. Here, we considered the case of filters induced by the integral form of the equations as well as by the truncation of the Fourier components of the nonlinear term. In fact, the truncation on the single velocity is recognized in terms of local filtering and the building of the quadratic product of such filtered velocity is therefore itself contained in a certain lattice of wavenumber components. The analysis of this other filter was addressed too. It was shown that the resulting filter on the continuous non-linear terms is quite different from that one we expected for the convective term. This fact is very important, since on a side we recognize the filtering effect on the resolved field on the other side we can analyse the effective residual part in the SGS term. Further analysis is required for understanding the effects of the sampling of the fields as well as to address the filtering induced by the discretization of the operators.

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