Unsteady Couette flow of viscoelastic fluids

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SUMMARY. The differential equation of an unsteady laminar motion for a linear viscoelastic fluid is solved analytically; the fluid fills the gap between two coaxial cylinders: the internal one moves whereas the external is fixed. The start-up is analysed in detail supposing that the internal cylinder suddenly rotates with an imposed fixed angular velocity. The memory of the fluid obeys to an exponential law $M\left(T\right)=\alpha^2\exp\left(-\beta T\right)$ where α and β are two parameters defining the viscoelastic behaviour of the fluid. The momentum equation is solved applying the Laplace transform with respect to time. The two parameters α and β, characterising the fluid, can be evaluated knowing the behaviour of the torque applied to the external cylinder as function of time.

1 INTRODUCTION

Many fluids, which are of interest in engineering, in food industry, or organic fluids like blood, have a fading memory. Melts and polymers in molten state show behaviour like that of Newtonian fluids in steady shear, but a suddenly imposed deformation causes a material dependent transient associated with the dynamics of the macro-molecular chains.

The physical idea of fading memory is that more recent ("remembered") deformations of a fluid have a greater effect on the present value of the stress. The shear stress is valuable as convolution between shear rate and memory.

The aim of this paper is to study the start-up of a viscoelastic fluid in Couette axisymmetric flow. An analogous problem has been solved for steady state flow in [1] using a Giesekus model to characterize the fluid. A start-up and a pulsatile plane Couette flow with fixed walls have been studied in [2] with upper-convected Maxwell and Oldroyd-B models. A plane Couette flow with a moving wall has been examined in [3] using a Giesekus fluid and the solution is discussed in [4].

An ample review on numerical methods to solve viscoelastic flow problems, using mainly the Oldroyd-B model, is found in [5] together with a comparison of the predictive capabilities of the viscoelastic solver with experimental results.

An extensive review on the theoretical background and physical interpretation of the damping function for non-linear viscoelastic function is in [6].

2 PROBLEM STATEMENT

The relation between shear stress τ and shear-rate $\dot{\gamma}$ for a viscoelastic fluid in Couette flow is

$$\tau\left(r,t\right)=\underset{z=0}{\overset{\infty}{\mathrm{F}}}\left[\dot{\gamma}\left(r,t-z\right)\right] \tag{1}$$

where F is a suitable continuous functional. For linear viscoelasticity the previous relation reduces to

$$\tau\left(r,t\right)=\int_0^t m\left(z\right)\dot{\bar{\gamma}}\left(r,t-z\right)dz=\int_0^t m\left(t-z\right)\dot{\bar{\gamma}}\left(r,z\right)dz \tag{2}$$

the function $m\left(z\right)$ can be considered the memory of the fluid. In order to represent the behaviour of a fluid $m_0=\int_0^\infty m\left(z\right)dz$ must be limited.

We consider a fluid which fills the gap between two coaxial circular cylinders: the internal one, whose radius is R_i, can rotate, whereas the external, of radius R_e, is fixed. Using cylindrical coordinates r,θ,z, the z-axis being the cylinder axis, and supposing a laminar axisymmetric flow, the momentum equation can be written as

$$\frac{\partial\tau}{\partial r}+2\frac{\tau}{r}=\rho\frac{\partial u}{\partial t} \tag{3}$$

where ρ is the fluid density, u the tangential velocity and t the time.

For a linear viscoelastic fluid

$$\tau=\int_0^t m\left(t-z\right)\dot{\bar{\gamma}}\left(r,z\right)dz \tag{4}$$

where the shear rate is $\dot{\bar{\gamma}}=\frac{\partial u}{\partial r}-\frac{u}{r}$ (Couette flow).

Introducing the dimensionless quantities $\eta=r/R_i$, $\eta_e=r/R_e$, $w=u/V$, $\theta=\tau R_i/m_0 V$, $\dot{\gamma}=\dot{\bar{\gamma}}R_i/V$, $T=tm_0/\rho R_i^2$, $M\left(T\right)=m\left(t\right)\frac{\rho R_i^2}{m_0^2}$, where V is a reference velocity, equation (3) becomes

$$\frac{\partial\theta}{\partial\eta}+2\frac{\theta}{\eta}=\frac{\partial w}{\partial T} \tag{5}$$

In steady motion the velocity $w_0\left(\eta\right)$ is

$$w_0\left(\eta\right)=\frac{W_0}{\eta_e^2-1}\left(\frac{\eta_e^2}{\eta}-\eta\right) \tag{6}$$

where W_0 is the tangential velocity at the internal wall.

Eq. (4) in dimensionless variables becomes

$$\theta=\int_0^T M\left(T-Z\right)\dot{\gamma}\left(\eta,Z\right)dZ \tag{7}$$

and substituting in (5)

$$\int_0^T M\left(T-Z\right)\left[\frac{\partial\dot{\gamma}\left(\eta,Z\right)}{\partial\eta}+\frac{2\dot{\gamma}}{\eta}\right]dZ=\frac{\partial w}{\partial T} \tag{8}$$

that is

$$\int_0^T M\left(T-Z\right)\left[\frac{\partial^2 w\left(\eta,Z\right)}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial w}{\partial\eta}-\frac{w}{\eta^2}\right]dZ=\frac{\partial w}{\partial T} \tag{9}$$

We examine the motion which arises in the fluid initially at rest when the internal cylinder suddenly begins to rotate with constant peripheral velocity W_0.

Taking the Laplace transform with respect to T of eq. (9), we obtain

$$\hat{M}(s)\left[\frac{\partial^2 \hat{w}(\eta,s)}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \hat{w}}{\partial \eta}-\frac{\hat{w}}{\eta^2}\right]=s\hat{w} \tag{10}$$

where the hat indicates transformed functions; eq. (10) can be rewritten as

$$\eta^2\frac{\partial^2 \hat{w}(\eta,s)}{\partial \eta^2}+\eta\frac{\partial \hat{w}}{\partial \eta}-\hat{w}\left(1-\frac{\eta^2 s}{\hat{M}}\right)=0 \tag{11}$$

Imposing the boundary conditions

$$\hat{w}(1,s)=\frac{W_0}{s} \qquad \hat{w}(\eta_e,s)=0 \tag{12}$$

the solution of (11) is

$$\hat{w}(\eta,s)=A(s)\,\mathrm{J}_1\left(i\eta\sqrt{s/\hat{M}}\right)+B(s)\,\mathrm{Y}_1\left(i\eta\sqrt{s/\hat{M}}\right) \tag{13}$$

where J_1 and Y_1 are the Bessel functions of order one, of first and second kind respectively. Imposing the boundary conditions (12) gives

$$A=\frac{1}{\Delta}\frac{W_0}{s}\mathrm{Y}_1\left(i\eta_e\sqrt{s/\hat{M}}\right) \tag{14}$$

$$B=-\frac{1}{\Delta}\frac{W_0}{s}\mathrm{J}_1\left(i\eta_e\sqrt{s/\hat{M}}\right) \tag{15}$$

where

$$\Delta=\mathrm{J}_1\left(i\sqrt{s/\hat{M}}\right)\mathrm{Y}_1\left(i\eta_e\sqrt{s/\hat{M}}\right)-\mathrm{Y}_1\left(i\sqrt{s/\hat{M}}\right)\mathrm{J}_1\left(i\eta_e\sqrt{s/\hat{M}}\right) \tag{16}$$

Fig. 1 shows the behaviour of $\Delta(\xi)=\mathrm{J}_1(\xi)\mathrm{Y}_1(\eta_e\xi)-\mathrm{Y}_1(\xi)\mathrm{J}_1(\eta_e\xi)$ for different values of η_e.

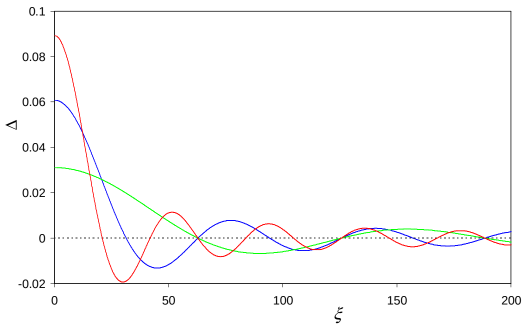

Figure 1: behaviour of Δ for different values of η_e:

red line $\eta_e=1.15$, blue line $\eta_e=1.10$, green line $\eta_e=1.05$

We can retrieve $w(\eta,T)$ inverting its Laplace transform $\hat{w}(\eta,s)$:

$$w\left(\eta,T\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sT}\hat{w}\left(\eta,s\right)\mathrm{d}\,s \tag{17}$$

The integral can be evaluated using the residue theorem:

$$w\left(\eta,T\right)=\sum \mathrm{Res}\left[e^{sT}\hat{w}\left(\eta,s\right)\right] \ . \tag{18}$$

The integrand $e^{sT}\hat{w}\left(\eta,s\right)$ has a first order pole in $s=0$ and a countable infinity of first order poles when $\Delta=0$. The equation $\Delta=0$ is solvable only numerically and admits a countable infinity of roots when $i\sqrt{s/\hat{M}}=\lambda_n$ (λ_n are real and can be ordered as an ascending sequence which tends to infinity), and then

$$\lambda_n^2\hat{M}\left(s_n\right)+s_n=0 \qquad\qquad n=1,2,3\ldots \tag{19}$$

When $n\to\infty$,

$$\lambda_{n+1}-\lambda_n\to\frac{\pi}{\eta_e-1} \tag{20}$$

For $s=0$ the following expression can be found:

$$\mathrm{Res}\left[e^{sT}\hat{w}\left(\eta,s\right)\right]_{s=0}=\lim_{s\to0} s\hat{w}\left(\eta,s\right)e^{sT}=\frac{W_0}{\eta_e^2-1}\left(\frac{\eta_e^2}{\eta}-\eta\right) \tag{21}$$

which gives the asymptotic steady velocity, whereas for $s=s_n$

$$\mathrm{Res}\left[e^{sT}\hat{w}\left(\eta,s\right)\right] \tag{22}$$

where

$$N_n\left(\eta\right)=W_0\left[\mathrm{Y}_1\left(\eta_e\lambda_n\right)\mathrm{J}_1\left(\eta\lambda_n\right)-\mathrm{Y}_1\left(\eta\lambda_n\right)\mathrm{J}_1\left(\eta_e\lambda_n\right)\right] \tag{23}$$

$$\begin{aligned}D_n\left(\eta\right)=s_n\Big\{&\mathrm{Y}_1\left(\eta_e\lambda_n\right)\mathrm{J}_0\left(\lambda_n\right)-\mathrm{J}_1\left(\eta_e\lambda_n\right)\mathrm{Y}_0\left(\lambda_n\right)+\\&+\eta_e\left[\mathrm{Y}_0\left(\eta_e\lambda_n\right)\mathrm{J}_1\left(\lambda_n\right)-\mathrm{J}_0\left(\eta_e\lambda_n\right)\mathrm{Y}_1\left(\lambda_n\right)\right]\Big\}\left.\frac{\partial f}{\partial s}\right|_{s_n}=F_n s_n\left.\frac{\partial f}{\partial s}\right|_{s_n}\end{aligned} \tag{24}$$

being $f=i\sqrt{s/\hat{M}}$.

We now suppose that the memory of the fluid can be expressed by the relation

$$M\left(T\right)=\alpha^2\exp\left(-\beta T\right) \tag{25}$$

where α and β ($\beta>0$) are suitable rheological parameters of the fluid, and then

$$\hat{M}\left(s\right)=\frac{\alpha^2}{s+\beta} \ , \tag{26}$$

$$f=\left(i/\alpha\right)\sqrt{s\left(s+\beta\right)} \ . \tag{27}$$

It results

$$s_n=-\frac{\beta}{2}\pm\sqrt{\frac{\beta^2}{4}-\alpha^2\lambda_n^2} \tag{28}$$

if $\dfrac{\beta^2}{4}-\alpha^2\lambda_n^2\geq 0$, s_n is real and negative and

$$\left.\frac{\partial f}{\partial s}\right|_{s_n}=\frac{1}{\alpha}\sqrt{\frac{\beta^2}{4\alpha^2\lambda_n^2}-1} \tag{29}$$

otherwise s_n is a complex number whose real part is always negative. The sequence λ_n is increasing and then a value n_1 of n exists, such that

$$\lambda_n>\frac{\beta}{2\alpha}\quad \text{for } n\geq n_1 \tag{30}$$

and thus

$$s_n=-\frac{\beta}{2}\pm i\sqrt{\alpha^2\lambda_n^2-\frac{\beta^2}{4}}=-\frac{\beta}{2}\pm ih_n \tag{31}$$

and the corresponding term of the solution oscillates with frequency $h_n/2\pi$.
Being

$$w\left(\eta,T\right)=\frac{W_0}{\eta_e^2-1}\left(\frac{\eta_e^2}{\eta}-\eta\right)+\sum_n\frac{N_n\left(\eta\right)}{D_n}\exp\left(s_nT\right) \tag{32}$$

the shear stress at the external cylinder, where $w=0$, is

$$\theta\left(\eta_e,T\right)=\int_0^T\alpha^2\exp\left[-\beta\left(T-Z\right)\right]\left.\frac{\partial w}{\partial\eta}\right|_{\eta_e}dZ \tag{33}$$

but

$$\left.\frac{\partial w}{\partial\eta}\right|_{\eta_e}=-\frac{2W_0}{\eta_e^2-1}+\sum_n\frac{1}{D_n}\left.\frac{\partial N_n\left(\eta\right)}{\partial\eta}\right|_{\eta_e}\exp\left(s_nT\right)=-\frac{2W_0}{\eta_e^2-1}+\sum_n C_n\exp\left(s_nT\right) \tag{34}$$

where

$$\begin{aligned}C_n&=\frac{1}{D_n}\left.\frac{\partial N_n\left(\eta\right)}{\partial\eta}\right|_{\eta_e}=\frac{W_0\lambda_n}{D_n}\left[\mathrm{Y}_1\left(\eta_e\lambda_n\right)\mathrm{J}_0\left(\eta_e\lambda_n\right)-\mathrm{Y}_0\left(\eta_e\lambda_n\right)\mathrm{J}_1\left(\eta_e\lambda_n\right)\right]=\\&=-\frac{2W_0}{\pi\eta_eD_n}\end{aligned} \tag{35}$$

and then

$$\begin{aligned}\theta\left(\eta_e,T\right)=\theta_e\left(T\right)&=\int_0^T\alpha^2\exp\left[-\beta\left(T-Z\right)\right]\left[-\frac{2W_0}{\eta_e^2-1}+\sum_n C_n\exp\left(s_nZ\right)\right]dZ=\\&=\alpha^2\exp\left(-\beta T\right)\left\{-\frac{2W_0}{\beta\left(\eta_e^2-1\right)}\left[\exp\left(\beta T\right)-1\right]+\right.\\&\left.+\sum_n\frac{C_n}{\beta+s_n}\left[\exp\left(\left(\beta+s_n\right)T\right)-1\right]\right\}\end{aligned} \tag{36}$$

which can be expressed as

$$\theta\left(\eta_e,T\right) = -\frac{2\alpha^2 W_0}{\beta\left(\eta_e^2-1\right)}\left[1-\exp\left(-\beta T\right)\right] \pm \alpha^2 \sum_1^{n_1-1} \frac{2W_0}{\pi \eta_e F_n \lambda_n \left|h_n\right|} \cdot$$
$$\cdot \left[\exp\left(s_n T\right) - \exp\left(-\beta T\right)\right] - \alpha^2 \sum_{n_1}^{\infty} \frac{4W_0}{\pi\eta_e F_n \lambda_n h_n} \sin\left(h_n T\right)\exp\left(-\beta T/2\right) \tag{37}$$

The torque applied to the external cylinder is

$$\mathsf{M}_e\left(T\right) = 2\pi\eta_e \theta_e\left(T\right) \tag{38}$$

where all terms but the first vanish as $T \to \infty$; the last terms decrease oscillating in time with period

$$P_n = 2\pi / h_n = 4\pi / \sqrt{4\alpha^2\lambda_n^2 - \beta^2} \tag{39}$$

When $T \to \infty$ the torque at the external wall becomes

$$\mathsf{M}_\infty = \lim_{T\to\infty} \mathsf{M}_e\left(\eta_e, T\right) = -\frac{4\pi\eta_e\alpha^2 W_0}{\beta\left(\eta_e^2-1\right)} \tag{40}$$

Knowing the geometry, i.e. the external radius η_e and the velocity W_0 of the internal cylinder, measuring M_∞ and the greatest period of oscillating terms $P_{n_1} = 4\pi / \sqrt{4\alpha^2\lambda_{n_1}^2 - \beta^2}$ we can evaluate the parameters of the fluid, α^2 and β. It results

$$\alpha^2 = -\frac{\mathsf{M}_\infty\left(\eta_e^2-1\right)}{4\pi\eta_e W_0}\beta \; ; \tag{41}$$

β satisfies the equation

$$\beta^2 + \frac{\lambda_{n_1}^2 \mathsf{M}_\infty\left(\eta_e^2-1\right)}{\pi\eta_e W_0}\beta + \frac{16\pi^2}{P_{n_1}^2} = 0 \tag{42}$$

i.e.

$$\beta = -\frac{\lambda_{n_1}^2 \mathsf{M}_\infty\left(\eta_e^2-1\right)}{2\pi\eta_e W_0} \pm \sqrt{\frac{\lambda_{n_1}^4 \mathsf{M}_\infty^2\left(\eta_e^2-1\right)^2}{4\pi^2\eta_e^2 W_0^2} - \frac{16\pi^2}{P_{n_1}^2}} \tag{43}$$

λ_{n_1} is the smallest λ_n which makes non negative the radical quantity of (43), i.e.

$$\lambda_{n_1}^2 \geq \frac{8\pi^2\eta_e W_0}{\left|\mathsf{M}_\infty\right| P_{n_1}\left(\eta_e^2-1\right)} \tag{44}$$

In eq. (43) the minus sign must be chosen: β must have the same value for every pair of values λ_n, P_n; the solution with plus sign increases increasing n, and than must be rejected. Then

$$\beta = -\frac{\lambda_n^2 \mathsf{M}_\infty\left(\eta_e^2-1\right)}{2\pi\eta_e W_0} - \sqrt{\frac{\lambda_n^4 \mathsf{M}_\infty^2\left(\eta_e^2-1\right)^2}{4\pi^2\eta_e^2 W_0^2} - \frac{16\pi^2}{P_{n_1}^2}} \tag{45}$$

If a period is not clearly identifiable, e.g. when the damping is very high, eq. (37) must be considered, expressing α and h_n as function of β and M_∞; measuring the value $\bar{M}_e$ of M_e at a fixed time $\bar{T}$, β and α can be evaluated.

3 NUMERICAL EXAMPLE

If the plot torque vs. time is like Fig. 2, a period is clearly recognizable and the rheological parameters α and β can be evaluated using a plot like Fig. 3. For example, supposing $\eta_e = 1.10$, Fig. 2 gives $M_\infty = -131$ and $T = 0.1$, and Fig. 3 allows to find $\alpha \simeq 2$, $\beta \simeq 2$.

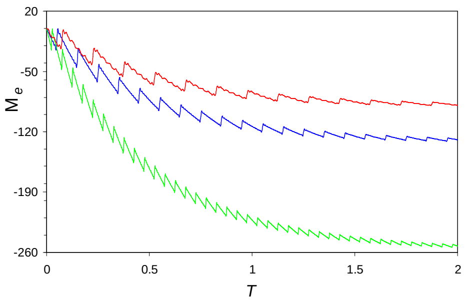

Figure 2: torque at the external cylinder vs. time:
red line $\eta_e = 1.15$, blue line $\eta_e = 1.10$, green line $\eta_e = 1.05$

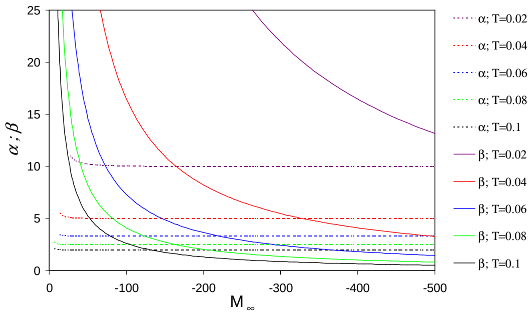

Figure 3: α and β vs. asymptotic torque for some values of period ($\eta_e = 1.10$)

When the behaviour in time of the torque is like Fig. 4, it is not possible to recognize any period; measuring the asymptotic torque M_∞, α can be expressed as a function of β using (41). At a fixed a time $\bar{T}$, equation (37) can then be considered as a function of β only, and the torque $\bar{M}_e = M_e(\bar{T})$ can be plotted: Fig. 5 has been plotted supposing $\eta_e = 1.10$, evaluating $M_\infty = -0.0033$ from Fig. 4 and choosing $\bar{T} = 0.8$. At time $\bar{T} = 0.8$ Fig. 4 gives $\bar{M}_e = -0.01045$, which, introduced in Fig. 5, allows to evaluate $\beta \simeq 2$ and, using (41) $\alpha \simeq 0.01$.

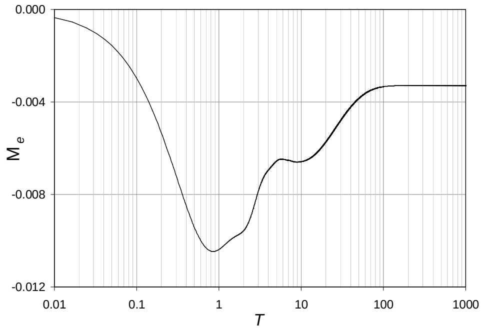

Figure 4: torque at the external cylinder vs. time in semi-logarithmic scale, for $\eta_e = 1.10$

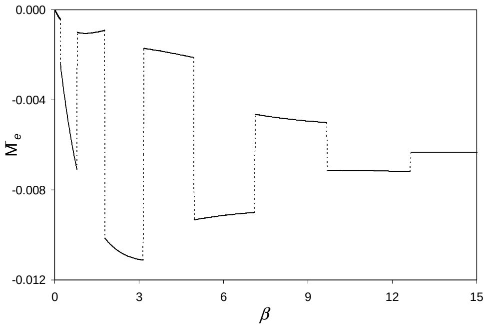

Figure 5: torque for $\eta_e = 1.10$ and $\overline{T} = 0.8$ vs. $\boldsymbol{\beta}$

4 CONCLUSIONS

The start-up of a viscoelastic fluid filling the gap between two coaxial cylinders is analyzed analytically supposing that the internal cylinder suddenly rotates with an imposed fixed peripheral velocity. The memory of the fluid obeys to an exponential law $M\left(T\right) = \alpha^2 \exp\left(-\beta T\right)$ where α and β are two parameters defining the behaviour of the fluid. The momentum equation is solved applying the Laplace transform with respect to time: the solution in the image domain has a countable infinity of first order poles and then can be inverted using the residue theorem. The solution is expressed as a convergent series, which as time tends to infinity approaches the steady distribution. The steady state is not reached monotonically as Newtonian fluids do, but with damped oscillations of different frequencies. The two parameters α and β, characterizing the fluid, can be evaluated knowing the behaviour of the torque as a function of time.

References

[1] Ravanchi, M.T., Mirzazadeh, M., Rashidi, F., “Flow of Giesekus viscoelastic fluid in a concentric annulus with inner cylinder rotation” *Int. J. Heat Fluid Flow* **28**, 838-845 (2007)

[2] Duarte, A.S.R., Miranda, A.I.P., Oliveira, P.J., “Numerical and analytical modeling of unsteady viscoelastic flows: The start-up and pulsating test case problems” *J. Non-Newtonian Fluid Mech.* **154**, 153-169 (2008)

[3] Raisi A., Mirzazadeh M., Dehnavi A.S., Rashidi F.; An approximate solution for the Couette-

Poiseuille flow of the Giesekus model between parallel plates. *Rheol Acta* **47**, 75-80 (2008)

[4] Daprà I., Scarpi G., “Couette-Poiseuille flow of the Giesekus model between parallel plates” *Rheol. Acta* **48**, 117-120 (2009)

[5] Baaijens, F.T.P., “Mixed finite element methods for viscoelastic flow analysis: a review” *J. Non-Newtonian Fluid Mech.* **79**, 361-385 (1998)

[6] Rolón-Garrido, V.H., Wagner, M.H. “The damping function in rheology” *Rheol. Acta* **48**, 245-284 (2009)